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On the Representability of a Class of Lexicographic Preferences

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Abstract

We present sufficient conditions for the non-representability of a class of lexicographic preferences. The proof of non-representability builds on the classical Debreu (1954) argument and its usefulness in generating counter examples is demonstrated through a series of examples.

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1 Introduction

Lexicographic preference is often the most accessed example of non-representability of a binary relation that is complete, transitive and monotone. The example originates in Debreu (1954)[1]and has received a formal textbook treatment in leading theory textbooks, for example, Mas-Colell, Whinston and Green (1995).

While the appeal of lexicographic reasoning is hardly contained by its use in generating counter-examples of non-representable preference orders, in this paper our interest is in this specific methodological use of a class of lexicographic preference orders. In particular our motivation is to provide a class of lexicographic orders and establish conditions on which such an order is non-representable. Our hope is that such a tool will find use in generating examples in decision theory. The argument presented here and the conditions sufficient for non-representability are derived from a similar result in Debreu (1954). We deal with the non-representation issue exclusively in contrast to the rich literature on the characterization of lexicographic preferences — see for example the classic by Fishburn (1975) and Mitra and Sen (2014) more recently.

One can think of our version of lexicographic preference as one that is defined on the space of "attributes". Following Lancaster (1966), if one interprets consumer behavior as an individual's attempt to choose the right mix of attributes in a consumption bundle and not just the bundle itself, then our preferences define a lexicographic ordering on what is considered the "right" mix. For instance, a consumer in choosing a car could have a value function defining the mix of safety and mileage in each car, and among those cars for which she is indifferent with regards to the consideration of safety and mileage, she chooses the one that gives the most mileage. In this example the consumer's preference is defined over the attributes of safety (x_1) and mileage (x_2) and his true preference over two attribute profiles (x_1, x_2) and (x'_1, x'_2) is a lexicographic ordering of the form $(v(x_1, x_2), x_2) \geq_L (v(x'_1, x'_2), x'_2)$, where v is the value she assigns to the mix of safety and mileage. This is the simple extension to the standard lexicographic case that we pursue, and demonstrate two conditions that guarantee non-representability.

As a road map of what lies ahead, we prove the main non-representability result in Proposition 1 (section 2) and demonstrate how failure of the sufficient conditions (**P1** and **P2**, stated in section 2) change the conclusion of Proposition 1. Finally in section 3 we demonstrate how the result of this note can be applied to concrete examples from the class of lexicographic orders under consideration.

[1]The use of the lexicographic order in topology and order theory has a long history. For an early order theoretical reference the reader can consult Sierpinski (1965, p. 221).

2 The Result

We focus our attention on the positive orthant of the two-dimensional Euclidean space[2]. Let $f : \mathbb{R}^2_+ \to \mathbb{R}_+$ and $g : \mathbb{R}^2_+ \to \mathbb{R}_+$ be two functions. We list the properties crucial to the analysis.

(P1) $f(a,a) \neq f(b,b)$ for $a, b \in \mathbb{R}_+$ and $a \neq b$.

(**P2**) There is an uncountable subset $U \subseteq \mathbb{R}_{++}$ such that for every $r \in U$ there is an $x \in \mathbb{R}^2_+$ with $x \neq (r,r)$ such that (i) $f(r,r) = f(x)$ and (ii) $g(x) \neq g(r,r)$.

Now define $\succsim$ by

$$(a,b) \succsim (a',b') \text{ iff } (f(a,b), g(a,b)) \geq_L (f(a',b'), g(a',b')) \tag{1}$$

where, $\geq_L$ is the standard lexicographic order. Recall that this means that whenever $f(a,b) > f(a',b')$ we have $(a,b) \succ (a',b')$ and if $f(a,b) = f(a',b')$, then $(a,b) \succsim (a',b')$ iff $g(a,b) \geq g(a',b')$. Denote the symmetric and asymmetric components of $\succsim$ $(\geq_L)$ by $\sim$ $(=_L)$ and $\succ$ $(>_L)$ respectively.

We will say that a binary relation $\succsim$ on $\mathbb{R}^2_+$ is *representable* if there is some function $u : \mathbb{R}^2_+ \to \mathbb{R}$ such that

$$x \succsim y \text{ iff } u(x) \geq u(y).$$

In **P1** and **P2** the diagonal $D = \{(x_1, x_2) : x_1 = x_2\}$ plays a crucial role. Condition **P1** states that the first component in the lexicographic ordering is sensitive to distinct element along the diagonal[3].

