E C O N O M I C S   B U L L E T I N

A Unified Theory of Implementation

Ismail Saglam
Bogazici University

Abstract

This paper unifies the theories of Nash implementation and Bayesian implementation. Environments considered are such that each agent's characteristics include, in addition to a specification of his private information, a commonly known type parameter, while both attributes are unknown to the designer. Each social choice correspondence (SCC) assigns a commonly known type vector to a social choice set. Conditions that fully characterize an implementable SCC in economic environments where agents are not satiated generalize and merge respective conditions in the complete information model of Danilov (1992) and the incomplete information model of Jackson (1991).

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1. Introduction

In this paper, we provide a unified framework in which the theories of both Nash implementation and Bayesian (Nash) implementation can be accommodated. We also discuss whether Danilov's (1992) notion of essential elements can be used to fill the gap between the necessary and sufficient conditions of implementation in noneconomic environments.

An environment is called economic if agents cannot be simultaneously satiated, and noneconomic otherwise. The problem of implementing social choice sets in both economic and noneconomic environments involving agents that have incomplete information about the state of the society is examined by Jackson (1991). He defines social choice functions from states to allocations, and social choice sets as collections of social choice functions. His contributions establish that social choice sets are Bayesian implementable only if they satisfy closure (C), incentive compatibility (IC), and Bayesian monotonicity (BM) conditions. Moreover, these three conditions are sufficient to implement a social choice set in any economic environment involving at least three agents.[1] Unfortunately, the same sufficiency result does not hold in noneconomic environments. Jackson shows that a social choice set in any noneconomic environment is implementable if it satisfies (C), (IC) and monotonicity-no-veto (MNV), a condition combining Bayesian monotonicity and no-veto conditions. Since (MNV) is not necessary, there exists a gap between necessary and sufficient conditions for Bayesian implementation in noneconomic environments. Jackson is quick to realize that Danilov's (1989) single condition, namely essential monotonicity (EM), that characterizes Nash implementable social choice correspondences can be helpful along this line.[2]

Danilov (1989, 1992) shows that any Nash implementable social choice correspondence (SCC) - from preferences to alternatives - is essentially monotone. Conversely, if a SCC is essentially monotone and there are at least three agents in the environment, then the SCC is implementable via Nash equilibria. Essential monotonicity is stronger than monotonicity, a necessary condition for Nash implementation. On the other hand, essential monotonicity is weaker than monotonicity + no-veto power, which are sufficient conditions of Nash implementation when there are at least three agents, a fact proved by Maskin (1977, 1999). Analogously, to reduce the gap in Bayesian implementation, we wish carefully translate (EM) to get a condition stronger than (BM) while weaker than (MNV).

The environment that we consider differs from that of Jackson in two aspects. First, an agent's characteristics include, in addition to a specification of his private information, a commonly known type parameter. The two attributes are both unknown to the designer. Second, instead of social choice sets, we deal with social choice correspondences assigning the commonly known types of individuals to social choice sets. Like in Jackson's model, however, each social choice function within a given social choice set maps the private type profiles to allocations. The problem of implementation is then to design a strategic outcome function whose equilibria for any environment coincides with the social choice correspondence.

[1]See, also, Matsushima (1990) for similar results in economic environments.

[2]A full characterization of necessary and sufficient conditions of Nash implementation is due to Moore and Repullo (1990). Danilov's (1989, 1992) single condition reduces Moore and Repullo's three conditions to one.

The distinction in the environment with regard to the previous literature has an important implication. In a single framework, we merge Nash implementation model and Bayesian implementation model. We show that conditions characterizing implementable social choice correspondences select, up to some required generalizations and modifications, from the respective conditions for Nash implementation and Bayesian implementation.

Any SCC in our framework is implementable only if it satisfies conditions generalizing Danilov's essential monotonicity and Jackson's closure, incentive compatibility and Bayesian monotonicity provided that the domain of preferences is sufficiently rich. In economic environments involving rich preference domains and at least three agents, the same conditions are also sufficient to fully implement a SCC. However, in noneconomic environments the sufficiency conditions must involve a generalized monotonicity-no-veto (GMNV) condition replacing generalized Bayesian monotonicity.

Two particular cases within our unified framework are of a special interest. In one extreme case in which the information set of each agent is a singleton, the model boils down to the Nash implementation model considered by Danilov. In the other extreme case in which the society is known to have a single type, the model coincides with Bayesian implementation model of Jackson.

The paper proceeds as follows: Section 2 introduces the environment that heavily borrows from Jackson (1991), and defines social choice correspondences. Section 3 provides the definitions that generalize and merge the notions in the Bayesian model of Jackson and the complete information model of Danilov. In Section 4 we describe the implementation problem, and in Section 5 we unify the theories of Nash implementation and Bayesian implementation.

2. Basic Structures

Environments

There are a finite number, N, of agents. Agent i has two attributes θ^i and s^i. The parameter θ^i is common knowledge while s^i is privately known by agent i. Henceforth, we will use the term *type* for θ^i and *information set* for s^i.

