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Pre-Decision Side-Bet Sequences

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Abstract

Risk-averse Expected Utility (EU) decision makers with wealth-dependent utility functions may find themselves indifferent between accepting and rejecting an indivisible risky prospect. Bell (1988) showed that under these circumstances it may be EU-enhancing for the decision maker to engage in an actuarially fair pre-decision side bet, accepting the indivisible risky prospect conditional upon winning the side bet. In this letter we show that decision makers restricted to actuarially unfair side bets may optimally engage in a sequence of individually EU-enhancing side bets. This contrasts with the case of actuarially fair side bets where only one such bet will be undertaken. The actuarially unfair side bet's optimal stake size can be a significant proportion of the individual's initial wealth. Nevertheless after losing the side bet wealth may still remain within the interval of interim (utility) convexity, whereupon it is optimal to place another side bet.

1   Introduction

Bell (1988) showed that for decision makers with wealth-dependent utility functions local convexity can emerge on an interim basis for globally risk-averse decision makers who are in possession of a standing offer to acquire a discrete, indivisible risky prospect. Global risk aversion notwithstanding, then, a normatively rational EU decision maker can benefit by increasing her risk exposure through a pre-decision side bet when within such a region of interim local convexity. This seemingly counterintuitive result applies in the neighborhood of any wealth level at which a risky prospect switches from being EU-diminishing to being EU-augmenting. Bell only considered an actuarially fair side bet and demonstrated that only one such bet will be undertaken. However we demonstrate in this letter that if the individual is restricted to actuarially unfair side bets, in the event of losing an actuarially unfair side bet, the decision-maker's wealth is diminished, but not necessarily by enough to fall out of the interval of interim convexity. Hence, a second optimally EU-enhancing side bet may be constructed, the stake of which is smaller than that of the first side bet, keeping the decision maker within the interval of interim convexity. When the decision maker is indifferent between her initial certain wealth and acquiring the indivisible risky prospect, not only will it be possible to construct a single actuarially unfair optimal side bet, but in general a sequence of individually optimal actuarially unfair side bets may be constructed, given continual availability of the side bets at ever-longer odds.

2   Single side bet

We follow Bell (1988) in illustrating the single-side-bet case with a logarithmic-utility example in which the decision maker's initial wealth is $w_0 = \pounds 10,000$. The decision maker is in possession of a standing offer to acquire the risky prospect $g_0 = (\pounds 10,000, \frac{1}{2}; -\pounds 5000, \frac{1}{2})$. We write the prospect in net-final-wealth terms as $g(g_0, w_0) = (\pounds 20,000, \frac{1}{2}; \pounds 5,000, \frac{1}{2})$. Define this to be the round-zero lottery $L_0 = g(g_0, w_0)$. Since

$$E[u(L_0)] = \tfrac{1}{2}\ln(\pounds 20,000) + \tfrac{1}{2}\ln(\pounds 5,000) \;=\; \ln(\pounds 10,000) = u(w_0) \tag{1}$$

the decision maker is indifferent between acquiring the risky prospect and sticking with her initial (certain) wealth.

Now let the decision maker consider a pre-decision side bet $g_1 = (\pounds 1,000, \frac{1}{2}; -\pounds 1,000, \frac{1}{2})$. Although g_1 and g_0 are stochastically independent, the decision maker considers a compound, conditional policy of only acquiring the round-zero lottery L_0 if the side bet proves successful. This compound lottery, which presumes the Reduction of Compound Lotteries (ROCL) axiom, may be written as follows.

$$L_1(g(g_0, w_0), g_1) = \left(\pounds 21,000, \tfrac{1}{4}; \; \pounds 6,000, \tfrac{1}{4}; \; \pounds 9,000, \tfrac{1}{2}\right) \tag{2}$$

Although the side bet g_1 is EU-diminishing in isolation due to risk aversion, when g_0 is implemented conditional upon successful resolution of g_1, the resulting compound lottery is EU-augmenting.

$$E[u(L_1)] = \tfrac{1}{4}\ln(\pounds 21,000) + \tfrac{1}{4}\ln(\pounds 6,000) + \tfrac{1}{2}\ln(\pounds 9,000) \;>\; \ln(\pounds 10,000) = u(w_0) \tag{3}$$

The Certainty Equivalent (CE) of this compound lottery is £10,002.[1] This is a marginal (£2) improvement, reflecting the arbitrary nature of the side bet g_1. By appropriate choice of the side bet, it will in principle be possible to improve upon this CE.

