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Matching with network: introduction behaviors with respect to various incentives

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Abstract

In this paper, we study a network-based matching problem, in which workers and firms could match with each other if there is a link between them. Based on this feature, workers strategically decide whether to introduce their linked workers (i.e., friends) to their linked firms. Crucially, we allow for distinct underlying incentives driving these introduction behaviors. The first is the individually rational incentive, which has been investigated by Ding (2023). The second is the external incentive, which has remained unexplored in the existing literature. This external incentive can be interpreted as either a psychological benefit or an external reward resulting from a successful introduction. Our findings regarding the external incentive provide novel insights into real-world introduction behaviors that cannot be explained by pure individual rationality. Furthermore, our results regarding the relationship between network efficiency and introduction-stability contrast with previous studies; specifically, we demonstrate that network efficiency and introduction-stability coincide under specific network structures.

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1. Introduction

Matching problems with network structure have been studied extensively in recent years (e.g., Afacan 2023, Ding 2023, Ostrovsky 2008, and Siedlarek 2022). One key consideration driving existing research is that matching can only be formed between connected players in a network. This feature is particularly prominent in decentralized matching markets. In this paper, we focus on a common real-world behavior that can endogenously reshape network structure: mutual introduction between connected players. Therefore, when taking introduction behavior into account, analyzing the incentives behind this behavior becomes a critical yet underexplored research question. Ding (2023) examines the setting where players care about their own gains from making introductions. Specifically, a player will introduce two of their linked players if doing so allows them to be matched to a strictly more preferred partner in the stable matching of the modified network, compared to their outcome in the original stable matching. However, in practice, many introduction behaviors fall outside this self-interested framework: we often observe introductions that are conducted solely to facilitate the matching of other players. This empirical observation motivates the present study.

2. Model

Consider a set of players $N = W \cup F = \{w_1, w_2, ..., w_n\} \cup \{f_1, f_2, ..., f_{n'}\}$ where $n \geq 2$ and $n' \geq 1$. W and F are finite and disjoint sets of workers and firms, respectively. Each player $i \in N$ is endowed with an ability level $a_i \in \mathbb{R}_{++}$. Let vectors $a_w = (a_{w_1}, a_{w_2}, ..., a_{w_n})$ and $a_f = (a_{f_1}, a_{f_2}, ..., a_{f_{n'}})$ denote the profiles of the endowments of workers and firms, respectively. For simplicity, we assume a strict ordering of ability levels for workers and firms, respectively. Formally, $a_w \neq a_{w'}$ for all $w, w' \in W$ with $w \neq w'$. Similarly, $a_f \neq a_{f'}$ for all $f, f' \in F$ with $f \neq f'$. Players are situated in a network represented by an $(n+n') \times (n+n')$ matrix $g = (g_{ij})_{i,j \in N}$, where $g_{ij} = 1$ means that players i and j have a link between them and $g_{ij} = 0$, otherwise. We assume the network is undirected, so that $g_{ij} = g_{ji}$, for all $i, j \in N$ and set $g_{ii} = 0$ for all $i \in N$. Let G denote the set of all possible networks for $|N|$ players. Players i and j are neighbors in g if $g_{ij} = 1$. The set of neighbors player i has in g is represented by $N_i(g) = \{j \in N : g_{ij} = 1\}$. A path in a network g is a sequence of distinct players $(i_0, i_1, ..., i_k)$, where $g_{i_{l-1} i_l} = 1$, for $l = 1, 2, ..., k$. Finally, consider a network g. If $g_{ij} = 0$, we use $g + ij$ to represent the network that is otherwise the same as g but with an additional link between i and j. Each firm $f \in F$, has its *capacity* $\mathbf{q}_f \in \mathbb{N}_{++}$, which is the positive number of available positions for workers. The vector list $(\mathbf{q}_f)_{f \in F}$ is the profile of capacities of all firms.

Given a network structure $g \in G$, players undergo a two-sided one-to-many matching process. The outcome μ_g of the process is a *matching* (mapping) from $W \cup F$ to $2^W \cup F$ such that

(i) $\mu_g(w) \in F \cup \{\varnothing\}$ for any $w \in W$,

(ii) for any $w \in W$ and $f \in F, \mu_g(w) = f$ if and only if $w \in \mu_g(f)$ and $w \in N_f(g)$,

(iii) $\mu_g(f) \subseteq W$ and $|\mu_g(f)| \leq \mathbf{q}_f$ for all $f \in F$.