Let Θ^i be the set of possible types of agent i. A type profile is a vector $\theta = (\theta^1, ..., \theta^N)$ and the set of all type profiles is $\Theta = \Theta^1 \times ... \times \Theta^N$. Let S^i describe the finite number of possible information sets of agent i. A *state* is a vector $s = (s^1, ..., s^N)$ and the set of states is $S = S^1 \times ... \times S^N$. Both the type profile and the state of the society are unknown to the designer.

Let A denote the set of feasible allocations. We assume A is fixed across states.

A *social choice function* is a map from states to allocations. The set of all social choice functions is $X = \{x | x : S \to A\}$.

Each agent i has a probability measure q^i defined on S.[3] It is assumed that if $q^i(s) > 0$ for some i and $s \in S$, then $q^j(s) > 0$ for all $j \neq i$. All agents agree on that T denotes the set of states which occur with positive probability, where $T = \{s \in S | q^i(s) > 0, \ \forall i\}$.

[3]For notational simplicity and with no loss of generality in our results, we assume that q^i is type-independent.

The sets Π^i are partitions of T defined by q^i. For a given information set $s^i \in S^i$, $\pi^i(s^i) = \{t \in S | t^i = s^i \text{ and } q^i(t) > 0\}$ denotes the set of states which agent i believes may be the true state. It is assumed that $\pi^i(s^i) \neq \emptyset$ for all i and $s^i \in S^i$. Let Π denote the finest partition which is coarser than each Π^i. For a given state $s \in S$, let $\pi(s)$ be the element of Π which contains s.

A preference is a linear order on X. The set of all preferences is denoted as $\mathcal{R}$. Each agent has preferences over social choice functions which have a conditional expected utility representation. Given $x, y \in X$, $s^i \in S^i$, and $\theta \in \Theta$, agent i's *weak preference relation* $R^i(s^i, \theta^i) \in \mathcal{R}$ is such that

$$xR^i(s^i, \theta^i)y \Leftrightarrow \sum_{s \in \pi^i(s^i)} q^i(s)U^i[x(s), s, \theta^i] \geq \sum_{s \in \pi^i(s^i)} q^i(s)U^i[y(s), s, \theta^i],$$

where $U^i : A \times S \times \Theta^i \to I\!R_+$ is a state and type dependent utility function. Preferences are complete and transitive. The strict preference and indifference relations associated with R^i are P^i and I^i, respectively.

An *environment* is a collection $[N, S, \Theta, A, \{q^i\}, \{U^i\}]$, whose structure is assumed to be common knowledge among agents.

Social Choice Correspondences

A *social choice correspondence* (SCC) is a nonempty subset $F \subset \Theta \times X$ (or $F : \Theta \Rightarrow X$). A SCC F assigns to every type profile $\theta \in \Theta$, a social choice set $F(\theta) \subset X$, i.e., a collection of social choice functions.

3. Definitions

Here, we generalize several notions in the Bayesian model of Jackson and the complete information model of Danilov.

Let $L(x, R^i(s^i, \theta^i))$ be the set of social choice functions to which agent i of type θ^i weakly prefers x at state s^i. This set is defined by $L(x, R^i(s^i, \theta^i)) = \{y \in X | xR^i(s^i, \theta^i)y\}$.

Definition 1: The social choice functions x and y are *equivalent* if $x(s) = y(s)$ for all $s \in T$. The social choice correspondences F and $\hat{F}$ are *equivalent* if for each θ and $x \in F(\theta)$ there exists $\hat{x} \in \hat{F}(\theta)$ which is equivalent to x, and for each θ and $\hat{x} \in \hat{F}(\theta)$ there exists $x \in F(\theta)$ which is equivalent to $\hat{x}$.

Definition 2: Let $x/_C z$ be a splicing of two social choice functions x and z along a set $C \in S$. The social choice function $x/_C z$ is defined by $[x/_C z](s) = x(s)\ \forall s \in C$, and $[x/_C z](s) = z(s)$ otherwise.

Definition 3: An environment is said to be *economic* if for any $z \in X$, $\theta \in \Theta$ and $s \in S$, there exist i and j $(i \neq j)$, $x \in X$ and $y \in X$ such that x and y are constant, $x/_C z \notin L(z, R^i(s^i, \theta^i))$ and $y/_C z \notin L(z, R^j(s^j, \theta^j))$ for all $C \subset S$ such that $s \in C$. An environment is called *noneconomic* if it is not economic.

DEFINITION 4: Let B and D be any disjoint sets of states such that $B \cup D = T$ and for any $\pi \in \Pi$ either $\pi \subset B$ or $\pi \subset D$. Consider a SCC F, and $\theta \in \Theta$. The social choice set $F(\theta)$ satisfies *closure* (C) if for any $x, y \in F(\theta)$, there exists $z \in F(\theta)$ such that $z(s) = x(s)$ $\forall s \in B$ and $z(s) = y(s)$ $\forall s \in D$. The SCC F satisfies *generalized closure* (GC) if for all θ, $F(\theta)$ satisfies (C).

Given a vector or vector of functions $v = (v^1, ..., v^N)$, the list $(v^{-i}, \tilde{v}^i)$ represents the vector $(v^1, ..., v^{i-1}, \tilde{v}^i, v^{i+1}, ..., v^N)$.

