Volume 43, Issue 3

A note on local uniqueness of equilibria: How isolated is a local equilibrium?

Stefano Matta
University of Cagliari

Abstract

The motivation of this note is to show how singular values affect local uniqueness. More precisely, Theorem 3.1 shows how to construct a neighborhood (a ball) of a regular equilibrium whose diameter represents an estimate of local uniqueness, hence providing a measure of how isolated a (local) unique equilibrium can be. The result, whose relevance in terms of comparative statics is evident, is based on reasonable and natural assumptions and hence is applicable in many different settings, ranging from pure exchange economies to non-cooperative games.

1   Introduction

Consider an equilibrium equation $f(p, q) = 0$, where p and q denote the unknowns and parameters, respectively. Local uniqueness and continuous (smooth) dependence of the unknowns on the parameters are at the heart of comparative statics. This means that, roughly speaking, the (locally unique) solution p changes continuously (smoothly) as the parameter q, belonging to the parameter set Q, varies under a continuous (smooth) perturbation.

More precisely, at a point p belonging to the solution set of q, there exists a neighborhood N of (p, q) such that the projection onto the second factor, $pr : (p, q) \to q$, restricted to N, is an homeomorphism (diffeomorphism).

This regularity property has been extensively studied in the literature in different settings, in particular after the introduction of the differentiable viewpoint in the seminal paper by Debreu (1970). The study of the equilibrium set $E = \{(p, q) | f(p, q) = 0\}$ and the properties of the projection $pr : E \to Q$, $(p, q) \mapsto q$, allows to establish natural connections between economic and mathematical properties: e.g., the surjectivity of pr (existence), the cardinality of the set $\{pr^{-1}(q)\}$ (uniqueness/multiplicity), the continuity of the correspondence of pr^{-1} (structural stability).

Under the very restrictive hypothesis of global uniqueness, i.e. when the set $pr^{-1}(q)$ is a singletone set for every parameter q, the whole solution set E is homeomorphic (diffeomorphic) to the parameter set.

Multiplicity, on the other hand, is deeply related to the set of singular values of the projection. Even if this set has measure zero in Q under certain assumptions, its components of codimension-one and codimension-two can be relevant. Consider, for example, redistribution policies, represented by continuous perturbations of the parameters. In such an instance, it is crucial not to cross the singular set to avoid catastrophes and undesiderable welfare effects, that can be determined by prices multiplicity. For example, Figure 1, taken from (Dierker, 1989), shows an example of two policies sharing the same target, but whose (different) outcomes depend on the order of use of resources.
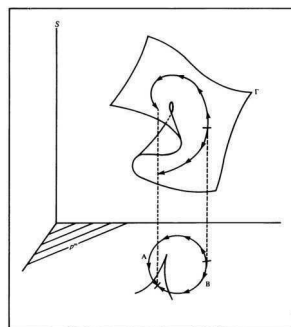

Figure 1: Two policies sharing the same target but with different outcomes (taken from (Dierker, 1989)).

In his remarkable survey of regular economies, Dierker (1982) emphasises the usefulness of local uniqueness and continuity of equilibrium price adjustment for the purposes of comparative statics, price computation and the study of core correspondence. Moreover, he points out that knowledge of the singularities of the projection is rather incomplete and much research has to be done in this direction: "A detailed study of the singularities of pr, however, seems to require a considerable amount of mathematical sophistication.

It appears desirable that, in addition to a general, abstract, mathematical study of the singularities of pr, some particular types of singularities be examined in their relation to specific economic phenomena." (Dierker, 1982).

The codimension-two components also play a key role, as highlighted in this note, since they have an impact on the topology of the parameter set. The intuition behind this is that they can be seen as holes in Q. The consequences of this property have not, to the best of my knowledge, been analyzed in the literature, which has traditionally focused on the codimension-one component, the only one that can disconnect Q.

The motivation of this short note is to highlight how the singular values can affect the size of local uniqueness. This information can be used to estimate how isolated a (local) unique equilibrium can be. More precisely, Theorem 3.1 shows a sufficient condition that enables to construct a neighborhood N such that the projection restricted to N is injective. This means that local uniqueness is satisfied in N.

This result can be applied to different settings, characterized by multiplicity and multiple roots associated to critical solutions. For example, one can appraise potential distorsive effects (e.g. transfer paradox Balasko (1978)) of redistributive policies Garrat and Goenka (1995) in exchange and production economies Balasko (2011). This is due to the fact that a common geometric structure is present. The same structure, with suitable changes, can be found, e.g., in the case of infinite economies Balasko (1997); Chichilnisky and Zhou (1998) or non-cooperative game theory Govindan and Wilson (2001).