For each worker $w \in W$, $\succsim_w$ is a preference relation over $F \cup \{\varnothing\}$ where $\varnothing$ is the option of being unmatched. Similarly, for each firm $f \in F$, $\succsim_f$ is a preference relation over $W \cup \{\varnothing\}$. If $f \succ_w \varnothing$, then firm f is said to be *acceptable* for worker $w \in W$. Similarly, if $w \succ_f \varnothing$, then worker w is said to be *acceptable* for firm $f \in F$. For simplicity, we assume that for all $w \in W$

and $f, f' \in F$ with $f \neq f'$, $f \succ_w f'$ if and only if $a_f > a_{f'}$, and $f \succ_w \varnothing$ if $a_f > 0$. Similarly, for all $f \in F$ and $w, w' \in W$ with $w \neq w'$, $w \succ_f w'$ if and only if $a_w > a_{w'}$, and $w \succ_f \varnothing$ if $a_w > 0$.

A matching μ_g is *stable* if

(i) $\mu_g(w) \succsim_w \varnothing$ for each $w \in W$,

(ii) $w' \succsim_f \varnothing$ for each $w' \in \mu_g(f)$ and $f \in F$,

(iii) if $f \succ_w \mu_g(w)$ and $f \in N_w(g)$, then $|\mu_g(f)| = \mathbf{q}_f$ and $w' \succ_f w$ for all $w' \in \mu_g(f) \subset N_f(g)$.

The conditions of stability of matching μ_g require that: first, each worker values their match at least as highly as the option of remaining unmatched, and firms follow the same preference rule as workers (conditions (i) and (ii)). Second, if a worker w is unmatched with a linked firm f, but prefers f to their match $\mu_g(w)$, then f already employs other linked workers, all of whom are strictly preferred by f to w (condition (iii)). Conditions (i) and (ii) are satisfied automatically for those matched workers and firms in μ_g under our assumptions above. Therefore, when assessing whether a matching is stable or not, condition (iii) is of essentially importance.

Consider a specific stable matching process: worker-proposing deferred acceptance mechanism with network g (WP-DA-g).

Step 1 Each worker applies to the firm that is ranked first in his/her preference list. Each firm rejects all $w \in W$ with $g_{fw} = 0$ and admits linked workers up to its capacity, following its preference. The remaining workers are rejected.

Step n, $n \geq 2$ Each worker rejected in the previous step applies to the most-preferred firm among those he/she has not yet applied to. Each firm receiving applications rejects all workers who have no link with it and considers the set of workers it admitted in the previous step together with the set of new acceptable applicants. From this set, the firm admits workers up to its capacity, following its preference. The remaining workers are rejected.

End The algorithm stops when no worker is rejected. Each worker is assigned to his/her final tentative firm. Any remaining workers are unassigned.

Following Gale and Shapley (1962) and Ding (2023), WP-DA-g generates an unique stable matching defined above for any $g \in G$.

Lemma 1. For any $g \in G$, there exists a unique stable matching μ_g^*, which can be obtained by WP-DA-g.

Now, we are ready to define preferences of workers over networks. Given $g \in G$ and the unique stable matching μ_g^*, let $U_g^*(w)$ denote the following set for each worker $w \in W$:

$$\left\{\mu_g^*(w') \neq \varnothing : w' \in N_w(g)\right\};$$

which is the set of linked workers who are matched with some firm in μ_g^* for worker $w \in W$. Let $\geq_w$ be the preference relation over the set of networks G of $w \in W$. Let $g =_w g'$ denote the case where w is indifferent between network g and g'. That is, $g =_w g'$ if and only if $\mu_g^*(w) = \mu_{g'}^*(w)$ and $|U_g^*(w)| = |U_{g'}^*(w)|$: worker w is matched with the same partner in μ_g^* and $\mu_{g'}^*$, and the number of linked and matched workers in μ_g^* is equal to the number of those in $\mu_{g'}^*$. A worker $w \in W$ strictly prefers a network g to g' which is written as $g >_w g'$ (i.e., $g \geq_w g'$ and $g \neq_w g'$) if and only if one of the following conditions hold

(i) $\mu_g^*(w) \succ_w \mu_{g'}^*(w)$,

(ii) $\mu_g^*(w) = \mu_{g'}^*(w)$ and $|U_g^*(w)| > |U_{g'}^*(w)|$.