DEFINITION 5: Given i, $x \in X$, and $t^i \in S^i$, define x_{t^i} by $x_{t^i}(s) = x(s^{-i}, t^i)$, $s \in S$. Consider a SCC F, and $\theta \in \Theta$. The social choice set $F(\theta)$ satisfies *incentive compatibility* (IC) if for all i, $x \in F(\theta)$, and $t^i \in S^i$,

$$x_{t^i} \in L(x, R^i(s^i, \theta^i)) \quad \forall s^i \in S^i.$$

The SCC F satisfies *generalized incentive compatibility* (GIC) if for all θ, $F(\theta)$ satisfies (IC).

DEFINITION 6: A *deception* for i is a mapping $\alpha^i : S^i \to S^i$. Let $\alpha = (\alpha^1, ..., \alpha^N)$ and $\alpha(s) = [\alpha^1(s^1), ..., \alpha^N(s^N)]$. The notation $x \circ \alpha$ represents the social choice function which results in $x[\alpha(s)]$ for each $s \in S$.

DEFINITION 7: Consider a SCC F, $\theta \in \Theta$, $x \in F(\theta)$ and a deception α. The social choice set $F(\theta)$ satisfies *Bayesian monotonicity* (BM) if whenever there is no social choice function in $F(\theta)$ which is equivalent to $x \circ \alpha$, there exists $i, s^i \in S^i$ and $y \in X$ such that

$$y \circ \alpha \notin L(x \circ \alpha, R^i(s^i, \theta^i)) \ \text{ while } \ y_{\alpha^i(s^i)} \in L(x, R^i(t^i, \theta^i)) \quad \forall t^i \in S^i.$$

The SCC F satisfies *generalized Bayesian monotonicity* (GBM) if for all θ, $F(\theta)$ satisfies (BM).

DEFINITION 8: A social choice function $z \in X$ satisfies the *no-veto hypothesis* (NVH) for α, θ and $D \subset T$, if for each $s \in D$ there exists i such that for each $j \neq i$ and $\tilde{z} \in X$ there is a set $C \subset D$ such that $s \in C$ and $\tilde{z} \circ \alpha /_C z \in L(z, R^j(s^j, \theta^j))$.

DEFINITION 9: Consider a SCC F, a deception α, and for each $\hat{\theta} \in \Theta$, $x \in F(\hat{\theta})$, and i, a set $B^i_{x,\hat{\theta}} \subset S^i$. Let $B_{x,\hat{\theta}} = B^1_{x,\hat{\theta}} \times ... \times B^N_{x,\hat{\theta}}$. Suppose that there exists $z \in X$ such that for each $\hat{\theta} \in \Theta$, $x \in F(\hat{\theta})$ and $s \in B_{x,\hat{\theta}}$, $z(s) = x \circ \alpha(s)$. Furthermore, suppose that z satisfies (NVH) for α, θ and $T - (\cup_{\hat{\theta} \in \Theta} \cup_{x \in F(\hat{\theta})} B_{x,\hat{\theta}})$. F satisfies *generalized-monotonicity-no-veto* (GMNV) if whenever there is no social choice function in $F(\theta)$ which is equivalent to z, there exist i, $\hat{\theta} \in \Theta$, $x \in F(\hat{\theta})$, $s \in \cup_{\bar{\theta} \in \bar{\Theta}_x} B_{x,\bar{\theta}}$ where $\bar{\Theta}_x = \{\theta : x \in F(\theta)\}$, and $y, \tilde{z}$, and $\bar{z} \in X$, such that $\bar{z}(t) = y \circ \alpha(t)$ when $t \in \cup_{\bar{\theta} \in \bar{\Theta}_x} B_{x,\bar{\theta}}$; $\bar{z}(t) = z(t)$ when $t^{-i} \in \cup_{\bar{\theta} \in \bar{\Theta}_{\bar{x}}} B^{-i}_{\bar{x},\bar{\theta}}$ for some $\bar{x}$ such that $\bar{x} \neq x$; and $\bar{z}(t) = \tilde{z} \circ \alpha(t)$ otherwise; and

$$\bar{z} \notin L(z, R^i(s^i, \theta^i)), \ \text{ while } \ y_{\alpha^i(s^i)} \in L(x, R^i(t^i, \theta^i)) \quad \forall t^i \in S^i.$$

If Θ is a singleton, every SCC is a social choice set; hence (GC), (GIC), (GBM) and (GMNV), respectively, reduce to the conditions (C), (IC), (BM) and (MNV) defined by Jackson (1991) for social choice sets.[4]

[4]For the intuition underlying the involving construction of (GMNV), see the last paragraph of the proof of Lemma 4 that provides sufficiency conditions to show that given any $\theta \in \Theta$ and any equilibrium σ in the game $G(M, \mu, \theta)$, there exists a social choice function $z \in F(\theta)$ which is equivalent to $\mu(\sigma)$.

Definition 10: Let i be an agent and $Y \subset X$. A social choice function $y \in Y$ is *essential* for i in set Y if $y \in F(\theta)$ for some $\theta \in \Theta$ and $L(y, R^i(s^i, \theta^i)) \subset Y$ for all $s^i \in S^i$.

