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Monotone comparative statics on semilattices

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Abstract

This paper examines to what extent monotone comparative statics analysis in the style of Milgrom and Shannon (1994) can be extended from lattices to semilattices (e.g., budget sets). A condition on preference orderings is found which is necessary and sufficient for a monotone response of the optimal choice from every subsemilattice to a perturbation of the preferences satisfying an appropriate single crossing condition. A strategic game model related to the consumption of positional goods is formulated where Nash equilibrium exists because the players' preferences satisfy our condition.

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1 Introduction

This paper strives to find out to what extent monotone comparative statics analysis in the style of Milgrom and Shannon (1994) can be extended from lattices to *semilattices* (e.g., budget sets). The results obtained are somewhat ambivalent. On the one hand, a condition on preference orderings is formulated which can claim a role similar to that of quasisupermodularity on lattices. On the other hand, it remains unclear whether preferences satisfying that condition can be found in interesting and important models.

The central result of Milgrom and Shannon (1994), Theorem 4, established conditions for a monotone response of the optimal choice from a lattice to a perturbation of both the preferences and the feasible set. Their Corollary 1 is about a perturbation of the feasible set alone (a "type B" monotone comparative statics problem in the terminology of Quah, 2007). An exhaustive analysis of "type B" problems on a lattice can be found in LiCalzi and Veinott (1992). Conditions for a monotone response of the optimal choice from every sublattice to a perturbation of the preferences ("type A" monotone comparative statics problems in the same terminology) were developed in Kukushkin (2013).

Quah (2007) restricted attention to the choice from subsets of a convex sublattice of $\mathbb{R}^n$, including budget sets, and obtained impressive results concerning "type B" problems. Barthel and Sabarwal (2018) modified his approach, dispensing, to an extent, with convexity requirement and allowing perturbations of both the preferences and the feasible set; however, their results still rely heavily on the structure of coordinate vector space and hence cannot be applied to arbitrary lattices or semilattices.

Here, we work in an abstract order-theoretic context, and the first thing to notice is that neither Veinott's "strong set order" (Topkis, 1978), nor the quasisupermodularity condition, to say nothing of Quah's (2007) constructions, can even be formulated in a semilattice. Accordingly, we consider two "order-like" relations between subsets of a semilattice: one is weaker than Veinott's order, the other stronger. As noted in Kukushkin (2013, p. 1047), neither relation could serve as a base for non-trivial "type B" monotone comparative statics; therefore, we have to restrict ourselves to "type A" problems.

Our main finding is a condition on preferences, "semiquasisupermodularity," so to speak, which ensures a monotone response of the optimal choice from every subsemilattice to a perturbation of the preferences satisfying an appropriate single crossing condition. (When applied to a preference relation on a lattice, our condition neither implies, nor is implied by quasisupermodularity.)

We also formulate a strategic game model related to the consumption of positional goods where the players' preferences satisfy our condition and hence a Nash equilibrium exists by an extension of Tarski's fixed point theorem due to Abian and Brown (1961). There is no need to assume, e.g., the concavity of preferences in the model. From the viewpoint of economics, however, the model can easily be criticised as intolerably stylized.

Section 2 reproduces more or less standard definitions, notation, and results. In Section 3, our replacement for quasisupermodularity is formulated and the main result, Theorem 1, is proven. Section 4 is about our strategic game model.

2 Basic Notions

The notions of a *partially ordered set* (*poset*), a *chain*, and a *lattice* are assumed to be commonly known. A *semilattice* is a poset A where there exists the *greatest lower bound* (*meet*), $x \wedge y$, for every $x, y \in A$. A *subsemilattice* of A is $X \subseteq A$ which contains $x \wedge y$ for every pair $x, y \in X$. A semilattice is *complete* if (i) the greatest lower bound, $\bigwedge X$, exists for every nonempty subset $X \subseteq A$, and (ii) the least upper bound, $\bigvee C$ or $\sup C$, exists for every chain $C \subseteq A$.