Condition **P2** is the crucial sufficient condition that makes preferences defined in (1) non-representable. Lexicographic orders of the form (1) will have intuitive appeal when the priority defining functions f and g are in some conflict. The precise nature of this conflict is what motivates condition **P2** and its structure is dictated by the proof of Proposition 1 (a generalization of the standard argument presented in Debreu 1954).

Proposition 1. *If f, g satisfy P1 and P2, then the preference order $\succsim$ as defined in (1) is not representable.*

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[2] While generalization to higher dimensions and more abstract spaces are possible they would not add much to our understanding of Proposition 1.

[3] Weak forms of strict monotonicity are sufficient for **P1**; for example if we require strict monotonicity only along the diagonal (Diagonal Pareto: $f(a,a) > f(b,b)$ whenever $a > b$) then **P1** would be satisfied. Sufficient conditions for representation of preferences that satisfy Diagonal Pareto are explored in Banerjee (2014).

PROOF: Suppose if possible, $\succsim$ be representable by a real valued function $u : \mathbb{R}^2_+ \to \mathbb{R}$. Let U be an uncountable subset as in property **P2**. For each $r \in U$ pick $c(r) \in B(r)$ where,

$$B(r) = \{x \in \mathbb{R}^2_+ : x \neq (r,r), f(r,r) = f(x) \text{ and } g(r,r) \neq g(x)\}. \tag{2}$$

By **P2**, the set $B(r)$ given in (2) is non empty, so there is a $c(r) \in B(r)$[4] and the inequality $g(c(r)) \neq g(r,r)$ must hold. Define:

$$x(r) = \begin{cases} c(r) \text{ if } g(r,r) > g(c(r)) \\ (r,r) \text{ otherwise.} \end{cases} \tag{3}$$

and

$$x'(r) = \begin{cases} c(r) \text{ if } g(r,r) < g(c(r)) \\ (r,r) \text{ otherwise.} \end{cases} \tag{4}$$

Now for any $r \in U$ we must have $x'(r) \succ x(r)$. To see this consider two cases: (a) $g(r,r) > g(c(r))$ and (b) $g(r,r) < g(c(r))$.

In (a) since $g(r,r) > g(c(r))$ it follows that $x'(r) = (r,r)$ (by (4)) and $x(r) = c(r)$ (by (3)). Now $f(r,r) = f(c(r))$ and $g(r,r) > g(c(r))$ implies (using (1)) that $x'(r) \succ x(r)$ holds.

In (b) since $g(r,r) < g(c(r))$, an argument similar to case (a) yields $x'(r) = c(r)$ and $x(r) = (r,r)$ which would again imply (using $f(r,r) = f(c(r))$ and the definition (1)) $x'(r) \succ x(r)$ as required.

Denote the non-degenerate interval $[u(x(r)), u(x'(r))]$ by $I(r)$. To complete the argument we need to show that whenever $q \neq r$ we must have $I(q) \cap I(r) = \emptyset$. Assume $q \neq r$, using **P1** we know that $f(r,r) \neq f(q,q)$. Two cases are possible (i) $f(r,r) < f(q,q)$ (ii) $f(r,r) > f(q,q)$.

In (i) we will compare $x'(r)$ with $x(q)$ and show that $x(q) \succ x'(r)$. Observe that

$$f(c(r)) = f(r,r) < f(q,q) = f(c(q)) \tag{5}$$

holds (the equalities follow from the definition of c and the inequality is the consequence of the assumption for case (i)). As $x(r)$ is either (r,r) or $c(r)$ and $x'(q)$ is either (q,q) or $c(q)$, (5) and (1) implies $x(q) \succ x'(r)$ as was needed. This establishes $I(q) \cap I(r) = \emptyset$ in case (i).

[4] On the collection $\mathcal{B} = \{B(r) : r \in U\}$ of non-empty sets there exists a choice function (using the Axiom of Choice) $c^* : \mathcal{B} \to \cup_{B \in \mathcal{B}} B$ such that $c^*(B) \in B$ for $B \in \mathcal{B}$. In recognition of the role played by the uncountable set U in this proof we are making the dependence on the elements of U explicit by writing $c(r) = c^*(B(r)) \in B(r)$.