Given the social choice correspondence F, the set of all essential elements for i in $Y \subset X$ is denoted by $Ess(F; i, Y)$ or simply $Ess(i, Y)$. Obviously $Ess(F; i, Y) \subset Y$, and if $Z \subset Y \subset X$ then $Ess(i, Z) \subset Ess(i, Y)$. Moreover, $Ess(F; i, X) = \cup_{\theta \in \Theta} F(\theta)$.

Definition 11: The SCC F satisfies *generalized-essential monotonicity* (GEM) if for $\theta, \hat{\theta} \in \Theta$, and $x \in F(\theta)$ the relations

$$Ess(F; i, L(x, R^i(t^i, \theta^i))) \subset L(x, R^i(t^i, \hat{\theta}^i)) \quad \forall t^i \in S^i \text{ and } \forall i$$

imply $x \in F(\hat{\theta})$.

Generalized-essential-monotonicity means that the social choice function x survives not only at an improvement of position of x at all states but also when its position gets nonessentially worse at some states of the society. In the case in which S is a singleton, (GEM) boils down to Danilov's essential monotonicity condition (EM).

Definition 12: Consider any linear order $\tilde{R} \in \mathcal{R}$. The environment satisfies *rich domain hypothesis* (RDH) if for each i there exists $\theta^i \in \Theta^i$ such that $R^i(s^i, \theta^i) = \tilde{R}$ for all $s^i \in S^i$.

4. Implementation

A *mechanism* is an action space $M = M^1 \times ... \times M^N$ and a map $\mu : M \to A$.

A *strategy* for agent i is a map $\sigma^i : S^i \to M^i$. Denote by Σ^i the set of all strategies for agent i, and define $\Sigma = \Sigma^1 \times ... \times \Sigma^N$.

For any $\sigma \in \Sigma$, $\mu(\sigma)$ represents the social choice function which results when σ is played.

Let θ be a type profile. A vector of strategies $\sigma \in \Sigma$ is a *Bayesian (Nash) equilibrium* in the game $G(M, \mu, \theta)$ if $\mu(\sigma^{-i}, \tilde{\sigma}^i) \in L(\mu(\sigma), R^i(s^i, \theta^i))$ for all i, s^i and $\tilde{\sigma}^i \in \Sigma^i$. In other words, $\mu(\sigma^{-i}, \Sigma^i) \subset L(\mu(\sigma), R^i(s^i, \theta^i))$ for all i and s^i.

Let $BE(\mu, \theta)$ be the set of all Bayesian equilibria in the game $G(M, \mu, \theta)$. Then the set of all *equilibrium outcomes* in this game is defined by $E(\mu, \theta) = \mu(BE(\mu, \theta))$.

A mechanism (M, μ) implements a social choice correspondence F if:

(i) for any $\theta \in \Theta$ and $x \in F(\theta)$ there exists an equilibrium $\sigma \in BE(\mu, \theta)$ with $\mu[\sigma(s)] = x(s)$ for all $s \in T$, and

(ii) for any $\theta \in \Theta$ and any equilibrium $\sigma \in BE(\mu, \theta)$ there exists $x \in F(\theta)$ with $\mu[\sigma(s)] = x(s)$ for all $s \in T$.

In other words, the mechanism (M, μ) implements F if $E(\mu)$ is equivalent to F. A social choice set F is *implementable* if there exists a mechanism (M, μ) which implements F.

5. Unifying Theories of Nash Implementation and Bayesian Implementation

This section begins with the description of essential elements for the equilibrium outcomes correspondence, $E(\mu)$. Next, we establish that $E(\mu)$ satisfies the condition (GEM) if the domain of preferences is sufficiently rich. These results actually extend similar results by Danilov (1992) obtained for the complete information case to our Bayesian framework.

LEMMA 1: *Assume the environment satisfies (RDH). Consider a mechanism* (M, μ), *a set* $Y \subset X$, *a social choice function* $y \in Y$ *and agent* i. *Then,* $y \in Ess(E(\mu); i, Y)$ *if and only if* $y = \mu(\sigma^{-i}, \sigma^i)$ *where* $\mu(\sigma^{-i}, \Sigma^i) \subset Y$.

LEMMA 2: *If the environment satisfies (RDH), then, for any mechanism* (M, μ), *the correspondence* $E(\mu)$ *satisfies (GEM).*

THEOREM 1: *Assume the environment is economic, satisfies (RDH) and* $N \geq 3$. *A SCC* F *is implementable if and only if there exists a SCC* $\hat{F}$ *which is equivalent to* F *and satisfies (GC), (GIC), (GEM), and (GBM).*

The assumptions that the environment is economic and $N \geq 3$ are only needed for the sufficiency part of the Theorem. If we drop the assumption that the environment is economic, we have the following sufficiency theorem.

THEOREM 2: *Assume the environment satisfies (RDH) and* $N \geq 3$. *A SCC* F *is implementable if there exists a SCC* $\hat{F}$ *which is equivalent to* F *and satisfies (GC), (GIC), (GEM), and (GMNV).*

Note here that when the type space Θ is finite, the environment $[N, S, \Theta, A, \{q^i\}, \{U^i\}]$ can be shown to have the same information structure and preferences as the environment $[N, \hat{S}, A, \{\hat{q}^i\}, \{\hat{U}^i\}]$ (a standart setting in Bayesian models) with $\hat{S}$, $\{\hat{q}^i\}$ and $\{\hat{U}^i\}$ appropriately defined. Thus, Theorems 1 and 2 could be proven as corollaries to the corresponding theorems in Jackson (1991) in situations where Θ is finite. This means that (GEM) is redundant when Θ is finite.