These two well-known theorems play important technical roles in the following.

Theorem A (Szpilrajn's Theorem)**.** *On every poset, there exists a linear order $\gg$ extending the basic order, i.e., $y > x \Rightarrow y \gg x$.*

Theorem B. *Let X be a complete semilattice and $r \colon X \to X$ be increasing i.e., $r(y) \geq r(x)$ whenever $y > x$. Then there exists a fixed point of r.*

A stronger version of Theorem A is proven in Dushnik and Miller (1941). Theorem B is a straightforward corollary of the main result of Abian and Brown (1961).

Our general framework is this: an agent has preferences over alternatives from a semilattice A. Those preferences are described by an *ordering* $\succ$, i.e., an irreflexive, transitive and *negatively transitive*, $z \not\succ y \not\succ x \Rightarrow z \not\succ x$, binary relation on A. Then the "non-strict preference" relation $\succeq$ defined by $y \succeq x \rightleftharpoons x \not\succ y$ is reflexive, transitive, and total. A *utility function* $u \colon A \to \mathbb{R}$ defines an ordering:

$$y \succ x \rightleftharpoons u(y) > u(x); \tag{1}$$

to make the converse statement true, one would have to allow an arbitrary chain instead of $\mathbb{R}$ in the definition of a utility function.

The set of all subsets of A is denoted $\mathfrak{B}_A$. Given $X \in \mathfrak{B}_A$, we define

$$M(X, \succ) := \{x \in X \mid \nexists y \in X\, [y \succ x]\} = \{x \in X \mid \forall y \in X\, [x \succeq y]\},$$

the set of *maximizers* of $\succ$ on X. The interpretation is that the agent has preferences over the whole A, but may be faced with the necessity to choose from a subset $X \in \mathfrak{B}_A$, in which case any alternative from $M(X, \succ)$ will do.

To be able to compare $M(X, \succ)$ with $M(X, \succ')$ ("type A" monotone comparative statics problems in the terminology of Quah, 2007) or $M(X, \succ)$ with $M(Y, \succ)$ ("type B" problems in the same terminology), one has to extend the order from A to $\mathfrak{B}_A$. There are several ways to do so on a lattice, see Veinott (1989) or LiCalzi and Veinott (1992), the most important among them being *Veinott's order* (Topkis, 1978):

$$Y \geq^{\mathrm{Vt}} X \rightleftharpoons \forall y \in Y\, \forall x \in X\, [y \wedge x \in X \;\&\; y \vee x \in Y]. \tag{2}$$

On a *semilattice*, $\geq^{\mathrm{Vt}}$ could not even be formulated. In the following, we consider just two relations on $\mathfrak{B}_A$:

$$Y \geq^{\wedge} X \rightleftharpoons \forall y \in Y \, \forall x \in X \, [y \wedge x \in X];$$
$$Y \ggg X \rightleftharpoons \forall y \in Y \, \forall x \in X \, [y \geq x].$$

Relation $\ggg$ can be defined on any poset; relation $\geq^{\wedge}$, just on semilattices. Both relations hold trivially if either Y or X is empty, which fact allows us to discuss monotonicity ignoring existence problem (Shannon, 1995, p. 213). If attention is restricted to *nonempty* subsets, then $\ggg$ is antisymmetric and transitive, but generally not reflexive; the relation $\geq^{\wedge}$ need not even be transitive.

Considering monotone comparative statics on lattices, Milgrom and Shannon (1994) singled out the properties of "quasisupermodularity" and "single crossing." Extending their definition beyond utility functions, we call an ordering on a lattice *quasisupermodular* iff these conditions hold:

$$\forall x, y \in A \left[x \succ y \wedge x \Rightarrow y \vee x \succ y \right]; \tag{3a}$$
$$\forall x, y \in A \left[y \succ y \vee x \Rightarrow y \wedge x \succ x \right]. \tag{3b}$$

Like Veinott's order (2), neither condition (3) could be even formulated on a semilattice.

