E c o n o m i c s B u l l e t i n

Equivalence between best responses and undominated strategies: a generalization from finite to compact strategy sets.

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Abstract

For games with expected utility maximizing players whose strategy sets are finite, Pearce (1984) shows that a strategy is strictly dominated by some mixed strategy, if and only if, this strategy is not a best response to some belief about opponents' strategy choice. This note generalizes Pearce's (1984) equivalence result to games with expected utility maximizing players whose strategy sets are arbitrary compact sets.

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1 Introduction

Rationalizability concepts, on the one hand, and *dominance solution concepts*, on the other hand, are solution concepts for strategic games which can be justified by the assumption that players involve in an internal process of reasoning which successively excludes *unreasonable* strategies[1]. While *rationalizability concepts* (Bernheim, 1984; Moulin, 1984; Pearce, 1984; Börgers, 1993; Ghirardato and Le Breton, 1997 and 2000) define strategies as unreasonable when they are not a best response to some *belief*, a strategy is unreasonable according to *dominance solution concepts* (e.g., Moulin, 1984; Milgrom and Roberts, 1990; Börgers, 1993) when it is dominated by another strategy.

At first glance, these two definitions of "unreasonable strategies" seemingly refer to rather different ideas. However, for two player-games with finite strategy sets, and under the assumption of expected utility maximizing players, Pearce (1984, Lemma 3) proves the remarkable result that a strategy is not strictly dominated by some mixed strategy, if and only if, this strategy is also a best response to some probability measure over opponents' strategies. Moreover, Pearce's (1984) result is easily extended to n-player games of expected utility maximizing players as long as strategy sets remain finite. As a consequence, Pearce's finding establishes equivalence for such games between the iterative solution concepts of *correlated rationalizability* and the *strict dominance solution with respect to mixed strategies*. Unfortunately, the restriction to finite strategy sets only does not permit for the application of Pearce's (1984) equivalence result to many strategic games of interest.

This note demonstrates that Pearce's (1984) equivalence result on unreasonable strategies can be extended to n-player games of expected utility maximizing players such that strategy sets are arbitrary compact subsets of a metric space. Moreover, under the assumption that best response correspondences are upper-hemicontinuous, it is shown for such games that *correlated rationalizability* and the *strict dominance solution with respect to mixed strategies* are equivalent strategic solution concepts.

2 Notation and Definitions

Given some set A, let $\triangle(A)$ denote the set of all probability measures on the σ-Algebra of Borel sets of A. For a finite set of players, I, let S_i denote the individual strategy set of player $i \in I$, and interpret a point $\sigma_i \in \triangle(S_i')$, with $S_i' \subseteq S_i$, as *mixed strategy*

[1]See, e.g., Pearce (1984), Tan and Werlang (1988), Guesnerie (2002) for an epistemic foundation of iterative solution concepts by the assumption that *it is common-knowledge among players that players do not choose unreasonable strategies.*

of player i with support on S_i'. In contrast, a point $\sigma_{-i} \in \triangle\left(S_{-i}'\right)$, with $S_{-i}' \subseteq S_{-i} = \times_{j \neq i} S_j$, is interpreted as *belief* of player i about her opponents' strategy choice in S_{-i}'. Moreover, suppose that there exists, for all players $i \in I$, some *utility representation* $U_i : \triangle(S_i) \times \triangle(S_{-i}) \to [0, a] \subset \mathbb{R}_+$ of player i's preferences such that, for all $s_i \in S_i$, $U_i(s_i, \cdot)$ is continuous on S_{-i}.

Definitions*:*

A strategy $s_i \in S_i$ is a best response to some belief on S_{-i}', if and only if, for some $\sigma_{-i} \in \triangle\left(S_{-i}'\right)$, $s_i \in f_i(\sigma_{-i})$ where $f_i : \triangle(S_{-i}) \to 2^{S_i}$ denotes player i's best response correspondence, i.e., $f_i(\sigma_{-i}) = \arg\max_{s_i \in S_i} U_i(s_i, \sigma_{-i})$.