In analyzing our first two results, two particular cases are of a special interest. First, consider an environment with a single state of the society. Then, the collection of SCC's which are equivalent to a SCC F consists simply of F itself, and every SCC satisfies (GC), (GIC), (EGBM) and (GMNV) regardless (RDH) holds. So, in both economic and noneconomic environments (GEM) becomes the unique sufficiency condition if S is a singleton. Moreover, (GEM) reduces to (EM) in such a case. Thus, we obtain the following result by Danilov (1992) as a straightforward corollary to our previous two theorems.

COROLLARY 1: Assume the environment satisfies (RDH), $\#S = 1$ and $N \geq 3$. A social choice correspondence F is implementable if and only if F satisfies (EM).

Consider now the other extreme case in which the type space contains a single element. In this case, (GEM) has no bite, whereas (GC), (GIC), (GBM) and (GMNV) reduce to (C), (IC), (BM) and (MNV), respectively. In addition, any SCC is a social choice set now. Thus, we obtain Theorem 1 and Theorem 2 in Jackson (1991) as separate corollaries to our first and second Theorems, respectively.

COROLLARY 2: *Assume the environment is economic,* $\#\Theta = 1$ *and* $N \geq 3$. *A SCC* F *is implementable if and only if there exists a SCC* $\hat{F}$ *which is equivalent to* F *and satisfies (C), (IC) and (BM).*

COROLLARY 3: *Assume* $\#\Theta = 1$ *and* $N \geq 3$. *A SCC* F *is implementable if there exists a SCC* $\hat{F}$ *which is equivalent to* F *and satisfies (C), (IC), and (MNV).*

Remark 1: Assume the environment is noneconomic and satisfies (RDH). The conditions (GC), (GIC), (GEM) and (GBM) are not sufficient for implementation.

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Appendix

Proof of Lemma 1: To show the "only if" part, let $y \in Ess(E(\mu); i, Y)$. Then $y \in E(\mu, \theta)$ for some θ such that $y = \mu(\hat{\sigma}^{-i}, \hat{\sigma}^i)$ where $\mu(\hat{\sigma}^{-i}, \Sigma^i) \subset L(y, R^i(s^i, \theta^i)) \subset Y$ for all $s^i \in S^i$. Conversely, let $y = \mu(\sigma^{-i}, \sigma^i)$ and $\mu(\sigma^{-i}, \Sigma^i) \subset Y$. Since the environment satisfies (RDH), let θ be such that $L(y, R^i(s^i, \theta^i)) = Y$ for all s^i and $y = \max_j R^j(s^j, \theta^j)$ for all $j \neq i$ and s^j. It is obvious that (σ^{-i}, σ^i) is Bayesian equilibrium in the game $G(M, \mu, \theta)$, and therefore $y \in Ess(E(\mu); i, Y)$. *Q.E.D.*

Proof of Lemma 2: Let σ be a Bayesian equilibrium in the game $G(M, \mu, \theta)$, and $x = \mu(\sigma)$. Then, $\mu(\sigma^{-i}, \Sigma^i) \subset L(x, R^i(t^i, \theta^i))$ for all i and t^i. By Lemma 1, $\mu(\sigma^{-i}, \Sigma^i) \subset Ess(E(\mu); i, L(x, R^i(t^i, \theta^i)))$ for all i and t^i. Let $\hat{\theta}$ be a type profile satisfying

$$Ess(F; i, L(x, R^i(t^i, \theta^i))) \subset L(x, R^i(t^i, \hat{\theta}^i))$$

for all i and t^i. It follows that $\mu(\Sigma^i, \sigma^{-i}) \subset L(x, R^i(t^i, \hat{\theta}^i))$, for all i and t^i. Therefore, σ is a Bayesian equilibrium in the game $G(M, \mu, \hat{\theta})$, and $x \in E(\mu, \hat{\theta})$. *Q.E.D.*

Proof of Remark 1: We extend Example 1 in Jackson (1991) in order to prove the claim in Remark 1. Consider the environment in which $N = 4$, $A = \{a, b\}$, $\Theta^i = \{\theta_1^i, \theta_2^i, \theta_3^i, \theta_4^i\}$, $S^i = \{s^i, t^i\}$, and $T = \{s_1 = (s^1, s^2, s^3, s^4); s_2 = (s^1, s^2, t^3, t^4); s_3 = (t^1, t^2, t^3, t^4)\}$, the partitions pictured below represent the information structure implied by T:

	States
Agents 1 and 2	$[s_1 \quad s_2]$ $[s_3]$
Agents 3 and 4	$[s_1]$ $[s_2 \quad s_3]$

The functional form of the utility functions of agents 1 and 2 is the same as is that of agents 3 and 4. The utilities representing the preferences are given below. Preferences satisfy rich domain hypothesis since the set $\times_{i=1}^{4}\{\theta_1^i, \theta_2^i, \theta_3^i\}$, alone, constitutes a rich domain.