Let $\succ$ and $\succ'$ be orderings on a poset A. Of various kinds of "single crossing" conditions only two are of use for us here:

$$\forall x, y \in A \left[y > x \ \& \ y \succ x \Rightarrow y \succ' x \right]; \tag{4a}$$
$$\forall x, y \in A \left[y > x \ \& \ y \succeq x \Rightarrow y \succ' x \right]; \tag{4b}$$

Either condition defines a transitive binary relation on the set of orderings on A. The first is reflexive, the second, generally, not.

Proposition 1 from Kukushkin (2013) states that (4a) holds if and only if $M(X, \succ') \geq^{\wedge} M(X, \succ)$ whenever $X \in \mathfrak{B}_A$ is a chain. Proposition 4 from Shannon (1995) states that (4b) holds if and only if $M(X, \succ') \ggg M(X, \succ)$ whenever $X \in \mathfrak{B}_A$ is a chain.

It is convenient for us in the following to write conditions (4) "backwards":

$$\forall x, y \in A \left[y < x \ \& \ y \succeq x \Rightarrow y \succeq' x \right]; \tag{5a}$$
$$\forall x, y \in A \left[y < x \ \& \ y \succeq x \Rightarrow y \succ' x \right]. \tag{5b}$$

It is easily checked that $\succ$ and $\succ'$ satisfy either one of conditions (4) if and only if $\succ'$ and $\succ$ satisfy the corresponding condition (5).

3 Main result

As already noted, neither $\geq^{\wedge}$, nor $\ggg$ could serve as a base for non-trivial "type B" monotone comparative statics theorems, e.g., if $Y \ggg X$, then $M(Y, \succ) \ggg M(X, \succ)$ regardless of what $\succ$ is. From the viewpoint of "type A" monotone comparative statics, the following condition upon an ordering $\succ$ on a semilattice A may pretend to the role of quasisupermodularity:

$$\forall x, y \in A \left[y \wedge x \succeq x \ \text{ or } \ y \wedge x \succeq y \right]. \tag{6}$$

Theorem 1. *Let A be a semilattice and $\succ$ be an ordering on A. Then the following statements are equivalent.*

1. *$\succ$ satisfies (6).*

2. *There holds $M(X, \succ) \geq^{\wedge} M(X, \succ')$ whenever X is a subsemilattice of A and $\succ'$ is an ordering on A satisfying (5a).*

3. *There holds $M(X, \succ) \ggg M(X, \succ')$ whenever X is a subsemilattice of A and $\succ'$ is an ordering on A satisfying (5b).*

Proof. Let (6) hold, $y \in M(X, \succ)$ and $x \in M(X, \succ')$. If (5a) holds, we have to show $y \wedge x \in M(X, \succ')$. Supposing the contrary, $x \succ' y \wedge x$, we have $x \succ y \wedge x$ by (5a), hence $y \wedge x \succeq y$ by (6), hence $x \succ y$, contradicting the optimality of y. Similarly, if (5b) holds, we have to show $y \geq x$. Otherwise, we would have $y \wedge x < x$; since $x \succeq' y \wedge x$, we have $x \succ y \wedge x$ by (5b), hence $y \wedge x \succeq y$ by (6) with the same contradiction.

Let (6) be violated: there are $x, y \in A$ such that $y \succ y \wedge x$ and $x \succ y \wedge x$. Without restricting generality, $y \succeq x$. First, we define $X := \{x, y, y \wedge x\}$ and $Y := \{z \in A \mid z \leq x\}$; our assumption imply $y \wedge x \notin M(X, \succ) \ni y \notin Y$. Second, we define an ordering $\succ''$ on A in this way. If $z', z \in Y$ or $z', z \in A \setminus Y$, then $z' \succ'' z \iff z' \succ z$; if $z \notin Y \ni z'$, then $z' \succ'' z$. On every equivalence class E of $\succeq''$, we pick a strictly increasing total order $\gg_E$, existing by the Szpilrajn theorem (Theorem A). Then we define $\succ'$ as a lexicography: $z' \succ' z$ if $z' \succ'' z$, or if z and z' belong to the same equivalence class E and $z \gg_E z'$.