The condition that a worker w prefers a network g to network g' is equivalent to one of the following: (i) worker w is matched with a strictly more preferred partner in μ_g^* than in $\mu_{g'}^*$; (ii) although worker w is matched with the same partner in μ_g^* and $\mu_{g'}^*$, the number of linked and matched workers in μ_g^* is larger than that in $\mu_{g'}^*$. Lemma 2 below confirms that $\geq_w$ is a well-defined (rational) preference relation over the set of networks G.

Lemma 2. $\geq_w$ is reflexive, complete, and transitive for any $w \in W$.

Proof. Reflexivity: pick arbitrary $g \in G$. Since for any $g \in G$, stable matching μ_g^* is unique, $\mu_g^*(w) = \mu_g^*(w)$ and $|U_g^*(w)| = |U_g^*(w)|$ for any $w \in W$. That is, $g =_w g$ for any $w \in W$. This in turn implies $g \geq_w g$.

Completeness: pick arbitrary $g, g' \in G$. Suppose that $\mu_g^*(w) = \mu_{g'}^*(w)$ for $w \in W$. Since both μ_g^* and $\mu_{g'}^*$ are unique, the numbers of linked workers who are matched with some firm at both matchings are determined uniquely. So, there are only three cases that need to be considered: $|U_g^*(w)| > |U_{g'}^*(w)|$, $|U_g^*(w)| < |U_{g'}^*(w)|$ and $|U_g^*(w)| = |U_{g'}^*(w)|$. The first case implies $g >_w g'$, which in turn implies $g \geq_w g'$. The second case implies $g <_w g'$, which in turn implies $g \leq_w g'$. The third case implies $g =_w g'$, which in turn implies both $g \geq_w g'$ and $g \leq_w g'$.

Suppose that $\mu_g^*(w) \neq \mu_{g'}^*(w)$. By assumptions of ability levels for firms and preference $\succsim_w$ of worker w, either $\mu_g^*(w) \succ_w \mu_{g'}^*(w)$ or $\mu_{g'}^*(w) \succ_w \mu_g^*(w)$ must hold. These two cases in turn imply $g \geq_w g'$ or $g \leq_w g'$, respectively.

Transitivity: pick arbitrary $g, g', g'' \in G$. Suppose that $g \geq_w g'$, $g' \geq_w g''$, and $\mu_g^*(w) = \mu_{g'}^*(w) = \mu_{g''}^*(w)$. $g \geq_w g'$ and $\mu_g^*(w) = \mu_{g'}^*(w)$ imply $|U_g^*(w)| \geq |U_{g'}^*(w)|$. $g' \geq_w g''$ and $\mu_{g'}^*(w) = \mu_{g''}^*(w)$ imply $|U_{g'}^*(w)| \geq |U_{g''}^*(w)|$. Combining these two conditions, we have $|U_g^*(w)| \geq |U_{g''}^*(w)|$. This in turn implies $g \geq_w g''$.

Suppose that $g \geq_w g'$, $g' \geq_w g''$, and $\mu_g^*(w) = \mu_{g'}^*(w) \neq \mu_{g''}^*(w)$. $g' \geq_w g''$ and $\mu_{g'}^*(w) \neq \mu_{g''}^*(w)$ imply that it cannot be either the case $g' =_w g''$ or the condition (ii) of $g' >_w g''$. Therefore, it must be the condition (i) of $g' >_w g''$ (i.e., $\mu_{g'}^*(w) \succ_w \mu_{g''}^*(w)$). Since $\mu_g^*(w) = \mu_{g'}^*(w)$, we have $\mu_g^*(w) \succ_w \mu_{g''}^*(w)$. That is, $g \geq_w g''$. The case $\mu_g^*(w) \neq \mu_{g'}^*(w) = \mu_{g''}^*(w)$ is similar.

Suppose that $g \geq_w g'$, $g' \geq_w g''$, and $\mu_g^*(w) \neq \mu_{g'}^*(w) \neq \mu_{g''}^*(w)$. $g \geq_w g'$ and $\mu_g^*(w) \neq \mu_{g'}^*(w)$ imply $\mu_g^*(w) \succ_w \mu_{g'}^*(w)$. $g' \geq_w g''$ and $\mu_{g'}^*(w) \neq \mu_{g''}^*(w)$ imply $\mu_{g'}^*(w) \succ_w \mu_{g''}^*(w)$. Combining these two conditions, we have $\mu_g^*(w) \succ_w \mu_{g''}^*(w)$. This in turn implies $g \geq_w g''$. □

3. Efficiency and introduction-stability of networks

We define Pareto efficiency below exclusively for workers. Since we only define workers' preferences over networks and allow introduction behaviors only for workers, we focus on the effect of introduction behaviors on workers' welfare.