	Agents 1 and 2		Agents 3 and 4	
	a	b	a	b
$U^i(., s_1, \theta_1^i)$	2	1	1	2
$U^i(., s_2, \theta_1^i)$	2	1	1	2
$U^i(., s_3, \theta_1^i)$	2	1	1	2
$U^i(., s_1, \theta_2^i)$	1	1	1	1
$U^i(., s_2, \theta_2^i)$	1	1	1	1
$U^i(., s_3, \theta_2^i)$	1	1	1	1
$U^i(., s_1, \theta_3^i)$	1	2	2	1
$U^i(., s_2, \theta_3^i)$	1	2	2	1
$U^i(., s_3, \theta_3^i)$	1	2	2	1
$U^i(., s_1, \theta_4^i)$	2	1	1	2
$U^i(., s_2, \theta_4^i)$	1	1	1	1
$U^i(., s_3, \theta_4^i)$	2	1	1	2

Consider the social choice set $F(\theta) = \{x, \bar{x}\}$ for all $\theta \in \Theta$, where $x(s) = a$ for all $s \in S$ and $\bar{x}(s) = b$ for all $s \in S$.

F satisfies (GEM) since F is constant on Θ. F satisfies (GC) since the common knowledge concatenation satisfies $\Pi = \{T\}$. Condition (GIC) is satisfied since x and $\bar{x}$ are constant. Since $x \circ \alpha = x$ and $\bar{x} \circ \alpha = \bar{x}$ for every deception α, it follows that for every $\theta \in \Theta$, $x \circ \alpha \in F(\theta)$ and $\bar{x} \circ \alpha \in F(\theta)$ for every deception α, and so (GBM) is satisfied.

Although F satisfies (GC), (GIC), (GEM) and (GBM), it is not implementable. To see this, suppose that a mechanism (M, μ) implements F. Let $\theta_4 = (\theta_4^1, \theta_4^2, \theta_4^3, \theta_4^4)$. Then there exist equilibrium sets of strategies $\sigma_x, \sigma_{\bar{x}} \in BE(\mu, \theta_4)$ resulting in x and $\bar{x}$ on T, respectively. Consider the set of strategies $\tilde{\sigma}$ defined by $\tilde{\sigma}^i(s^i) = \sigma_x(s^i)$ and $\tilde{\sigma}^i(t^i) = \sigma_{\bar{x}}(t^i)$. Since each agent i is completely indifferent at (s_2, θ_4^i), $\tilde{\sigma}$ is an equilibrium. Notice that $\mu[\tilde{\sigma}(s_1)] = a$ and $\mu[\tilde{\sigma}(s_3)] = b$. However, there is no social choice function in $F(\theta_4)$ which coincides with $\mu[\tilde{\sigma}]$ on T, which is a contradiction. Therefore, F is not implementable. Q.E.D.

The proofs of Theorem 1 and Theorem 2 closely follow the respective proofs in Jackson (1991) established for social choice sets.

Proof of Theorem 2: The following mechanism, which slightly extends the mechanism proposed by Jackson for social choice sets, implements the SCC F if the conditions of Theorem 2 are met. Let $\bar{S} = \max_i \#S^i$ and $n = N + N\bar{S}$. Let $V = \{0, 1, ..., \bar{S}^2\}^n$. Thus $v \in V$ is an $(N + N\bar{S})$-dimensional vector such that each entry is an integer between 0 and $\bar{S}^2$. Let $M^i = \{m^i \in \Theta \times S^i \times \cup_\theta F(\theta) \times \{\emptyset \cup V\} \times X \times \{\emptyset \cup X\} \,|\, m_3^i \in F(m_1^i)\}$ and $M = M^1 \times ... \times M^N$. Partition M into sets:

$$\begin{aligned} d_0 &= \{m \in M | \exists x \in F(\theta) \text{ s.t. } m^j = (\theta, \cdot, x, \emptyset, \cdot, \emptyset)\ \forall j\}, \\ d_1^i &= \{m \in M | m \notin d_0,\ \exists x \in F(\theta) \text{ s.t. } m^j = (\theta, \cdot, x, \emptyset, \cdot, \emptyset)\ \forall j \neq i \\ &\qquad \text{and } m^i = (\cdot, \cdot, x, \cdot, \cdot, \emptyset) \text{ or } (\cdot, \cdot, \bar{x}, \cdot, \cdot, \cdot)\}, \\ d_2^i &= \{m \in M | \exists x \in F(\theta) \text{ s.t. } m^j = (\theta, \cdot, x, \emptyset, \cdot, \emptyset)\ \ \forall j \neq i \end{aligned}$$

$$\text{and } m^i = (\cdot,\cdot,x,\cdot,\cdot,y)\},$$

$$d_3 \;=\; \{m \in M | m \notin d_1 \cup d_2\}.$$

Let $d_2 = \cup_i d_2^i$ and $d_1 = \cup_i d_1^i$.