Now, $\succ'$ is a total order on A, and both $z' > z$ and $z' \succeq' z$ can only hold together when $z' \succ z$; therefore, (5b) holds on A. On the other hand, $M(X, \succ') = \{x\}$; hence even $M(X, \succ) \geq^{\wedge} M(X, \succ')$ does not hold, to say nothing of $M(X, \succ) \ggg M(X, \succ')$. □

Remark. When A is a lattice, (6) obviously implies (11a) from Kukushkin (2013). In this respect, our Theorem 1 is in perfect accord with Proposition 15 from that paper.

The following, rather trivial, statement has a direct analog about a quasisupermodular ordering on a lattice. The next statement, not so trivial, has no analog at all in lattice optimization.

Proposition 2. *Let A be a semilattice and $\succ$ be an ordering on A satisfying (6). Let X be a subsemilattice of A such that $M(X, \succ) \neq \emptyset$. Then $M(X, \succ)$ is a subsemilattice of A.*

Proposition 3. *Let A be a semilattice, $\succ$ be an ordering on A satisfying (6), and $\succ'$ an ordering on A satisfying (5a). Then $\succ'$ satisfies (6).*

Proof. Let $x \succ' y \wedge x$. Then $x > y \wedge x$ and hence $x \succ y \wedge x$ by (5a) [or rather by (4a) with the reversed roles of $\succ$ and $\succ'$]; hence $y \wedge x \succeq y$ by (6). If $y > y \wedge x$, then $y \wedge x \succeq' y$ by (5a). If $y = y \wedge x$, there is nothing to prove. We see that (6) holds for $\succ'$ as well. □

In the terminology of Kukushkin (2013, p. 1048), condition (6) is "downward-looking." It seems natural to ask about "upward-looking" conditions on semilattices. The answer is rather disheartening: the following conditions emerge.

$$\forall x, y \in A \left[[x > y \wedge x < y] \Rightarrow y \wedge x \succeq x\right]. \tag{7a}$$

$$\forall x, y \in A \left[[x > y \wedge x < y] \Rightarrow y \wedge x \succ x\right]. \tag{7b}$$

Obviously, (7b) ⇒ (7a) ⇒ (6).

Proposition 4. *Let A be a semilattice and $\succ$ be an ordering on A. Then $\succ$ satisfies (7a) if and only if $M(X, \succ') \geq^{\wedge} M(X, \succ)$ whenever X is a subsemilattice of A and $\succ'$ is an ordering on A satisfying (4a).*

Proposition 5. *Let A be a semilattice and $\succ$ be an ordering on A. Then $\succ$ satisfies (7b) if and only if $M(X, \succ') \ggg M(X, \succ)$ whenever X is a subsemilattice of A and $\succ'$ is an ordering on A satisfying (4b).*

Both proofs, quite similar to that of Theorem 1, are omitted.

Proposition 6. *Let $\succ$ be an ordering on a semilattice $A := \{\langle x^1, \ldots, x^m \rangle \in \mathbb{R}^m_+ \mid \sum_{k=1}^{m} p_k x^k \leq B\}$ $(\forall k \in \{1, \ldots, m\}\, [p_k > 0])$ with the order*

$$y \geq x \rightleftharpoons \forall k \in \{1, \ldots, m\}\, [y^k \geq x^k]. \tag{8}$$

Then:

1. *$\succ$ satisfies (7a) if and only if $\forall x, y \in A\, [x > y \Rightarrow y \succeq x]$;*
2. *$\succ$ satisfies (7b) if and only if $\forall x, y \in A\, [x > y \Rightarrow y \succ x]$.*

The straightforward proof is omitted.