Definition 1. A network $g \in G$ is **Pareto efficient** if there does not exist a $g' \in G$ such that $g' \geq_w g$ for all $w \in W$, and $g' >_w g$, for some $w \in W$.

Given a network g, consider a worker $w \in W$, who has two neighbors, a worker $w' \in W$ and a firm $f \in F$ (i.e., $w', f \in N_w(g)$) with $g_{w'f} = 0$. Worker w can introduce w' to f, and thus change the network structure from g to $g+w'f$. We assume that worker w initiates the introduction if $g+w'f >_w g$. As we have defined $g >_w g'$ with conditions (i)$\mu_g^*(w) \succ_w \mu_{g'}^*(w)$, and (ii)$\mu_g^*(w) = \mu_{g'}^*(w)$ and $|U_g^*(w)| > |U_{g'}^*(w)|$, we clarify here when a worker $w \in W$ will initiate an introduction. Condition (i) of $g+w'f >_w g$ has been studied by Ding (2023). It states that worker w will introduce another worker to the firm if they can benefit from the introduction by being matched with a strictly more preferred partner in $\mu_{g+w'f}^*$ than in μ_g^*. In contrast, condition (ii) is the core incentive condition studied in this paper. It captures a distinct incentive: if the introduction increases the number of their linked and matched workers (e.g., friends) in $\mu_{g+w'f}^*$ compared to μ_g^*, while leaving w's own match unchanged, w will initiate the introduction. Therefore, condition (ii) can be explained by two real-world scenarios: first, altruistically motivated workers refer job opportunities to their close friends who are searching for employment; second, external reward schemes are provided to incentivize introduction behavior. We sometimes refer to condition (i) as individually rational incentive (IR-incentive) and condition (ii) as external incentive (EX-incentive) in the remaining content of this paper.

Definition 2. A network $g \in G$ is **introduction-stable** if there does not exist a worker $w \in W$, who has two neighbors, a worker $w' \in W$, a firm $f \in F$ (i.e., $w', f \in N_w(g)$) with $g_{w'f} = 0$ such that $g+w'f >_w g$.

3.1 Result about network efficiency

Proposition 1. Suppose $\sum_{f\in F} \mathbf{q}_f = |W|$. A network $g \in G$ is Pareto efficient if and only if $|\{ w \in W : \mu_g^*(w) = \varnothing \}| = 0$ and $g_{ww'} = 1$ for all $w, w' \in W$ with $w \neq w'$.

Proof. Suppose $g \in G$ is Pareto efficient and $|\{ w \in W : \mu_g^*(w) = \varnothing \}| > 0$. That is, $g \in G$ is Pareto efficient, while there exists some $w \in W$ such that $\mu_g^*(w) = \varnothing$. By assumption, $\sum_{f\in F} \mathbf{q}_f = |W|$, we know that there exists some $f \in F$ such that $|\mu_g^*(f)| < \mathbf{q}_f$. By creating a link between w and f, we change network from g to $g+wf$. We know that $\mu_g^* = \mu_{g+wf}^*$, except that $\mu_{g+wf}^*(w) = f$, since $f \succ_w \varnothing$, $w \succ_f \varnothing$, and $|\mu_g^*(f)| < \mathbf{q}_f$. That is, $\mu_{g+wf}^*(w') = \mu_g^*(w')$ and $|U_{g+wf}^*(w')| \geq |U_g^*(w')|$ for all $w' \in W \backslash \{w\}$, and $\mu_{g+wf}^*(w) \succ_w \mu_g^*(w)$. They imply that $g+wf \geq_{w'} g$ for all $w' \in W \backslash \{w\}$ and $g+wf >_w g$. A contradiction is shown.

Now, we know that $|\{ w \in W : \mu_g^*(w) = \varnothing \}| = 0$ must hold. Suppose there exist two workers $w, w' \in g$ such that $g_{ww'} = 0$. Suppose $|U_g^*(w)| = l$ and $|U_g^*(w')| = m$ where $l, m \geq 0$. By creating a link between w and w', we change network from g to $g+ww'$. We know that $\mu_g^* = \mu_{g+ww'}^*$, since workers can only be matched with firms. That is, $\mu_g^*(w) = \mu_{g+ww'}^*(w)$ for all $w \in W$. However, $|U_{g+ww'}^*(w)| = l+1 > |U_g^*(w)| = l$ and $|U_{g+ww'}^*(w')| = m+1 > |U_g^*(w')| = m$. They imply $g+ww' >_w g$ and $g+ww' >_{w'} g$, respectively. A contradiction is shown.