Define the payoff function $\mu : M \to X$ by

$$\begin{array}{lll} \mu(m) = x(m_2), & m \in d_0 \cup d_1, & \\ \mu(m) = y(m_2), & m \in d_2^i \quad \text{and} & y_{m_2^i} \in L(x, R^i(t^i,\theta^i)) \text{ for all } t^i \in S^i, \\ \mu(m) = x(m_2), & m \in d_2^i \quad \text{and} & y_{m_2^i} \notin L(x, R^i(t^i,\theta^i)) \text{ for some } t^i \in S^i, \\ \mu(m) = m_5^{i^*}(m_2), & m \in d_3, & \end{array}$$

where i^* is determined as follows: Let $I^* = \{i | m_4^i \neq \emptyset\}$ and for $i \in I^*$ denote m_4^i by v^i. Let $J(i)$ be the number of $j \in I^*$ such that $v_l^i = v_i^j$ for an integer l where $N+(j-1)\bar{S} < l \leq N+j\bar{S}$. If there exists $i \in I^*$ such that $J(i) > J(k)$ for all $k \in I^*$, then $i^* = i$, otherwise $i^* = 1$.

Remark 4: For any i and σ there exists $v^i \in V$ such that such that $\tilde{\sigma}^i$, where $\tilde{\sigma}_4^i(s^i) = v^i$ for all s^i and $\tilde{\sigma} = \sigma$ otherwise, is such that $i^* = i$ whenever $[\sigma^{-i}, \tilde{\sigma}^i](s) \in d_3$.

The following lemmas establish Theorem 2.

Lemma 3: *If F satisfies (GIC), then for each θ and $x \in F(\theta)$ there is a set of strategies σ which form an equilibrium to the game $G(M,\mu,\theta)$ such that $\mu(\sigma) = x$.*

Proof: Given an arbitrary $\theta \in \Theta$, $x \in F(\theta)$, we consider σ defined by $\sigma^i(s^i) = (\theta, s^i, x, \emptyset, \cdot, \emptyset)$. Notice that $\mu[\sigma(s)] = x(s)$ for all $s \in S$. We verify that σ is an equilibrium by showing that there are no improving deviations. Consider a deviation $\tilde{m}^i$ by i at $s^i \in S^i$.

If $\tilde{m}^i = (\tilde{\theta}, \tilde{s}^i, x, \cdot, \cdot, \emptyset)$ or $\tilde{m}^i = (\tilde{\theta}, \tilde{s}^i, \bar{x}, \cdot, \cdot, \cdot)$ then $[\sigma^{-i}(s^{-i}), \tilde{m}^i] \in d_0 \cup d_1$ (where it is possible that $\tilde{\theta} = \theta$ and $\tilde{s}^i = s^i$). The resulting allocation is $x_{\tilde{s}^i}$ (on $\pi^i(s^i)$). From (GIC) we know that this is not improving.

If $\tilde{m}^i = (\tilde{\theta}, \tilde{s}^i, x, \cdot, \cdot, y)$, then $[\sigma^{-i}(s^{-i}), \tilde{m}^i] \in d_2$ (where it is possible that $\tilde{\theta} = \theta$ and $\tilde{s}^i = s^i$). If $y_{\tilde{s}^i} \in L(x, R^i(t^i,\theta^i))$ for all $t^i \in S^i$, then the allocation is $y_{\tilde{s}^i}$ (on $\pi^i(s^i)$), which is not improving. Otherwise the allocation is $x_{\tilde{s}^i}$ (on $\pi^i(s^i)$), which is not improving by (GIC).

Lemma 4: *If F satisfies (GC), (GEM) and (GMNV), then for each set of strategies σ which form an equilibrium to the game $G(M,\mu,\theta)$ there exists $z \in F(\theta)$ which is equivalent to $\mu(\sigma)$.*

Proof: Let σ be an equilibrium to $G(M,\mu,\theta)$ and let α describe the announcement of s (m_2 as a function of s) under σ. For each i, $\hat{\theta} \in \Theta$ and $x \in F(\hat{\theta})$, let $B^i_{x,\hat{\theta}} = \{s^i : \sigma^i(s^i) = (\hat{\theta}, \alpha^i(s^i), x, \emptyset, ., \emptyset)\}$.

Since σ is an equilibrium, $\mu(\sigma)$ satisfies (NVH) for α, θ and $T - (\cup_{\hat{\theta}\in\Theta} \cup_{x\in F(\hat{\theta})} B_{x,\hat{\theta}})$. This is seen as follows. Suppose that $\mu(\sigma)$ does not satisfy (NVH) for α, θ and $T - (\cup_{\hat{\theta}\in\Theta} \cup_{x\in F(\hat{\theta})} B_{x,\hat{\theta}})$. Then there exist $s \in T - (\cup_{\hat{\theta}\in\Theta} \cup_{x\in F(\hat{\theta})} B_{x,\hat{\theta}})$, j, and z^j such that $z^j \circ \alpha \,/_C \mu(\sigma) \notin L(\mu(\sigma), R^j(s^j,\theta^j))$ for all $C \subset T - (\cup_{\hat{\theta}\in\Theta} \cup_{x\in F(\hat{\theta})} B_{x,\hat{\theta}})$ such that $s \in C$. Since the failure of (NVH) guarantees the existence of two such agents, and since $s \notin (\cup_{\hat{\theta}\in\Theta} \cup_{x\in F(\hat{\theta})} B_{x,\hat{\theta}})$, j can be chosen such that $\sigma(s) \notin d_1^j \cup d_2^j$. Let $\tilde{m}^j$ be the same as $\sigma^j(s^j)$

