E C O N O M I C S   B U L L E T I N

A new characterization of absolute qualified majority voting

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Abstract

We show that the class of absolute qualified majority voting rules are the only ones to satisfy Anonymity, Neutrality, Monotonicity, Weak Pareto and Decisiveness Non-Equivalence. When there are two alternatives x and y, the latter axiom states that if an individual voting for y can improve the result of x by abstaining, then it is not the case that an individual abstaining can improve the result of x by voting for x.

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1 Introduction

In a recent article, [Houy, 2006] proposed an original characterization of the class of voting rules that we will call M_k. These voting rules were introduced by [Fishburn, 1973] and [Saari, 1990] and have been more recently studied by [Llamazares, 2006], and [Sanver, 2006]. When there are two alternatives x and y, x (resp. y) is chosen according to M_k if the number of votes in favor of x (resp. y) outnumbers by k the number of votes in favor of y (resp. x). These voting rules ensure that the majority obtained by the winner is significant enough. The characterization given in [Houy, 2006] uses the usual axioms of Anonymity, Neutrality, Monotonicity, Weak Pareto and the original axiom of Decisiveness Equivalence. This axiom states that individuals voting for y are decisive when abstaining if and only if individuals abstaining are decisive when voting for x.

In real politics, when an important decision has to be made and when one wants to be sure that the majority obtained by the winner is significant enough, absolute qualified majority voting rules M^k are often implemented. When there are two alternatives, x and y, x (resp. y) is chosen according M^k if the number of votes in favor of x (resp. y) is greater than k. For instance, in the European Council, decisions are made according to M^k voting rules with different values of k, depending on the issue. In this paper, we show that the class of M^k voting rules can be characterized by the axioms of Anonymity, Neutrality, Monotonicity, Weak Pareto and Decisiveness Non-Equivalence. This last axiom is a form of contradiction of Decisiveness Equivalence since it states that if individuals voting for y are decisive when abstaining, then individuals abstaining are not decisive when voting for x. An independent study, [Aşan and Sanver, 2006], showed that absolute qualified majority voting rules are the only ones to satisfy Anonimity, Neutrality and Maskin Monotonicity.[1]

2 Notation

Let $N = \{1, ..., n\}$ be the set of individuals in the society. Each individual expresses his choice between two alternatives, x and y. The vector of all the individuals' votes is a voting configuration, $V = (V_1, ..., V_n) \in \{-1, 0, 1\}^n =$

[1] Another characterization is given in [Austen-Smith and Banks, 1999] and restated in [Sanver, 2006].

$\mathcal{V}$. If $V_i = 1$ (resp. $V_i = -1$), individual i votes for x (resp. y). If $V_i = 0$, individual i abstains. Let σ be a permutation of N, V_σ is the voting configuration defined by $V_\sigma = (V_{\sigma(1)}, ..., V_{\sigma(n)})$. $V + (i, t)$ is the voting configuration defined by $V + (i, t) = (V_1, ..., V_{i-1}, t, V_{i+1}, ..., V_n)$. Then, $V + (i, t)$ is the voting configuration obtained from V when individual $i \in N$ changes his vote for $t \in \{-1, 0, 1\}$. For $V \in \mathcal{V}$, we define $n^+(V) = \#\{i \in N/V_i = 1\}$, $n^-(V) = \#\{i \in N/V_i = -1\}$ and $n^0(V) = \#\{i \in N/V_i = 0\}$, where $\#J$ is the cardinality of the set J.

A voting rule C is a mapping from $\mathcal{V}$ onto $\{-1, 0, 1\}$. Then, $C(V)$ is the alternative chosen by the society when V is the voting configuration.

We say that i is (-10)decisive at V for C if and only if $V_i = -1$ and $C(V + (i, 0)) > C(V)$. Then, i is (-10)decisive if he can positively influence the social choice by changing his vote from -1 to abstention. Analogously, we say that i is (01)decisive at V for C if and only if $V_i = 0$ and $C(V + (i, 1)) > C(V)$.

We now give some axioms for voting rules.

Axiom 1 (Anonymity, A)
The voting rule C satisfies Anonymity if and only if for all σ, permutation of N, $\forall V \in \mathcal{V}, C(V_\sigma) = C(V)$.

Axiom 2 (Neutrality, N)
The voting rule C satisfies Neutrality if and only if $\forall V \in \mathcal{V}, C(-V) = -C(V)$.[2]

Axiom 3 (Monotonicity, M)
The voting rule C satisfies Monotonicity if and only if $\forall V \in \mathcal{V}$ and $\forall i \in N$ such that $V_i \leq 0$, $C(V + (i, V_i + 1)) \geq C(V)$.

Axiom 4 (Weak Pareto, WP)
The voting rule C satisfies Weak Pareto if and only if $C(1, ..., 1) = 1$ and $C(-1, ..., -1) = -1$.

Axiom 5 (Decisiveness Equivalence, DE)
The voting rule C satisfies Decisiveness Equivalence if and only if $\forall V \in \mathcal{V}$ and $\forall i, j \in N$ such that $V_i = 0$ and $V_j = -1$,
i is (01)decisive at V for C if and only if j is (-10)decisive at V for C.