This paper is organized as follow. Section 2 recalls the main definitions and tools. Section 3 is devoted to the proof of the main result.

2   Definitions

In this section the main definitions and properties are outlined for the reader's convenience. For an introduction to covering spaces, the reader is referred to Massey (1991).

The analysis is focused on the projection map $pr : E \to Q$. The set Q denotes the parameter set and E is the equilibrium set, i.e. $E = \{(p, q)| f(p, q) = 0\}$, where $f(p, q) = 0$ represents an equilibrium condition to be satisfied by the unknowns p, given the constraints and the primitives of the model.

Suppose that both E and Q are smooth manifolds where we can measure the length of curves and the distance between points. More precisely, given a piecewise differentiable curve $\tilde{\gamma} : [0, 1] \to E$ joining two points $\tilde{x}_1$ and $\tilde{x}_2$ in E, namely $\tilde{\gamma}(0) = \tilde{x}_1$ and $\tilde{\gamma}(1) = \tilde{x}_2$, we can compute the length of $\tilde{\gamma}$ denoted by $L_E(\tilde{\gamma})$ and the distance

$$d_E(\tilde{x}_1, \tilde{x}_2) = \inf_{\tilde{\gamma} \in \tilde{\Gamma}_{\tilde{x}_1, \tilde{x}_2}} L_E(\tilde{\gamma}). \tag{1}$$

between $\tilde{x}_1$ and $\tilde{x}_2$, where $\tilde{\Gamma}_{\tilde{x}_1, \tilde{x}_2}$ denotes the set of piecewise differentiable curves with endpoints $\tilde{x}_1$ and $\tilde{x}_2$. Similarly, we can compute the length $L_Q(\gamma)$ of a piecewise differentiable curve $\gamma : [0, 1] \to Q$ joining two points x_1 and x_2 of Q and their distance $d_Q(x_1, x_2)$.

In this paper we assume that the following properties are satisfied:

(i) pr is smooth;

(ii) pr is proper;

(iii) pr is a length decreasing map, i.e.

$$L_E(\tilde{\gamma}) \geq L_Q(pr \circ \tilde{\gamma}), \ \forall \tilde{\gamma} : [0, 1] \to E. \tag{2}$$

Consequently, by (1), pr is distance decreasing, namely

$$d_E(\tilde{x}, \tilde{y}) \geq d_Q(pr(\tilde{x}), pr(\tilde{y})), \ \forall \tilde{x}, \tilde{y} \in E. \tag{3}$$

(iv) the ball centered at a point $\tilde{x} \in E$ (resp. $x \in Q$) of radius $\tilde{r}$ (resp. r), namely $B_{\tilde{r}}(\tilde{x}) = \{\tilde{y} \in E \mid d_E(\tilde{y}, \tilde{x}) < \tilde{r}\}$ (resp. $B_r(x) = \{y \in Q \mid d_Q(y, x) < r\}$) is connected.

All these are reasonable and natural assumptions. Smoothness is an approximation of continuity under suitable topologies, properness also has a nice economic meaning: for example, it represents the idea of scarcity and desirability. Assumption (iii) and (iv) have a very intuitive meaning. Indeed it is expected that a projection does not increase the length of curves and that the metrics chosen satisfy the connectedness property. As a specific and natural example, one can take a Riemannian metric g_Q on Q (if Q is a subset of some Euclidean space, g_Q can be the flat metric as is the case, e.g., in literature related to the Edgeworth box) and the pull-back metric $g_E = pr^* g_Q$ on E. Then pr is length decreasing if we endow Q and E with the length functions associated to these metrics.

In this setting, the application of the inverse function theorem (IFT) and Sard's theorem to pr leads to standard regularity properties enjoyed by an open and dense subset of the parameter space, denoted by $\mathcal{R}$. More precisely, it is not hard to prove that $pr_{|pr^{-1}(\mathcal{R})} : pr^{-1}(\mathcal{R}) \to \mathcal{R}$ is a covering map (see (Loi and Matta, 2010, Proposition 2.2)). Indeed the properness of pr is a sufficient condition to turn a surjective local diffeomorphism into a covering map (see, e.g., (Loi and Matta, 2010, Proposition 2.3)). We recall that a *covering map* between two topological spaces $\tilde{X}$ and X is a continuous surjective map such that each $x \in X$ admits a *well-covered* neighborhood, i.e., each $x \in X$ has an open neighborhood U such that $p^{-1}(U)$ is a disjoint union of open sets in $\tilde{X}$, each of which is mapped by p homeomorphically onto U.