except that $\tilde{m}_4^j = v^j$ as defined in Remark 4, and $m_5^j = z^j$. Let C be the set of $t \in \pi^j(s^j)$ such that $[\sigma^{-j}(t^{-j}), \tilde{m}^j] \in d_3$. The outcome on C is thus $z^j \circ \alpha$. Furthermore, $s \in C$, since $[\sigma^{-j}(s^{-j}), \tilde{m}^j] \in d_3$, and $C \subset T - (\cup_{\hat{\theta}\in\Theta} \cup_{x\in F(\hat{\theta})} B_{x,\hat{\theta}})$. From the design of $\tilde{m}^j$ it follows that if $t \in \pi^j(s^j)$ and $t \notin C$, then $[\sigma^{-j}(t^{-j}), \tilde{m}^j]$ leads to the same outcome as σ. Hence, the outcome of the deviation is $z^j \circ \alpha$ on $C \cap \pi^j(s^j)$ and $\mu(\sigma)$ otherwise. This is improving for j, which contradicts the fact that σ is an equilibrium.

It has been established that $\mu(\sigma)$ satisfies (NVH) for α, θ, and $T - (\cup_{\hat{\theta}\in\Theta} \cup_{x\in F(\hat{\theta})} B_{x,\hat{\theta}})$. Next, (MNV) is applied to find a social choice function in $F(\theta)$ which is equivalent to $\mu(\sigma)$.

Suppose that there does not exist a social choice function in $F(\theta)$ which is equivalent to $\mu(\sigma)$. By $(GMNV)$ there exist i, $\hat{\theta} \in \Theta$, $x \in F(\hat{\theta})$, $y, \tilde{z}, \bar{z}$ and $s^i \in \cup_{\bar{\theta}\in\bar{\Theta}_x} B^i_{x,\bar{\theta}}$, where $\bar{\Theta}_x = \{\theta : x \in F(\theta)\}$, such that $\bar{z}(s) = y \circ \alpha$ when $s \in \cup_{\bar{\theta}\in\bar{\Theta}_x} B_{x,\bar{\theta}}$; $\bar{z}(s) = \mu[\sigma(s)]$ when $s^{-i} \in \cup_{\bar{\theta}\in\bar{\Theta}_{\bar{x}}} B^{-i}_{\bar{x},\bar{\theta}}$ for some $\bar{x}$ such that $\bar{x} \neq x$; and $\bar{z}(s) = \tilde{z} \circ \alpha$ otherwise; and such that $\bar{z} \notin L(\mu(\sigma), R^i(s^i, \theta^i))$, while $y_{\alpha^i(s^i)} \in L(x, R^i(t^i, \theta^i))$ $\forall t^i \in S^i$. Therefore i is better off submitting $[\hat{\theta}, \alpha^i(s^i), x, v^i, \tilde{z}, y]$ (where v^i is defined in Remark 4) whenever s^i is observed, since the resulting outcome is $\bar{z}$ on $\pi^i(s^i)$. This is shown as follows: The deviation puts the action in d_2^i for all $s \in \cup_{\bar{\theta}\in\bar{\Theta}_x} B_{x,\bar{\theta}}$, and the outcome is $y \circ \alpha$. The action is in d_1^i for all $s \in \pi^i(s^i)$ such that $s^{-i} \in \cup_{\bar{\theta}\in\bar{\Theta}_{\bar{x}}} B^{-i}_{\bar{x},\bar{\theta}}$ for some $\bar{x}$ such that $\bar{x} \neq x$, and the outcome remains $\mu[\sigma(s)]$. For any other $s \in \pi^i(s^i)$ the deviation puts the action in d_3 with $i^* = i$ and the outcome $\tilde{z} \circ \alpha(s)$. Thus the outcome is $\bar{z}$ on $\pi^i(s^i)$ which is strictly preferred by i to $\mu(\sigma)$ on $\pi^i(s^i)$. This contradicts that σ is an equilibrium, and so the supposition was wrong. Q.E.D.

PROOF OF THEOREM 1: The sufficiency part follows from Theorem 2. In an environment which satisfies (E), (NVH) can never be satisfied. Therefore given (GC), (GMNV) and (GBM) are equivalent. The necessity part of the theorem is now checked.

Let μ implement F and define $\hat{F}$ such that $\hat{F}(\theta) = \{x | x = \mu(\sigma)$ for some equilibrium σ in the game $G(M, \mu, \theta)\}$. From the definition of implementation $\hat{F}$ is equivalent to F. It is obvious that $\hat{F}$ satisfies (GC). Consider any $\theta \in \Theta$. $\hat{F}(\theta)$ satisfies (IC) and (BM), by the proof of Theorem 1 in Jackson (1991). So, $\hat{F}$ satisfies (GIC) and (GBM). Since the environment satisfies (RDH), $E(\mu)$ satisfies (GEM) by Lemma 2. Thus, $\hat{F}$ satisfies (GEM) since $\hat{F} = E(\mu)$. Q.E.D.