[2] For $V \in \mathcal{V}$, $-V$ is defined by $-V = (-V_1, ..., -V_n)$.

Axiom 6 (Decisiveness Non-Equivalence, DNE)
The voting rule C satisfies Decisiveness Non-Equivalence if and only if $\forall V \in \mathcal{V}$ *and* $\forall i, j \in N$ *such that* $V_i = 0$ *and* $V_j = -1$,
i is (01)decisive at V for C implies that j is not (-10)decisive at V for C.

The first four axioms are usual in Voting Theory and we omit their interpretation. The fifth axiom has been introduced in [Houy, 2006]. It states that there is equivalence between (01)decisiveness and (-10)decisiveness whenever it is possible. On the contrary, the sixth axiom states that whenever an individual is (01)decisive, no individual is (-10)decisive at the same voting configuration. Obviously, this axiom also states that whenever an individual is (-10)decisive, no individual is (01)decisive at the same voting configuration. Then, according to DNE, at a voting configuration, either the individuals abstaining or the individuals voting for the alternative that is not socially chosen can be decisive. However, the two categories of individuals cannot be decisive simultaneously. We will further discuss DE and DNE after we give the results.

We define the two voting rules studied in this article. The first one is the well-known absolute qualified majority voting.

Definition 1 (M^k)
Let k be an integer in $[n/2, n-1]$. The voting rule M^k is defined by $\forall V \in \mathcal{V}$,

$$M^k(V) = \begin{cases} 1 & \text{if } n^+(V) > k \\ -1 & \text{if } n^-(V) > k \\ 0 & \text{otherwise.} \end{cases}.$$

The second one is majority voting based on difference of votes. It is sometimes denoted relative majority rules in the literature.

Definition 2 (M_k)
Let k be an integer in $[0, n-1]$. The voting rule M_k is defined by $\forall V \in \mathcal{V}$,

$$M_k(V) = \begin{cases} 1 & \text{if } n^+(V) > n^-(V) + k \\ -1 & \text{if } n^-(V) > n^+(V) + k \\ 0 & \text{otherwise.} \end{cases}.$$

The first theorem, proved in [Houy, 2006], states that the class of voting rules M_k are the only ones to satisfy Anonymity, Neutrality, Monotonicity, Weak Pareto and Decisiveness Equivalence.

Theorem 1 ([Houy, 2006])
The voting rule C satisfies A, N, M, WP and DE if and only if it is M_k for some integer $k \in [0, n-1]$.

In the same line, we can show that the class of voting rules M^k are the only ones to satisfy Anonymity, Neutrality, Monotonicity, Weak Pareto and Decisiveness Non-Equivalence.

Theorem 2
The voting rule C satisfies A, N, M, WP and DNE if and only if it is M^k for some integer $k \in [n/2, n-1]$.

Moreover, we can state that the axioms given in Theorem 2 are independent for any $n \geq 2$.

Proposition 1
Let us have $n \geq 2$. Axioms A, N, M, WP and DNE are independent.

Notice that Theorems 1 and 2 imply that a voting rule satisfies A, N, M, WP, DNE and DE if and only if it is $M^{n-1} = M_{n-1}$. Then, a voting rule satisfying A, N, M, WP and different from M^{n-1} cannot satisfy both DNE and DE. It is straightforward to check that if $n = 2$, then a voting rule satisfying A, N, M, WP satisfies either DNE or DE. However, if $n \geq 3$, there exist some voting rules satisfying A, N, M, WP and satisfying DE nor DNE. To prove it, consider the following voting rule: $\forall V \in \mathcal{V}$, $M_{-DE,DNE}(V) = \begin{cases} 1 & \text{if } n^+(V) > 0 \text{ and } n^-(V) = 0 \\ -1 & \text{if } n^-(V) > 0 \text{ and } n^+(V) = 0 \\ 0 & \text{otherwise.} \end{cases}$.

Let us now discuss further axioms DE and DNE. Axiom DE imposes some equal treatment of the voters. Indeed, DE imposes that abstainers and non-abstainers have the same power in the sense that if abstainers can influence the outcome of the social vote by increasing their vote, *i.e.* they are decisive, so are non-abstainers by abstaining. On the contrary, DNE imposes that there is always inequality of treatment between non-abstainers and abstainers. If the latter are decisive by increasing their vote, then the former are not by abstaining. As we showed, M^k voting rules are characterized by DNE together with some usual axioms. In this sense, M^k voting rules are characterized by the unequal treatment of abstainers and non-abstainers. On the contrary, M_k voting rules are characterized by DE, hence by the equal treatment of abstainers and non-abstainers. Of course, M_k and M^k coincide only for $k = n - 1$. Only in this case, abstainers are decisive when no other voter can be. Then, both DNE and DE are logically satisfied. This certainly sheds a new light on absolute qualified majority voting.