The economic meaning behind the covering property is that it represents the well-known property of smooth selection of equilibria. This is crucial for comparative statics analysis and characterizes the regular values $\mathcal{R}$ of the projection. Its complement $Q \setminus \mathcal{R}$ represents the projection of the critical equilibria. In a general equilibrium framework, this set is a disjoint union of closed smooth manifolds: for a deep analysis, the reader is referred to Balasko (1978, 1992, 2011) and Loi and Matta (2016). We will assume in the sequel that the set of singular values, denoted by $\Sigma = \cup \Sigma^k$, is the union of closed manifolds, possibly singular, of codimension k, with k greater or equal to 1. Hence Σ^1 denotes the component which disconnects the parameter set. Literature is usually focused on Σ^1 as the main cause of catastrophes. But Σ^2 affects $\mathcal{R}$'s topology in a relevant way, as we will see in Theorem 3.1 (see also Remark 3.1 below).

3   Main result

Let $x \in \mathcal{R}$ and let Γ_x be the set of non contractible loops $\gamma : [0, 1] \to \mathcal{R}$, $\gamma(0) = \gamma(1) = x$. Notice that the image of γ lies in a connected component K_x of $\mathcal{R}$ containing x. Let

$$m_x = \inf_{\gamma \in \Gamma_x} L_{d_Q}(\gamma) \geq 0 \tag{4}$$

be the nonnegative number which measures the length of the minimal loop of all the non contractible closed curves in K_x based at x. If no Σ^2 component belongs to the connected component of $Q \setminus \Sigma^1$ containing x, then $m_x = 0$. It could be said in a suggestive way that Σ^2 are "holes" in K_x iff $m_x \neq 0$.

Let us also define

$$d_x = d_Q(x, \Sigma^1) = \inf_{\sigma \in \Sigma^1} d_Q(x, \sigma). \tag{5}$$

The positive number d_x measures the distance between x and the boundary of K_x, which corresponds to Σ^1. This distance is well-defined since Σ^1 is closed in Q, being the projection of a closed set via a proper map.

The following theorem uses the information so far to construct a neighborhood N of $\tilde{x}$ such that $pr_{|N}$ is injective, i.e. where one has local uniqueness.

Theorem 3.1 *Let us assume assumptions (i)-(iv) above are satisfied. Let $x \in \mathcal{R}$ and $\tilde{x} \in pr^{-1}(x)$. Let r_x be the positive real number defined as*

$$r_x = \begin{cases} 3d_x & \text{if } m_x = 0 \\ \min(m_x, d_x) & \text{if } m_x \neq 0. \end{cases}$$

where m_x and d_x are given by (4) and (5) respectively. Then $pr_{|B_{\frac{r_x}{3}}(\tilde{x})}$ is injective.

Proof: Denote by $\tilde{K}_x$ the connected component of $pr^{-1}(\mathcal{R})$ containing $\tilde{x}$. Then, being E and Q smooth manifolds, by (Massey, 1991, Ch. 5, Lemma 2.1) $pr_{|\tilde{K}_x} : \tilde{K}_x \to K_x$ is a covering map. If $m_x = 0$ then K_x is simply-connected and hence $pr_{|\tilde{K}_x}$ (Massey, 1991, Ch. 5, Exercise 6.1) is a diffeomorphism. Notice that by (iv) $B_{d_x}(x)$ is connected and hence $B_{d_x}(x) \subset K_x$. Moreover since by (3) pr is distance decreasing one gets that $pr\left(B_{d_x}(\tilde{x})\right) \subset B_{d_x}(x) \subset K_x$ and hence $B_{d_x}(\tilde{x}) \subset \tilde{K}_x$. It follows that $pr_{|B_{\frac{r_x}{3}}(\tilde{x})} = pr_{|B_{d_x}(\tilde{x})}$ is injective.

Let $m_x \neq 0$. Assume, by contradiction, that $\tilde{x}_1$ and $\tilde{x}_2$ are two distinct points in $B_{\frac{r_x}{3}}(\tilde{x})$, $r_x = \min(m_x, d_x)$, such that $pr(\tilde{x}_1) = pr(\tilde{x}_2) = x$. Let $\tilde{\Gamma}_{\tilde{x}_1, \tilde{x}_2}$ denote the set of piecewise differentiable curves $\tilde{\gamma} : [0, 1] \to B_{\frac{r_x}{3}}(\tilde{x})$ with endpoints $\tilde{x}_1$ and $\tilde{x}_2$. Notice that $\tilde{\Gamma}_{\tilde{x}_1, \tilde{x}_2} \neq \emptyset$ by assumption (iv). Then, for each $\tilde{\gamma} \in \tilde{\Gamma}_{\tilde{x}_1, \tilde{x}_2}$, the curve