Suppose $|\{ w \in W : \mu_g^*(w) = \varnothing \}| = 0$ and $g_{ww'} = 1$ for all $w, w' \in g$ with $w \neq w'$. From these two conditions, we know that $|U_g^*(w)| = |W|$ for all $w \in W$. Note that only the creation of links between workers and firms needs to be considered here. Assume that we create a link between an arbitrary $w \in W$ and $f \in F$, which changes the network from g to $g+wf$. If $\mu_g^* = \mu_{g+wf}^*$, then we conclude that g is Pareto-efficient. If $\mu_g^* \neq \mu_{g+wf}^*$, then $w \in \mu_{g+wf}^*(f)$, since the only difference between g and $g+wf$ is the addition of the link between w and f. This implies that there exists some $w' \in \mu_g^*(f)$ such that $w' \notin \mu_{g+wf}^*(f)$. Worker w' is worse off because they are rejected by f during the WP-DA-$g+wf$ process, which forces them to propose to a less preferred firm. Therefore, we conclude that g is Pareto-efficient. $\square$

3.2 Example

To facilitate a clearer understanding of our results regarding introduction-stability, we provide an example in this section. Suppose $W = \{w_1, w_2, w_3, w_4, w_5\}, F = \{f_1, f_2\}$, and their ability levels are consistent with their indexes. Capacity of firm f_1 is given by $\mathbf{q}_{f_1} = 1$ and capacity of firm f_2 is given by $\mathbf{q}_{f_2} = 2$. Consider following network g:

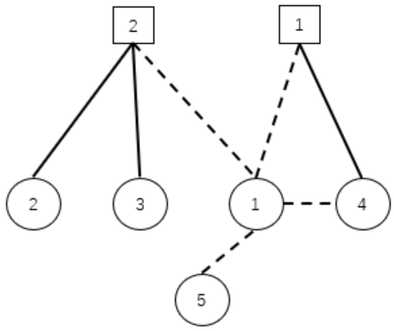

Figure 1: network g.

Workers and firms are indexed by cycles and squares, respectively. Their abilities are indexed by the number inside their shapes. Both dashed and solid lines represent links between players. Moreover, solid lines also represent μ_g^* which can be obtained by WP-DA-g.

Consider the IR-incentive of worker w_1. If w_1 introduces worker w_4 to firm f_2, this changes the network from g to $g + w_4 f_2$. Network $g + w_4 f_2$ can be visualized on the left side of Figure 2. We know that $\mu^*_{g+w_4 f_2}(w_1) = f_1$ by running WP-DA-$g + w_4 f_2$. Since $\mu^*_{g+w_4 f_2}(w_1) = f_1 \succ_{w_1} \varnothing = \mu_g^*(w_1)$, we know that $g + w_4 f_2 >_{w_1} g$. The path in g shown on the right side of Figure 2, which is critical for worker w_1 to initiate the introduction, is formally defined in Ding (2023) (i.e., two-step decreasing ability alternating path). Our results with respect to IR-incentive are consistent with Ding (2023).

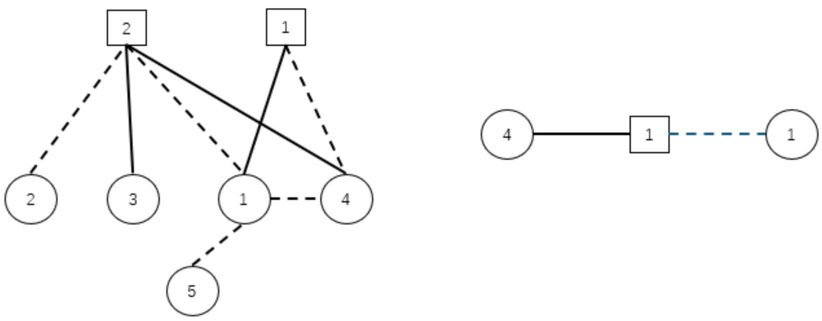

Figure 2: network $g + w_4 f_2$ and essentially important path in g.