An EU-augmenting side bet can be constructed only if the decision maker's wealth falls within the interval of interim convexity. The bounds of this interval also determine the payoffs and probabilities of the optimal side bet. Figure 1 illustrates the decision-maker's utility function $u(\cdot)$ without the risky prospect and $v(\cdot)$ with the risky prospect. The common tangent to $u(\cdot)$ and $v(\cdot)$ identifies the lower and upper bounds of the interval of interim convexity (w_0', w_0''). The solution method for identifying these bounds analytically is presented in Appendix A. This yields the numerical values $(w_0', w_0'') = (9037.16, 10774.6)$. Given the decision-maker's assumed wealth in this example, an EU-augmenting side bet is possible since $10,000 \in (9037.16, 10774.6)$, and the unconstrained optimal side bet is given by $g_1^* = (w_0'' - w_0, \frac{w_0 - w_0'}{w_0'' - w_0'}; \ w_0' - w_0, \frac{w_0'' - w_0}{w_0'' - w_0'}) = (£774.6, 0.554; \ -£962.84, 0.446)$. With this optimal pre-decision side bet g_1^*, the compound lottery becomes

$$L_1(g(g_0, w_0), g_1^*) = (£20774.6, 0.277; \ £5774.6, 0.277; \ £9037.16, 0.446) \tag{4}$$

and the decision maker increases her utility

$$E[u(L_1(g(g_0, w_0), g_1^*))] \ > \ E[u(L_1(g(g_0, w_0), g_1))] \ > \ u(w_0) \tag{5}$$

to the CE wealth level of £10,052.8, which is clearly greater than the £10,002 of the arbitrary side bet g_0.

3   Sequence of actuarially unfair side bets

We now demonstrate the new result that a sequence of side bets may be optimal. Here we focus on actuarially unfair 'wager' side bets based on European Roulette. The (negative) expected return per unit staked at European Roulette is $\overline{\mu} = -\frac{1}{37}$. Odds are denoted a. For a single-number bet in European roulette, the payout odds are 35:1, that is $a = \frac{35}{1}$. The associated win probability is $p = \frac{1+\mu}{1+a}$, which for the single-number bet is $p = (1 + \frac{-1}{37})/(1 + \frac{35}{1}) = \frac{1}{37}$. Here for illustrative purposes and to illustrate the length of the betting sequence we assume that odds are offered at all probabilities with a negative expected rate of return of $-\frac{1}{37}$. Meanwhile, the interim-convexity bounds remain as above.[2] To determine the optimal round-one counterparty-constrained optimal side bet, we maximize
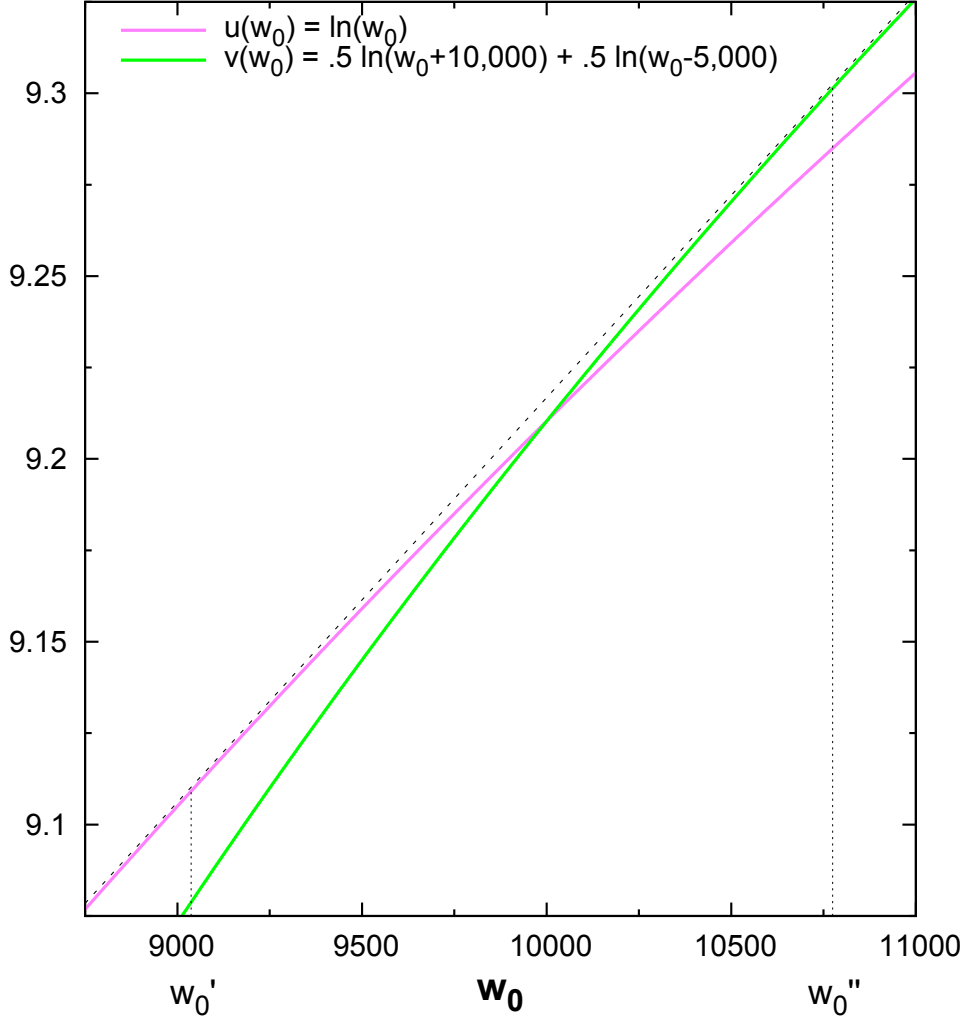
$$p\left[\frac{1}{2}\ln(w_0 + 10,000 + sa) + \frac{1}{2}\ln(w_0 - 5,000 + sa)\right] + (1-p)\ln(w_0 - s) - \ln(w_0) \tag{6}$$