$$\gamma = pr \circ \tilde{\gamma} : [0, 1] \to K_x$$

is a piecewise differentiable loop based on x. By (Massey, 1991, Ch. 5, Theorem 4.1) each of these loops is non-contractible and hence by (4) one has $L_{d_Q}(\gamma) \geq m_x$. It follows, by (1) and (2), that

$$d_E(\tilde{x}_1, \tilde{x}_2) = \inf_{\tilde{\gamma} \in \tilde{\Gamma}_{\tilde{x}_1, \tilde{x}_2}} L_{d_E}(\tilde{\gamma}) \geq \inf_{\gamma \in \Gamma_x} L_{d_Q}(\gamma) = m_x \geq r_x.$$

On the other hand, by the triangle inequality,

$$d_E(\tilde{x}_1, \tilde{x}_2) \leq d_E(\tilde{x}_1, \tilde{x}) + d_E(\tilde{x}, \tilde{x}_2) = \frac{r_x}{3} + \frac{r_x}{3} = \frac{2}{3}r_x < r_x,$$

yielding the desired contradiction. $\qquad\square$

Remark 3.1 The key idea behind the proof of the theorem is, roughly speaking, as follows. If K_x is the connected component of $\mathcal{R}$ containing x, the set Σ^2 are "holes"

inside K_x. If K_x did not contain holes, then it would be simply connected and hence the solution set would be singletone (global uniqueness) for every parameter in K_x, if one restricts pr to the connected component $\tilde{K}_x$ of $pr^{-1}(\mathcal{R})$ containing $\tilde{x}$. On the other hand, if K_x contains holes then $m_x \neq 0$, namely one has to be careful with the non-contractible loops by taking the ball centered at $\tilde{x}$ of radius $\min(d_x, m_x)$.

Remark 3.2 The example of continuous perturbations provided in the Introduction highlights the role played by the codimension-one component of the singular values in determining "catastrophes", in a smooth selection setting. The main idea of the paper is to explain how the codimension-two component affects the size of local uniqueness. In particular, a variation (discrete or continuous) of the parameter affects d_x and m_x, and hence this size: therefore, the main result, by estimating the loneliness of the equilibrium, can also provide a clue to the magnitude of potential effects caused by a shift in the equilibrium price brought about by a change in parameters.

Remark 3.3 The diameter of the ball $\frac{r_x}{3}(\tilde{x})$ represents a sufficient condition that ensures injectivity but it is does not maximize the size of the ball (if $m_x \neq 0$). It can be seen as an estimate of local uniqueness and a measure of how isolated is a local equilibrium, providing us with a possible answer to the question raised in the title of this note.

References

Balasko, Y. (1978) "The transfer problem and the theory of regular economies" *International Economic Review* **19**, 687-694.

Balasko, Y. (1978) "Equilibrium analysis and envelope theory" *Journal of Mathematical Economics* **5**, 153-72.

Balasko, Y. (1992) "The set of regular equilibria" *Journal of Economic Theory* **58**, 1-8.

Balasko, Y. (1997) "The natural projection approach to the infinite-horizon model" *Journal of Mathematical Economics* **27**, 251-265.

Balasko, Y. (2011) *General Equilibrium Theory of Value*, Princeton University Press: NJ.

Chichilnisky, G., Zhou, Y. (1998) "Smooth infinite economies" *Journal of Mathematical Economics* **28**, 27-42.

Debreu, G. (1970) "Economies with a finite set of equilibria" *Econometrica* **38**, 387-92.

Dierker, E. (1982) "Regular Economies", in *The Handbook of Mathematical Economics, vol. II*, edited by K.J. Arrow and M.D. Intriligator, North-Holland Publishing Company: Amsterdam.

Dierker, E. (1989) "Regular Economies" in *The New Palgrave General Equilibrium*, edited by J. Eatwell, M. Milgate and P. Newman, The Macmillan Press Limited: London.

Garratt, R., Goenka, A. (1995) "Income redistributions without catastrophes" *Journal of Economic Dynamics and Control* **19**, 441-455.

Govindan, S., Wilson, R. (2001) "Direct Proofs of Generic Finiteness of Nash Equilibrium Outcomes" *Econometrica* **69**, 3, 765-769.

Loi, A. and S. Matta (2010) "A note on the structural stability of the equilibrium manifold" *Journal of Mathematical Economics* **46**, 591-594.

Loi, A. and S. Matta (2016) "On the topology of the set of critical equilibria" *International Journal of Economic Theory* **12**, 107-126.

Massey, W.S. (1991) *A Basic Course in Algebraic Topology.* Springer-Verlag: New York.