Although network $g + w_4 f_2$ is introduction-stable with respect to IR-incentive, it fails to be introduction-stable with respect to EX-incentive. Consider whether worker w_1 will introduce worker w_5 and firm f_2 in network $g + w_4 f_2$. If they do so, network $g + w_4 f_2$ is changed to $g + w_4 f_2 + w_5 f_2$. Network $g + w_4 f_2 + w_5 f_2$ can be visualized as following:

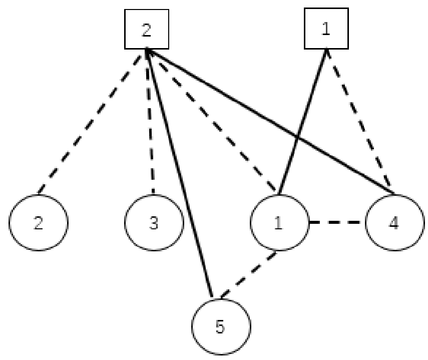

Figure 3: network $g+w_4f_2+w_5f_2$.

Let g' denote network $g+w_4f_2$. Worker w_1 will introduce firm w_5 to firm f_2, since $\mu^*_{g'}(w_5)=\varnothing$ but $\mu^*_{g'+w_5f_2}(w_5)=f_2$ (i.e., $|U^*_{g'+w_5f_2}(w_1)|=2>1=|U^*_{g'}(w_1)|$). Combining with $\mu^*_{g'}(w_1)=\mu^*_{g'+w_5f_2}(w_1)=f_1$, we know that $g'+w_5f_2>_{w_1}g'$. There are two points worthwhile to be noted. First, worker w_5 has a higher ability level than worker w_3 who has the lowest ability among those workers who are matched with firm f_2 in network g'. Second, if there exists a link between workers w_1 and w_3 in g (i.e., $g_{w_1w_3}=1$), worker w_1 will still introduce worker w_4 to firm f_2 with respect to IR-incentive. However, they will not introduce worker w_5 to firm f_2 with respect to EX-incentive. One possible interpretation from the perspective of worker w_1 is as follows: if I (w_1) introduce my friend w_5 to firm f_2, w_5 will get a new job. However, my other friend w_3 will lose their job. w_3 could feel angry with me, and I will lose this friend. Since I do not want to lose the friendship, I decide not to carry out this introduction.

3.3 Results about EX-incentive

In this section, we present our results regarding the introduction-stability of network with respect to EX-incentive. The following Lemma is immediately.

Lemma 3. Consider network g and three players, i.e., two workers $w,w'\in W$ and one firm $f\in F$, such that $w',f\in N_w(g)$ and $g_{w'f}=0$. If $\mu^*_g(w')=\varnothing$ and $|\mu^*_g(f)|<\mathbf{q}_f$, then worker w will introduce w' and f with respect to their EX-incentive.

Proof. Suppose $\mu^*_g(w')=\varnothing$ but $|\mu^*_g(f)|<\mathbf{q}_f$. By assumption, $\mu^*_g(w')=\varnothing$, we know that worker w' is rejected by any firm during the WP-DA-g process. Particularly, by assumptions, $g_{w'f}=0$ and $|\mu^*_g(f)|<\mathbf{q}_f$, worker w' is rejected by firm f, since there is no link between them in network g. Also by assumption, $|\mu^*_g(f)|<\mathbf{q}_f$, we know there is no enough linked workers for firm f to fill its capacity in network g. Therefore, by introducing w' to f, we know that $\mu^*_{g+w'f}(w)=\mu^*_g(w)$ for all $w\in W\setminus\{w'\}$ and $\mu^*_{g+w'f}(w')=f$. Therefore, $\mu^*_{g+w'f}(w)=\mu^*_g(w)$ and $|U^*_{g+w'f}(w)|=|U^*_g(w)|+1>|U^*_g(w)|$ for the worker w. These in turn imply $g+w'f>_w g$. That is, worker w will introduce w' to f with respect to his/her EX-incentive. $\square$

As we obtain a trivial case from Lemma 3, we turn our attention to a non-trivial case. That is, we study when a worker will introduce their job-seeking friend to a fully staffed firm.

Given network g and unique stable matching μ^*_g, for each firm $f\in F$, let $\min\{a_w\}_{w\in\mu^*_g(f)}$ denote the minimal ability level of worker who is matched with firm f among those workers being matched with f in unique stable matching if $\mu^*_g(f)\neq\varnothing$. By our assumption of strict ordering of ability levels for workers, $w \in \mu_g^*(f)$ with $\min\{a_w\}_{w\in\mu_g^*(f)}$ is determined uniquely for each $f \in F$ if $\mu_g^*(f) \neq \varnothing$.