A strategy $s_i \in S_i$ is strictly dominated on S_{-i}' by some mixed strategy $\sigma_i \in \triangle(S_i)$, if and only if, $U(\sigma_i, s_{-i}) > U(s_i, s_{-i})$ for all $s_{-i} \in S_{-i}'$. Moreover, for given $S_{-i}' \subseteq S_{-i}$, let $g_i : S_{-i}' \to 2^{S_i}$ collect all strategies $s_i \in S_i$ that are not strictly dominated on S_{-i}' by some mixed strategy $\sigma_i \in \triangle(S_i')$.

The two alternative notions of "unreasonable strategies" - i.e., *a strategy is unreasonable, if and only if, it is not a best response to some belief* versus *a strategy is unreasonable, if and only if, it is strictly dominated by a mixed strategy* - give rise to (seemingly) alternative iterative solution concepts which successively exclude unreasonable strategies:

Definition *(*Pearce, 1984*): The set of correlated rationalizable strategies for a game G is defined as $R(G) = \bigcap_{k=0}^{\infty} \lambda^k$ such that $\lambda^k = \times_{i=1}^{I} \lambda_i^k$ with*

$$\lambda_i^k = \bigcup_{\sigma_{-i} \in \triangle\left(\lambda_{-i}^{k-1}\right)} f_i(\sigma_{-i})$$

and $\mu_{-i}^0 = S_{-i}$.

Definition: *The strict dominance solution with respect to mixed strategies of game G is defined as $D(G) = \bigcap_{k=0}^{\infty} \vartheta^k$ such that $\vartheta^k = \times_{i=1}^{I} \vartheta_i^k$ with*

$$\vartheta_i^k = \bigcup_{s_{-i} \in \vartheta_{-i}^{k-1}} g_i(s_{-i})$$

and $\vartheta_{-i}^0 = S_{-i}$.

3 Results

Pearce derives his equivalence result (Pearce, 1984, Lemma 3) by a saddlepoint argument for zero-sum games with mixed-strategy spaces (see also Lemma 3.2.1. and 3.2.2. for bimatrix games in van Damme, 1991). Fudenberg and Tirole (1996) suggest that a direct application of the separating hyperplane theorem for finite normed spaces might offer a shortcut to Pearce's proof; and indeed the following proposition is based on the more general Hahn-Banach Theorem (see, e.g., p. 157 in Berge, 1997).

Proposition. *Given a game G such that, for all $i \in I$,*
(A1) player i is an expected utility maximizer, and
(A2) S_i is a compact subset of some metric space.
Then strategy $s_i \in S_i$ is not strictly dominated on S_{-i} by some mixed strategy $\sigma_i \in \triangle(S_i)$, if and only if, there is some belief $\sigma_{-i} \in \triangle(S_{-i})$ such that s_i is a best response to σ_{-i}.

Before the proposition is formally proved, observe that, by Berge's (1997) maximum theorem, the sets λ^k and ϑ^k are compact for all $k \in \mathbb{N}$ under the assumption of upper-hemicontinuous best response correspondences and compact strategy sets. Thus, by the proposition, $\lambda^k = \vartheta^k$ for all $k \in \mathbb{N}$.

Corollary. *Given a game G such that, for all $i \in I$,*
(A1) player i is an expected utility maximizer,
(A2) S_i is a compact subset of some metric space, and
(A3) the best response correspondence f_i is upper-hemicontinuous.
Then the strict dominance solution of G with respect to mixed strategies coincides with the set of correlated rationalizable strategies of G, i.e., $D(G) = R(G)$.