Proposition 6 does not hold on *all* semilattices, but this is a very weak solace.

4 A strategic game model

Let there be a finite set N of agents, each of which spends her budget B_i on several goods, some of them consumed privately, some openly. Thus, a strategy of player $i \in N$ is a vector $x_i = \langle x_i^0, x_i^1, \ldots, x_i^m \rangle$, where x_i^0 denotes money spent on private consumption and x_i^k ($k \in \{1, \ldots m\}$) money spent on the consumption of positional good k. The set of strategies is

$$X_i := \{\langle x_i^0, x_i^1, \ldots, x_i^m \rangle \in \mathbb{R}^{m+1}_+ \mid \sum_{k=0}^{m} x_i^k = B_i\}.$$

We denote $X_N := \prod_{i\in N} X_i$; given $i \in N$, we denote $X_{-i} := \prod_{j\in N\setminus\{i\}} X_j$.

We denote $K := \{0, 1, \dots m\}$ and $K^* := \{1, \dots m\}$. For each $i \in N$ and $k \in K$, there is a function $v_i^k(x_i^k)$, expressing the internal satisfaction obtained from the consumption of the corresponding good; the consumption profiles of other agents exert a negative influence, $\bar{v}_i(x_{-i})$, "poisoning" satisfaction derived from the consumption of positional goods. All kinds of satisfaction are strictly complementary to one another; thus, the overall utility function of agent i on X_N is

$$u_i(x_N) = \min\{v_i^0(x_i^0), \min_{k\in K^*} v_i^k(x_i^k) - \bar{v}_i(x_{-i})\}. \tag{9}$$

We assume each v_i^k ($k \in K$) to be increasing and upper semicontinuous in x_i, while $\bar{v}_i(x_{-i})$ to be increasing and lower semicontinuous in x_{-i} (in both cases, not necessarily *strictly* increasing).

We define a partial order on X_i by (8) (note that x_i^0 is ignored). Each X_i becomes a complete semilattice; X_N, as well as each X_{-i}, is also a complete semilattice in the Carthesian product of all those orders.

Proposition 7. *For every $i \in N$ and $x_{-i} \in X_{-i}$, the ordering on X_i defined, in the sense of (1), by $u_i(\cdot, x_{-i})$ satisfies (6).*

Proof. Since x_{-i} is fixed throughout, we may simplify notations by replacing $v_i^k(x_i^k) - \bar{v}_i(x_{-i})$ for $k \in K^*$ with just $v_i^k(x_i^k)$, i.e., assuming that $u_i(x_N) = \min_{k\in K} v_i^k(x_i^k)$. Given $x_i, y_i \in X_i$, we denote $M^+ := \{k \in K^* \mid x_i^k > y_i^k\}$ and $M^- := \{k \in K^* \mid x_i^k < y_i^k\}$. If x_i and y_i are comparable in the basic order, then (6) obviously holds; therefore, we may assume that $M^+ \neq \emptyset \neq M^-$. Denoting $z_i := y_i \wedge x_i$, we have $x_i > z_i < y_i$; hence $v_i^0(x_i^0) \le v_i^0(z_i^0) \ge v_i^0(y_i^0)$. Let $x_i \succ z_i$; then $\mathrm{Argmin}_{k\in K} v_i^k(z_i^k) \subseteq M^+$. Since $y_i^k = z_i^k$ for each $k \in M^+$, we have $\mathrm{Argmin}_{k\in K} v_i^k(y_i^k) \subseteq M^+ \cup \{0\}$, and hence $z_i \succeq y_i$. □

Proposition 8. *Let $i \in N$ and let $y'_{-i}, y_{-i} \in X_{-i}$ be such that $y'_{-i} > y_{-i}$. Let $\succ$ and $\succ'$ be orderings on X_i defined by $u_i(\cdot, x_{-i})$ as in (9) with $x_{-i} := y_{-i}$ and $x_{-i} := y'_{-i}$, respectively. Then (4a) holds on X_i.*