by appropriate choice of stake s and payout odds a. As the win probability p is a function of a and the fixed parameter $\overline{\mu}$, there are no other undetermined parameters. This is solved by setting $s = 573.33$ and $sa = 732.77$ – i.e. smaller than in the unconstrained, actuarially fair problem – from which follows that $a = 732.77/573.33 = 1.27809$ and consequently $p = (1 - (1/37))/(1 + 1.27809) = 0.42710$. Due to the house advantage $\overline{\mu} = \frac{-1}{37}$, the expected value of this side bet is $|\overline{\mu}|s = £15.50$ less than in the unconstrained problem,

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[1] $E[u(L_1)] = u(£10,002)$

[2] The round-one side bet's bounds depend on round-zero parameters alone.

Figure 1: Common tangent to $u(\cdot)$ and $v(\cdot)$ which determines the bounds of the interval of interim convexity (w_0', w_0'') for the round-one side bet.

i.e. $E[w_0 + psa - (1-p)s] = \pounds10,000 - \pounds15.50 = \pounds9,984.50$.

$$L_1(g(g_0, w_0), g_1^*|_{\overline{\mu}}) = (\pounds20732.77, 0.21355; \ \pounds5732.77, 0.21355; \ \pounds9,426.67, 0.5729) \quad (7)$$

With the optimal counterparty-constrained pre-decision side bet $g_1^*|_{\overline{\mu}}$, the CE of this compound lottery (7) becomes $\pounds10,030.69$, in which the $\pounds30.69$ increase over the pre-side-bet CE is 58% of the $\pounds52.80$ increase achieved with the actuarially fair side bet g_1^*. Still, with the counterparty-constrained optimal side bet $g_1^*|_{\overline{\mu}}$ the compound lottery $L_1(g(g_0, w_0), g_1^*|_{\overline{\mu}})$ nevertheless proves to be EU-augmenting.

$$E[u(L_1(g(g_0, w_0), g_1^*))] \quad > \quad E[u(L_1(g(g_0, w_0), g_1^*|_{\overline{\mu}}))] \quad > \quad E[u(L_1(g(g_0, w_0), g_1))] \quad > \quad u(w_0)$$
$$(8)$$