A Proof of Theorem 2

Checking that any M^k with $k \in [n/2, n-1]$ satisfies A, N, M, WP and DNE is straightforward.

For the proof of the sufficiency part of Theorem 2, we will need the following lemmas. They are usual and we omit the proofs.

Lemma 1
If the voting rule C satisfies A, then $C(V)$ depends only on $n^+(V)$, $n^0(V)$ and $n^-(V)$.

Lemma 2
If the voting rule C satisfies A, N and M, then ($C(V) \geq 0$ if $n^+(V) \geq n^-(V)$) and ($C(V) \leq 0$ if $n^-(V) \geq n^+(V)$).

By Lemma 1, we know that, for a voting rule satisfying A, the whole relevant information is contained in $n^+(V)$ and $n^-(V)$ (obviously, $n^0(V) = n - n^+(V) - n^-(V)$. Then, for a voting rule C and a voting configuration V, we will define $c(n^+(V), n^-(V)) = C(V)$.

By Lemma 2 and by N, it is enough to show that if C satisfies A, N, M, WP and DNE, there exists an integer $k < n$ such that $\forall V \in \mathcal{V}, n^+(V) > k \Leftrightarrow C(V) = 1$. Let us have $V_1 \in \mathcal{V}$ such that $C(V_1) = 1$ and $\forall V \in \mathcal{V}$ such that, $n^+(V) < n^+(V_1)$ or ($n^+(V) = n^+(V_1)$ and $n^-(V) > n^-(V_1)$) imply $C(V) \leq 0$. By WP, such a V_1 exists. By Lemma 2, $n^+(V_1) > n^-(V_1)$ and then $n^+(V_1) \geq 1$. Moreover, by WP, $n^+(V_1) \leq n$.

Let us set $k = n^+(V_1) - 1$. Let $V_2 \in \mathcal{V}$.

If $n^+(V_2) < k+1$, $C(V_2) \leq 0$ follows by definition of V_1.

If $n^+(V_2) = k+1$ and $n^-(V_2) \leq n^-(V_1)$, $C(V_2) = 1$ follows by A and M. Let us show that we cannot have $n^+(V_2) = k+1$, $n^-(V_2) > n^-(V_1)$ and $C(V_2) \leq 0$. If such a V_2 exists, by definition of V_1, $c(n^+(V_1), n^-(V_1) + 1) \leq 0$ (this is well-defined since $n^+(V_1) \geq 1$). By what we just showed, $c(n^+(V_1) - 1, n^-(V_1)) \leq 0$. Then, by Neutrality, $c(n^-(V_1) + 1, n^+(V_1)) \geq 0$, $c(n^-(V_1), n^+(V_1) - 1) \geq 0$ and $c(n^-(V_1), n^+(V_1)) = -1$. But this contradicts DNE. Then, if $n^+(V_2) = k+1$, $C(V_2) = 1$ (This with Lemma 2 implies that $k \geq n/2$).

If $n^+(V_2) \geq k+1$, by M, we have $c(n^+(V_2), n^-(V_2)) \geq c(k+1, n^-(V_2))$. By what we showed above, $c(k+1, n^-(V_2)) = 1$ and then, $C(V_2) = 1$.

B Proof of Proposition 1

Let us have $n \geq 2$.

A, N, WP, DNE $\not\Rightarrow$ M: Let us define the voting rule C_{-M} by $\forall V \in \mathcal{V}$,

$$C_{-M}(V) = \begin{cases} 1 & \text{if } n^+(V) = n \text{ or } (n^+(V), n^-(V)) = (0,1) \\ -1 & \text{if } n^-(V) = n \text{ or } (n^+(V), n^-(V)) = (1,0) \\ 0 & \text{otherwise.} \end{cases}.$$

It is left to the reader to check that C_{-M} satisfies A, N, DNE and WP but does not satisfy M.

A, N, M, DNE $\not\Rightarrow$ WP: Let us define the voting rule C_{-M} by $\forall V \in \mathcal{V}$,

$$C_{-WP}(V) = 0.$$

It is left to the reader to check that C_{-WP} satisfies A, N, M and DNE but does not satisfy WP.

N, WP, M, DNE $\not\Rightarrow$ A: Let us define the voting rule C_{-A} by $\forall V \in \mathcal{V}$,

$$C_{-A}(V) = V_1.$$

It is left to the reader to check that C_{-A} satisfies N, DNE, M and WP but does not satisfy A.

A, N, WP, M $\not\Rightarrow$ DNE: It is left to the reader to check that M_{n-2} satisfies A, N, M and WP but does not satisfy DNE.

A, WP, M, DNE $\not\Rightarrow$ N: Let us define the voting rule C_{-N} by $\forall V \in \mathcal{V}$,

$$C_{-N}(V) = \begin{cases} 1 & \text{if } n^+(V) = n \text{ or } (n^+(V), n^-(V)) = (n-1,0) \\ -1 & \text{if } n^-(V) = n \\ 0 & \text{otherwise.} \end{cases}.$$

It is left to the reader to check that C_{-N} satisfies A, M, DNE and WP but does not satisfy N.

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