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Equitable exchange for multiplicative gambles

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Abstract

Peters (02019) argues for reframing economic theory considering ergodicity. He contends that the traditional economic models of expected wealth from an investment are unrealistic and fail to capture what happens over time unless the wealth dynamic is additive rather than multiplicative. His motivating example is a gamble on the outcome of a fair coin in which your wealth increases by 50% if the coin lands heads and decreases by 40% if tails. Expected change in wealth for Peters's paradigmatic gamble (PPG) is positive, but most players accepting it have a decrease in expected wealth. Peters and Gell-Mann (02016) contrast such multiplicative iterated gambles, where the amount won or lost each time is proportional to the amount bet, with additive gambles in which, “the expectation value would reflect how the individual fares over time”. While others have noted the economics literature that the analysis in Peters (02019) ignores, this brief paper looks at a simpler fundamental problem with the Peters analysis. Specifically, I examine an assumption central to the analysis presented by Peters and, using only basic mathematics, show it to be false. By considering other parameter choices I show that iterated multiplicative gambles can have expectations that reflect how individuals (not just ensembles) fare over time, and I set out a sufficient criterion for when that is so.

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Peters (02019) argues for reframing economic theory based on considerations of ergodicity. He contends that the traditional economic models of expected wealth from an investment are unrealistic, that they fail to capture what happens over time unless the wealth dynamic is additive rather than multiplicative, that expected utility was introduced in 01738[1] by Daniel Bernoulli as a psychology-based correction to problems in the initial model (such as the St. Petersburg Paradox), and that by following this psychological rather than physical correction, economics took a step in a wrong direction that it has been headed in ever since.

Doctor et al. (02020a,b) have argued that the failure lies in Peters's analysis, that it just ignores an extensive body of research on risk and economic modeling, a literature that also accounts for dynamic settings. This brief paper makes no attempt to weigh in on these points. Though illuminating to invoke overlooked relevant developments from the established economic literature, there is a more directly comprehensible problem with the Peters analysis. Specifically, I examine an assumption central to the analysis presented by Peters and, using only simple mathematics, show it to be false.

The motivating example Peters (02019) examines to argue for his view is a "simple gamble" on the outcome of a fair coin in which your wealth increases by 50% if the coin lands heads and decreases by 40% if tails.

$$\Delta x = \begin{cases} \Delta x_H = +0.5x, & P_H = 1/2 \\ \Delta x_T = -0.4x, & P_T = 1/2 \end{cases}$$

The main claimed observation is that the expected change in wealth for such a gamble is positive, but if time average and ergodicity are considered, we get a different, more illuminating model of the situation. Consider a player who repeatedly accepts the gamble, where x in each repetition is the amount that has resulted from the previous gamble. This is an example of what Peters and Gell-Mann (02016) call a gamble with multiplicative repetition. In Peters's paradigmatic gamble (PPG) a small fraction of individuals obtain a significant increase in wealth over time. But most individuals will have a decrease in wealth. Peters and Gell-Mann (02016) contrast this with gambles having additive dynamics, where the amount won or lost each time is not proportional to the amount bet, but is, e.g., a constant. In such gambles, "the expectation value would reflect how the individual fares over time".

[1]I follow herein the date conventions of the Long Now Foundation (https://www.longnow.org/).

This brief paper makes no attempt to broadly address when temporal dynamics and/or ergodicity play a useful role in economic modeling. I only intend to note that iterated multiplicative gambles can have expectations that reflect how individuals (not just ensembles) fare over time and to set out a sufficient criterion for when that is so. The multiplicative versus additive nature of PPG is thus not sufficient to generate the main observations it is intended to motivate. Further, it is not actually necessary to consider the iteration or order of the gambles an individual takes to determine how that individual fares but simply the multiplicative parameters and the proportion of wins and losses.

An iterated gamble with multiplicative dynamics can be prudent to accept for individuals if it has different payoff parameters from those of PPG. To see this, it is useful to consider the two-envelope paradox (aka "the exchange paradox"), to which PPG is similar: You are offered a pair of envelopes and told that one contains twice as much money as the other. You choose one envelope, open it, and see that it contains, e.g., \$100. Given that the first envelope contains \$100, the other envelope must contain either \$50 or \$200, and either case is equally possible. Thus, to switch is to risk \$50 to gain \$100. Put another way, the expected value of switching is

$$(1/2 \times \$50) + (1/2 \times \$200) = \$125$$

Given the opportunity to exchange an expected value of \$100 for one of \$125, it seems to make obvious sense to switch. The problem that leads some to call this a "paradox" is that you could have employed this reasoning no matter which envelope you opened. So, no matter what value you saw, the expected value of the other envelope is always 1.25 times that value. This problem has many variants and an extensive literature—see, e.g., reference lists of (Syverson 02010) or (Wikipedia contributors 02020). My focus, however, is not the paradox per se, but its relation to PPG.