Proposition 2. Consider network g and three players, i.e., two workers $w, w' \in W$ and one firm $f \in F$, such that $w', f \in N_w(g)$ and $g_{w'f} = 0$. Suppose $\mu_g^*(w') = \varnothing$ and $|\mu_g^*(f)| = \mathbf{q}_f$. For worker w to introduce w' and f with respect to his/her EX-incentive, then one of the followings must hold:

(i) whenever $w \in \mu_g^*(f)$, $a_{w'} > \min\{a_w\}_{w\in\mu_g^*(f)}$, $a_w \neq \min\{a_w\}_{w\in\mu_g^*(f)}$, and

$w'' \notin N_w(g)$ with $w'' \neq w'$, $a_{w''} = \min\{a_w\}_{w\in\mu_g^*(f)}$ and $\mu_{g+w'f}^*(w'') = \varnothing$;

(ii) whenever $w \notin \mu_g^*(f)$, $a_{w'} > \min\{a_w\}_{w\in\mu_g^*(f)}$, and

$w'' \notin N_w(g)$ with $w'' \neq w'$, $a_{w''} = \min\{a_w\}_{w\in\mu_g^*(f)}$ and $\mu_{g+w'f}^*(w'') = \varnothing$.

Proof. Suppose $w \in \mu_g^*(f)$ but $a_{w'} \leq \min\{a_w\}_{w\in\mu_g^*(f)}$. By our assumption of strict ordering of ability levels for workers, it is sufficient to consider the case $a_{w'} < \min\{a_w\}_{w\in\mu_g^*(f)}$. Then, we know that worker w' will be rejected by firm f during the WP-DA-$g + w'f$ process. Since the difference between network $g + w'f$ and network g is only the addition of a link between w' and f, we know that $\mu_{g+w'f}^*(w) = \mu_g^*(w)$ for all $w \in W$. Therefore, $\mu_{g+w'f}^*(w) = \mu_g^*(w)$ and $|U_{g+w'f}^*(w)| = |U_g^*(w)|$ together imply $g + w'f =_w g$ (i.e., $g + w'f \not\succ_w g$). That is, worker w has no EX-incentive to initiate the introduction.

Suppose $a_{w'} > \min\{a_w\}_{w\in\mu_g^*(f)}$ but $a_w = \min\{a_w\}_{w\in\mu_g^*(f)}$. Then, we know that worker w will be rejected by firm f, and thus applies to a less preferred firm than f during the WP-DA-$g + w'f$ process. That is, $\mu_{g+w'f}^*(w) \neq \mu_g^*(w)$. At this point, we know that worker w has no EX-incentive to initiate the introduction (i.e., $g + w'f \not\succ_w g$ with respect to EX-incentive of worker w).

Suppose $w'' \in N_w(g)$ with $w'' \neq w'$, $a_{w''} = \min\{a_w\}_{w\in\mu_g^*(f)}$ and $\mu_{g+w'f}^*(w'') = \varnothing$. By assumption of ability levels for workers: $a_w \neq a_{w'}$ for all $w, w' \in W$ with $w \neq w'$, Condition $a_{w''} = \min\{a_w\}_{w\in\mu_g^*(f)}$ implies that $w'' \in \mu_g^*(f)$. Then, we know that worker w'' will be rejected while both w and w' will not be rejected by firm f during the WP-DA-$g + w'f$ process. That is, $\mu_{g+w'f}^*(w) = \mu_g^*(w)$. Since the difference between network $g + w'f$ and network g is only the addition of a link between w' and f, we know that $\mu_{g+w'f}^*(w) = \mu_g^*(w)$ for all $w \in W \setminus \{w', w''\}$. However, since $\mu_g^*(w') = \varnothing$, $\mu_{g+w'f}^*(w') = f$, $\mu_g^*(w'') = f$, $\mu_{g+w'f}^*(w'') = \varnothing$ and $w, w'' \in N_w(g)$, we know that $|U_{g+w'f}^*(w)| = |U_g^*(w)|$ which in turn implies $g + w'f =_w g$ (i.e., $g + w'f \not\succ_w g$). Condition (ii) is similar to condition (i). □

3.4 Relationship between Efficiency and introduction-stability

In the model where only IR-incentive is considered (Siedlarek 2022 and Ding 2023), efficiency and introduction-stability do not imply each other. Compared with previous studies, we show that under specific network structures, the two properties do imply each other when EX-incentive is present.