Proof. Let $y_i > x_i$ and $u_i(y_i, y_{-i}) > u_i(x_i, y_{-i})$. Since $x_i^0 > y_i^0$, we have $v_0(x_i^0) \ge v_i^0(y_i^0)$ and hence $u_i(x_i, y_{-i}) = v_i^k(x_i^k) - \bar{v}_i(y_{-i}) < v_i^0(y_i^0)$ for a $k \in K^*$, while $v_i^h(y_i^h) > v_i^k(x_i^k)$ for all $h \in K^*$. Since $y'_{-i} > y_{-i}$, we have $u_i(x_i, y'_{-i}) \le v_i^k(x_i^k) - \bar{v}_i(y'_{-i}) < v_i^0(y_i^0)$ as well; moreover, $v_i^h(y_i^h) - \bar{v}_i(y'_{-i}) > v_i^k(x_i^k) - \bar{v}_i(y'_{-i}) \ge u_i(x_i, y'_{-i})$ for all $h \in K^*$. Therefore, $u_i(y_i, y'_{-i}) > u_i(x_i, y'_{-i})$. □

Proposition 9. *Every strategic game satisfying assumptions formulated in this section possesses a Nash equilibrium.*

Proof. For every $i \in N$ and $x_{-i} \in X_{-i}$, the set of the *best responses*, $R_i(x_{-i}) := \mathrm{Argmax}_{x_i\in X_i} u_i(x_i, x_{-i})$ is a subsemilattice of X_i by Propositions 2 and 7. Since u_i is upper semicontinuous, $R_i(x_{-i})$ is nonempty and compact; hence it is a complete subsemilattice and hence there exists $\bigwedge R_i(x_{-i})$, which we denote $r_i(x_{-i})$. By Theorem 1 and Propositions 7 and 8, there holds $R_i(y_{-i}) \geq^{\wedge} R_i(x_{-i})$ whenever $y_{-i} \geq x_{-i}$. Therefore, r_i is increasing, and hence the Cartesian product of all r_i is an increasing mapping $X_N \to X_N$. Now Theorem B applies, establishing the existence of a fixed point, i.e., a Nash equilibrium. □

Remark. It is worth noting that Propositions 7–9 do not need concavity of the utility functions. Moreover, everything remains valid for a discrete version of the model, i.e., if $\mathbb{R}_+^{m+1}$ in (9) is replaced with $\mathbb{N}^{m+1}$. Although these games do not satisfy the definition from Vives (1990), they may be called games with strategic complementarities in an extended sense.

The minimum aggregation in a utility function, i.e., the "absolute complementarity" of components, is not met in economic models very often; however, it is not exceptionally rare either. Galbraith (1958, Chapter XVIII) effectively viewed it as most natural in the evaluation of tradeoffs between public and private consumption ("social balance"). The minimum aggregation of local utilities was employed in the model of Germeier and Vatel' (1974), as well as in a wide class of related models; it ensures even the existence of a strong Nash equilibrium (Harks et al., 2013; Kukushkin, 2017). However, the general context there is different from ours (closer to Galbraith's): "fellow travelers" rather than consumers of positional goods.

Conclusion

Technically speaking, this paper achieves its declared objective: a condition on a preference relation on a semilattice is formulated that plays in "type A" monotone comparative statics problems exactly the same role as quasisupermodularity on lattices. The condition is employed to show the existence of Nash equilibrium in a game-theoretic model.

On the other hand, it is impossible to deny that the model of Section 4 is rather artificial. If a reader will interpret the findings of this paper as showing that any attempt to extend monotone comparative statics analysis in pure order-theoretic terms, as in Theorem 4 of Milgrom and Shannon (1994), beyond lattices is essentially hopeless, well, so be it.

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