Consider an iterated version of the two-envelope gamble in which you are repeatedly presented the outcome of the previous round and offered to keep it or exchange it for an envelope that contains either twice or half this amount. As in PPG, the amount in the first envelope is always known by you, either because this is your initially chosen investment or because it is the result of the previous gamble. In this two-envelope gamble (TEG) there is thus no paradox arising from the possibility that you might have first looked in the other envelope and seen the other amount. TEG is the same as PPG, differing only in the multiplicative parameters for winning and losing.

$$\Delta x = \begin{cases} \Delta x_H = +x, & P_H = 1/2 \\ \Delta x_T = -0.5x, & P_T = 1/2 \end{cases}$$

In PPG the majority of players over time lose money, but not in TEG, *even though the wealth dynamic is multiplicative rather than additive.* You could still have more tails than heads in a run of accepted gambles and thus lose money. But if you lose in one round, you can make that loss right back in the next round. In contrast, a player losing in one round of PPG will need at least two more rounds to get back at least to where she started.

More generally, both PPG and TEG are multiplicative exchange gambles. Call a (multiplicative) exchange gamble *equitable* if in an iterated chain of such gambles, one is guaranteed to at least break even if the proportion of losing gambles is no more than the proportion of winning gambles. If the multiplicative factor in a win is w and in a loss l, then for an exchange gamble to be equitable, it is easy to see that $(1+w)(1-l) \geq 1$. In the idiom of Peters (02019), $x + \Delta x_H$ must be greater than or equal to $1/(x + \Delta x_T)$; it is not sufficient that Δx_H merely be larger than $-\Delta x_T$. (I assume you can only lose what you gamble each round, so $-x \leq \Delta x_T \leq 0$.) In PPG, $1.5 < 1.\bar{6} = 1/.6$. In TEG, $2 = 1/.5$, so TEG is just at the threshold of equitability.

Like PPG, the basic two-envelope paradox is often compared to the St. Petersburg paradox. Many authors have noted that the amount that might be in the envelopes is not specified and thus that the expectation is unbounded. Syverson (02010), however, describes how to reproduce the paradox selecting from a fixed bounded set of envelope pairs. Regardless, as Peters is focused on being more realistic, we can do so by restricting to finite initial wealth values and finite chains of offered gambles. This is reflective of the number and size of gambles any single living player could back and iteratively contemplate, and can be given upper bounds via various combinations, e.g., by giving a maximum possible final payout. (In combination these restrictions are thus also reflective of "house limits". Of course any house openly offering genuinely equitable exchanges, even with limits, would quickly lose money.)

Define an n-round (w, l) exchange gamble to be an iterated offering of n gambles as I have been describing, where the initial value you hold is x_0 and, for $i = 0, \ldots, n-1$,

$$\Delta x_i = \begin{cases} \Delta x_{i,H} = +wx_i, & P_H = 1/2 \\ \Delta x_{i,T} = -lx_i, & P_T = 1/2 \end{cases}$$

Given an n-round (w, l) exchange gamble with initial value x_o, let h be the number of heads in a given run of the gamble. The expected payoff from such an n-round gamble, is then $(1+w)^h(1-l)^{n-h}x_0$. So, for an n-round PPG, expected payoff is $1.5^h \times .6^{n-h} \times x_0$. The majority of n-round acceptances of PPG reflect a loss (those with a proportion of heads less than roughly 55.7%). In equitable exchange gambles, that is no longer true, and the growth rate is nonetheless multiplicative rather than additive. Further, since the order of the coin flips does not matter, time is not actually an important component. Nothing significant changes if the experiment is not based on n iterated flips of the same fair coin, but the simultaneous flipping of n fair coins. Realistically, there may be a psychological significance to the iteration that will affect successive choices. But there is nothing inherent to the physical description of PPG that requires a temporal component.

This does not show considerations of time or ergodicity are uninformative or unimportant in economic models. I have, however, shown simply by considering different parameter values that a gamble having multiplicative rather than additive dynamics is by itself insufficient to motivate Peters's conclusions and further that there is nothing inherently temporal or ordered to the determination of when a collection of gambles with a multiplicative payoff is equitable. If Peters's analysis is getting at anything useful, then we must look further for a firm foundation.

References

Doctor, J. N., P. P. Wakker, and T. V. Wang (02020a). Economists' views on the ergodicity problem. *Nature Physics 16*, 1168.

Doctor, J. N., P. P. Wakker, and T. V. Wang (02020b). Supplementary information on: "economists' views on the ergodicity problem". https://static-content.springer.com/esm/art%3A10.1038%2Fs41567-020-01106-x/MediaObjects/41567_2020_1106_MOESM1_ESM.pdf. Online; Retrieved May 28 02021.

Peters, O. (02019). The ergodicity problem in economics. *Nature Physics 15*(12), 1216–1221.

Peters, O. and M. Gell-Mann (02016). Evaluating gambles using dynamics. *Chaos: An Interdisciplinary Journal of Nonlinear Science 26*(2).

Syverson, P. (02010). Opening two envelopes. *Acta Analytica 25*(4), 479–498.

Wikipedia contributors (02020). Two envelopes problem. https://en.wikipedia.org/wiki/Two_envelopes_problem. Online; Retrieved Dec. 28 02020.