Proposition 3. Suppose $\sum_{f\in F} \mathbf{q}_f = |W|$. A network $g \in G$ is introduction-stable if g is Pareto-efficient and g is introduction-stable with respect to IR-incentive.

Proof. By Proposition 1, g is Pareto efficient, then $|\{w \in W : \mu^*_g(w) = \varnothing\}| = 0$ and $g_{ww'} = 1$ for all workers $w, w' \in W$ with $w \neq w'$. That is, $|U^*_g(w)| = |W|$ for all $w \in W$. We know that g is introduction-stable with respect to EX-incentive. Combining with assumption that g is introduction-stable with respect to IR-incentive, g is introduction-stable. □

Proposition 3 states that no worker will initiate an introduction if every worker has already found a job and cannot find a better one through introduction.

Proposition 4. Suppose $\sum_{f \in F} \mathbf{q}_f = |W|$, $g_{ww'} = 1$ for all $w, w' \in W$ with $w \neq w'$, and there exists a worker $w \in W$, such that $g_{wf} = 1$ for all firms $f \in F$. A network $g \in G$ is Pareto efficient if g is introduction-stable.

Proof. Suppose g is introduction-stable. By Proposition 1 and our assumption, $g_{ww'} = 1$ for all $w, w' \in W$ with $w \neq w'$, it is sufficient to show that it must be the case $|\{w \in W : \mu^*_g(w) = \varnothing\}| = 0$. Suppose $|\{w \in W : \mu^*_g(w) = \varnothing\}| \neq 0$. Then, there exists some worker $w' \in W$, such that $\mu^*_g(w') = \varnothing$. By assumption $\sum_{f \in F} \mathbf{q}_f = |W|$, there exists some firm $f \in F$, such that $|\mu^*_g(f)| < \mathbf{q}_f$ and $g_{w'f} = 0$. By assumption of network structure, there exists a worker $w \in W$ with $w \neq w'$, such that $g_{wf} = 1$ and $g_{ww'} = 1$. If w introduces w' to f, and thus changes the network from g to $g + w'f$, then $\mu^*_g(w) = \mu^*_{g+w'f}(w)$ for all $w \in W \setminus \{w'\}$ and $\mu^*_{g+w'f}(w') = f$. Since $\mu^*_g(w') = \varnothing$, $|U^*_{g+w'f}(w)| = |U^*_g(w)| + 1 > |U^*_g(w)|$ in turn implies $g + w'f >_w g$. Therefore, g is not introduction-stable. We obtain a contradiction. □

Combining the above two results, we obtain Corollary 1 below.

Corollary 1. Suppose $\sum_{f \in F} \mathbf{q}_f = |W|$, $g_{ww'} = 1$ for all $w, w' \in W$ with $w \neq w'$, and there exists a worker $w \in W$, such that $g_{wf} = 1$ for all firms $f \in F$. A network $g \in G$ is Pareto efficient and g is introduction-stable with respect to IR-incentive if and only if g is introduction-stable.[1]

4. Concluding remarks

In this paper, we study matching problem with network where workers have various incentives to initiate introductions. Workers are individually rational, meaning that they seek to secure better jobs for themselves (i.e., the IR-incentive). Concurrently, external rewards for successful matchings motivate workers to find jobs for their friends (i.e., the EX-incentive). The primary contribution of this paper lies in the analysis of the EX-incentive. In Section 3.3, we demonstrate that for a worker to introduce a job-seeking friend to a well-staffed firm, the condition that this introduction harms neither the worker themselves nor their other friends necessarily holds. Furthermore, due to the EX-incentive, network efficiency and introduction-stability imply each other under specific network structures.

[1] Some may criticize that the assumption in Proposition 4 and Corollary 1 is too strong. Particularly, it could be too strong that there is a worker connected to all firms. However, the assumption can be modified as following: for all $w' \in W$, $f \in F$ with $\mu^*_{g-w'f}(w') = \varnothing$ and $|\mu^*_{g-w'f}(f)| < \mathbf{q}_f$, there exists a $w \in W \setminus \{w'\}$ such that $g_{wf} = 1$ where $g - ij$ represents the network that is otherwise the same as g but there is no link between i and j.

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