4   Sequences of individually optimal side bets

When side bets are constrained to be actuarially unfair, the round-one side-bet stake is smaller than the difference between initial wealth w_0 and the lower bound of the interval of convexity w_0'. Hence, after losing the side bet $g_1^*|_{\overline{\mu}}$ in the round-one compound lottery $L_1(g(g_0, w_0), g_1^*|_{\overline{\mu}})$ and therefore the associated stake $s_{1\overline{\mu}}^* = \pounds 573.33$, the decision-maker's wealth is reduced to $w_1 = \pounds 9,426.67 \in (w_0', w_0'')$, which is within the interval of convexity. Consequently the decision-maker can benefit from a further side-bet round. In turn the stake of the optimal round-two side bet $s_{2\overline{\mu}}^* = \pounds 135.25$ is also smaller than $w_1 - w_0'$. Upon losing the side bet of the round-two compound lottery $L_2(g(g_0, w_1), g_2^*|_{\overline{\mu}})$, the decision-maker's wealth is reduced to $w_2 = \pounds 9,291.43 \in (w_0', w_0'')$, which is within the interval of convexity. Again the decision maker can benefit from a further side-bet round. Table 1 presents this sequence of individually EU-augmenting optimal side bets. These side bets have been computed under the assumption that the minimum stake-size increment is 0.01 (one penny). The round-four side bet's $\mathrm{CE}_{4\overline{\mu}}^*$ is larger than w_3 in the fourth decimal digit. Similarly $\mathrm{CE}_{5\overline{\mu}}^* > w_4$ in the sixth decimal digit, and $\mathrm{CE}_{6\overline{\mu}}^* > w_5$ in the eighth decimal digit. Notice that availability of ever-longer-odds side bets is necessary for extending the sequence, which nevertheless is truncated to six rounds due to the discrete, one-penny stake-size increment.

Table 1: Parameters of optimal EU-augmenting round-i side bets in European roulette ($\overline{\mu} = \frac{-1}{37}$).

i	w_{i-1}	$s_{i\overline{\mu}}^*$	$a_{i\overline{\mu}}^*$	$|\overline{\mu}|s_{i\overline{\mu}}^*$	$\mathrm{CE}_{i\overline{\mu}}^*$	$|\overline{\mu}|s_{i\overline{\mu}}^* \prod_{j=0}^{i-1} q_j$
1	10,000.00	573.33	1.28	15.49	10,030.69	15.49
2	9,426.67	135.25	9.85	3.66	9,428.03	2.10
3	9,291.42	19.41	75.58	0.52	9,291.45	0.27
4	9,272.01	1.80	827.27	0.05	>9,272.01	0.03
5	9,270.21	0.23	6482.14	0.01	>9,270.21	0.00
6	9,269.98	0.03	49702.21	0.00	>9,269.98	0.00
$\sum$		730.05		19.73		17.89

4.1   Maximally unfair case

Here we ask, what is the most *disadvantageous* per-unit-bet expected return $\underline{\mu}$ that the decision maker will still find EU enhancing?

Two of the parameters determining $\underline{\mu}$ may be treated as exogenous for present purposes. The first captures the decision maker's 'just-noticeable difference', operationalized as the number of significant digits to which EU and ϵ are measured. The second captures the house minimum-stake-size increment (here assumed to be 1 penny). Operationally we maximize $|\underline{\mu}|$ subject to (i) the discrete 1-penny minimum-stake-size increment and (ii) the requirement that the round-i side bet be EU-enhancing ($w_{i-1} - \mathrm{CE}_{i\underline{\mu}}^* > 0$) at

Table 2: Parameters of individually optimal EU-augmenting round-i side bets for maximally disadvantageous, but still EU-augmenting mean return ($\underline{\mu} = -0.19$).

i	w_{i-1}	$s^*_{i\mu}$	$a^*_{i\mu}$	$|\underline{\mu}|s^*_{i\mu}$	$CE^*_{i\mu}$	$|\underline{\mu}|s^*_{i\mu}\prod_{j=0}^{i-1}q_j$
1	10,000.00	1.11	45.338	0.21	>10,000.00	0.21
2	9,998.89	0.90	75.998	0.17	>9,998.89	0.17
3	9,997.99	0.16	564.422	0.03	>9,997.99	0.03
4	9,997.83	0.07	1146.683	0.01	>9,997.83	0.01
$\sum$		2.24		0.42		0.42

any arbitrarily fine-grained just-noticeable difference. Table 2 presents the sequence of individually EU-enhancing side bets which follows the first-round stake and mean-return combination $(s^*_{1\mu}, \underline{\mu}) = (1.11, -0.19)$.

5 A tie-breaking rule?

Since the side-bet strategy breaks the decision maker's indifference between initial wealth w_0 and the indivisible risky prospect L_0, it is natural to query the extent to which the side-bet strategy can be understood as a *tie-breaking rule*. And if this conception is valid, what function can a side bet perform once the first side-bet is played out, and wealth deviates from the indifference level w_0?[3]

This is an incisive question that demands careful revisitation of the tie-breaking literature. A tie-breaking rule may be understood "as a second strict and transitive preference relation that the agent consults only when he is indifferent" (Kimya 2017, p. 140). Crucially, however, a tie-breaking rule does not alter the decision maker's underlying preference ordering. The indifference class remains unchanged by the tie-breaking rule, as does the rest of the preference ordering above and below the indifference class.