Proof of the proposition: Since preferences of expected utility maximizing players obey monotonicity with respect to first order stochastic dominance, the if-part is obvious. Turn to the only-if part and note that a strategy s_i is not strictly dominated by some mixed strategy σ_i, if and only if, there exists for all $\sigma_i \in \triangle(S_i)$ some $s_{-i} \in S_{-i}$, dependent on σ_i, such that

$$U_i(\sigma_i, s_{-i}) \leq U_i(s_i, s_{-i})$$

Now define the set

$$V(\sigma_i) = \{x : S_{-i} \to \mathbb{R} \mid x \in \mathbb{C}[S_{-i}], x(s_{-i}) < U_i(\sigma_i, s_{-i}) \text{ for all } s_{-i} \in S_{-i}\}$$

where $\mathbb{C}[S_{-i}]$ equipped with the sup norm is the space of all continuous real valued functions with domain S_{-i}. Observe that when s_i is not strictly dominated by some mixed strategy σ_i, the function $U_i(s_i, \cdot) : S_{-i} \to \mathbb{R}_+$ is, by construction, not a point in the set

$$V = \bigcup_{\sigma_i \in \triangle(S_i)} V(\sigma_i)$$

The set $V \subseteq \mathbb{C}[S_{-i}]$ is nonempty, convex and open, and it contains, by lemma 4.43 in Aliprantis and Border (1994), an *internal point* as prerequisite for the application of the Basic Separating Hyperplane Theorem (Theorem 4.42 in Aliprantis and Border, 1994; also known as Hahn-Banach Theorem, see p. 157 in Berge, 1997). By this theorem, there exists some linear functional $T \in (\mathbb{C}[S_{-i}])^*$ - with $(\mathbb{C}[S_{-i}])^*$ denoting the norm dual of $\mathbb{C}[S_{-i}]$ - which separates the singleton $\{U_i(s_i, \cdot)\}$ from V, i.e.,

$$T(x) < T(\{U_i(s_i, \cdot)\}) \text{ for all } x \in V \tag{1}$$

where the direction of the inequality is due to the fact that the values $x(s_{-i})$ can be chosen arbitrarily small.

Let $\mathfrak{B}(S_{-i})$ denote the set of all finite Borel measures on S_{-i} equipped with the *variation norm.* Since S_{-i} is a compact subset of a metric space, the Riesz-Markov characterization theorem (see Theorem 11.41 and Corollary 11.44 in Aliprantis and Border, 1994) implies the existence of a mapping $\Lambda : \mathfrak{B}(S_{-i}) \to (\mathbb{C}[S_{-i}])^*$ with

$$(\Lambda\mu)(y) = \int_{S_{-i}} y d\mu$$

such that $\mathfrak{B}(S_{-i})$ and $(\mathbb{C}[S_{-i}])^*$ are *isometric.* As a consequence, there exists a finite Borel measure μ on S_{-i} such that the system of inequalities (1) can be equivalently written as

$$\int_{S_{-i}} x d\mu < \int_{S_{-i}} U_i(s_i, \cdot) d\mu \text{ for all } x \in V \tag{2}$$

where continuity of x and $U_i(s_i, \cdot)$ on S_{-i} ensure that these integrals are well defined. Moreover, since μ is finite, we can normalize μ to obtain a probability measure σ_{-i}, with $\sigma_{-i}(S'_{-i}) = \frac{\mu(S'_{-i})}{\mu(S_{-i})}$ for the Borel sets $S'_{-i} \subseteq S_{-i}$, such that (2) is equivalent to

$$\int_{S_{-i}} x d\sigma_{-i} < \int_{S_{-i}} U_i(s_i, \cdot) d\sigma_{-i} \text{ for all } x \in V \tag{3}$$

Since, by construction, any value $U_i(\sigma_i, s_{-i})$ can be approached, arbitrarily close, by the value $x(s_{-i})$ of some function $x \in V$, the system of inequalities (3) implies, for all $\sigma_i \in \triangle(S_i)$,

$$\int_{S_{-i}} U_i(\sigma_i, s_{-i}) d\sigma_{-i} \leq \int_{S_{-i}} U_i(s_i, s_{-i}) d\sigma_{-i}$$

In words: given belief σ_{-i}, player i's expected utility of choosing strategy s_i is maximal on S_i. Thus, when s_i is not strictly dominated by some mixed strategy $\sigma_i \in \triangle(S_i)$ there exists some belief $\sigma_{-i} \in \triangle(S_{-i})$ such that s_i is a best response to σ_{-i} for an expected utility maximizing player.□

4 References

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