Meanwhile, Bell's (1988) side-bet strategy constitutes a new 'act' – a non-redundant mapping from the set of states to the set of consequences – which increases the decision maker's utility above that delivered by initial wealth alone or by directly undertaking the indivisible round-zero lottery L_0. As a new act, it alters the set of available non-redundant choice options, and thus it also alters the decision maker's underlying preference ordering. Hence, Bell's (1988) side-bet strategy is not a 'tie-breaking rule' in a formal sense as it is understood in the literature.

A careful reading of Bell (1988) shows that the side-bet strategy differs from a tie-breaking rule in a second sense: "This strategy works in the *neighborhood* of any wealth level at which alternative switches from being unattractive to being attractive" (emphasis added, p. 797). The side-bet strategy offers the prospect of enhanced expected utility when wealth is in the non-degenerate *interval of interim convexity*.

The possibility for side-bet *sequences* arises precisely because the decision maker's optimal side-bet's stake is *smaller* when wagering opportunities are unfair ($\mu < 0$), and therefore even if she loses the side bet, she remains *within* the interval of interim convexity.

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[3]We thank an anonymous Reviewer for raising the connection with tie-breaking rules.

6   Conclusion

In this note we revisit Bell's (1988) finding that a pre-decision side bet can be a rational, EU-enhancing strategy for determining whether to take on a large, indivisible, risky prospect. When pre-decision side bets are constrained to be actuarially unfair, unlike in Bell's presentation, the side bet's optimal stake size is biased downward. Upon losing the side bet, the decision maker's wealth consequently remains within the interval of interim convexity, instead of being ejected to its lower boundary. Hence, the decision maker rationally engages in a further side-bet round. Under the assumptions of Bell's (1988) example, we find that the decision maker rationally engages in up to four side-bet rounds (in one case, six rounds). For each successive optimal side bet, the stake becomes smaller and the required odds become longer. This ever-longer-odds requirement may hinder implementation of extended side-bet sequences in practice. Nevertheless side-bet sequences are surprisingly general from a theoretical standpoint, as they arise in all families of non-linear, non-exponential risk-averse utility functions.[4] Together with Bell's (1988) seminal result, the present finding expands the range of empirical phenomena that can be explained within the framework of normative rationality. However, it also introduces a further set of methodological considerations that must be confronted in the design of risk-aversion-elicitation procedures and in the empirical estimation of risk-aversion coefficients.

References

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Kimya, M. (2017) "Nash implementation and tie-breaking rules" *Games and Economic Behavior* **102**, 138–146.

Quiggin, J. (1993) *Generalized Expected Utility Theory: The Rank-Dependent Model.* Kluwer Academic: Boston, MA.

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[4]We have found that this effect may be replicated with Rank-Dependent Expected Utility (Quiggin, 1993).

A   Side-bet bounds

The tangent line to $u(\cdot)$ at $(w_0', u(w_0'))$, where $u(w_0) = \ln(w_0)$, is

$$y = u(w_0') + u'(w_0')(w_0 - w_0') \tag{9}$$

$$= \ln(w_0') + \frac{w_0 - w_0'}{w_0'} \tag{10}$$

while the tangent line to $v(\cdot)$ at $(w_0'', v(w_0''))$, where $v(w_0) = \frac{1}{2}\ln(w_0 + 10,000) + \frac{1}{2}\ln(w_0 - 5,000)$, is given by

$$y = u(w_0'') + u'(w_0'')(w_0 - w_0'') \tag{11}$$

$$= \frac{1}{2}\left(\ln(w_0'' + 10,000) + \ln(w_0'' - 5,000) + \frac{w_0 - w_0''}{w_0'' + 10,000} + \frac{w_0 - w_0''}{w_0'' - 5,000}\right) . \tag{12}$$

In order for these lines to be the same (i.e. a common tangent to $u(\cdot)$ and $v(\cdot)$) then they must share the same slope

$$\frac{1}{w_0'} = \frac{1}{2(w_0'' + 10,000)} + \frac{1}{2(w_0'' - 5,000)} \tag{13}$$

and vertical intercept

$$\ln(w_0') - 1 = \frac{1}{2}\left(\ln(w_0'' + 10,000) + \ln(w_0'' - 5,000) - \frac{w_0''}{w_0'' + 10,000} - \frac{w_0''}{w_0'' - 5,000}\right) . \tag{14}$$

Solving these two equations computationally yields $(w_0', w_0'') = (9037.16, 10774.6)$.