In (ii) we will compare $x'(q)$ and $x(r)$ and show that $x(r) \succ x'(q)$. Note that in case (ii)

$$f(c(r)) = f(r,r) > f(q,q) = f(c(q))$$

must hold. Using an argument similar to case (i) we can show that $x(r) \succ x'(q)$ is true. This implies $I(q) \cap I(r) = \emptyset$ for case (ii).

In conclusion, the interval $I(q) \cap I(r) = \emptyset$ holds whenever $r \neq q$ and $r, q \in U$. So we have established a one-to one correspondence between a collection of non-degenerate, non-intersecting intervals of $\mathbb{R}$ (a countable collection) and the set U (an uncountable set). This is a contradiction which proves that $\succsim$ is not representable. ■

Remarks.

We demonstrate that Proposition 1 fails to hold when we violate each of the three conditions (**P1, P2(i)** and **P2(ii)**). Three simple examples of the form (1) are presented towards that goal. In each example the set U from condition **P2** is $\mathbb{R}_{++}$.

(i) Proposition 1 fails without P1: Let $f : \mathbb{R}^2_+ \to \mathbb{R}$ be defined as $f(x) = 5$ and for all $x \in \mathbb{R}^2_+$ and let $g : \mathbb{R}^2_+ \to \mathbb{R}$ be defined by $g(x) = 6 + x_1 + x_2$. It is easy to see that **P2** is satisfied but **P1** fails to hold. The resultant order $\succsim$ is representable, since in this case $x \succsim y$ iff $g(x) \geq g(y)$ must hold true.

(ii) Proposition 1 fails without P2 (i): Let $f : \mathbb{R}^2_+ \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} \max\{(x_1/x_2), (x_2/x_1)\} & \text{if } x_1 \neq x_2 \text{ and } (x_1, x_2) \in \mathbb{R}^2_{++} \\ [x_1/(x_1+1)] & \text{if } x_1 = x_2 \text{ and } (x_1, x_2) \in \mathbb{R}^2_{++} \\ 0 & \text{otherwise} \end{cases}$$

and let $g : \mathbb{R}^2_+ \to \mathbb{R}$ be defined by $g(x) = -[f(x)]$. We verify that condition **P1** holds. Note that along the diagonal f is a strictly increasing function in x_1 which implies that **P1** must be satisfied. To show that condition **P2(i)** fails it is enough to note that $f(r,r) < 1$ for any $r > 0$ and $f(x) > 1$ for any $x \in \mathbb{R}^2_{++}$ with $x_1 \neq x_2$. Condition **P2(ii)** is satisfied, since $g(r,r) = -[f(r,r)] > -1 > -[f(x)] = g(x)$ for all $x \in \mathbb{R}^2_{++}$ and $r > 0$. Hence the resultant order $\succsim$ satisfies **P1** but not **P2**. However observe that $\succsim$ is representable since whenever $x \succsim y$ we must have $f(x) \geq f(y)$ and conversely.

(iii) Proposition 1 fails without P2 (ii): Let $f : \mathbb{R}^2_+ \to \mathbb{R}$ be defined as $f(x) = x_1 + x_2$ and $g : \mathbb{R}^2_+ \to \mathbb{R}$ by $g(x) = f(x)$. We can verify that properties **P1** and **P2(i)** are satisfied but **P2(ii)** fails. The resultant order $\succsim$ is representable, since in this case $x \succsim y$ iff $f(x) \geq f(y)$ must hold true.

3 Examples

We conclude the analysis with three examples of non-representable preferences.

Example 1: We demonstrate that the standard lexicographic order, an example often cited as an example in the discussion of non-representable preferences in a standard introductory graduate Microeconomic Theory course (see Mas-Colell, Whinston and Green 1995, p. 46) meets the conditions required by Proposition 1. Take $f(x_1, x_2) = x_1$ and $g(x_1, x_2) = x_2$ for all $(x_1, x_2) \in \mathbb{R}^2_+$. As f is strictly increasing along the diagonal condition **P1** must hold. To verify condition **P2**, for $r \in U$ take $x = (r, r+1)$. Then $f(x) = f(r, r) = r$ and $g(x) = r + 1 \neq g(r, r) = r$. So we can apply Proposition 1 and conclude that $\succsim$ in this case is not representable.

Example 2: Define $f : \mathbb{R}^2_+ \to \mathbb{R}$ and $g : \mathbb{R}^2_+ \to \mathbb{R}$ as follows: $f(x_1, x_2) = x_1 + x_2$ for $(x_1, x_2) \in \mathbb{R}^2_+$; $g(x_1, x_2) = [x_1 x_2/(x_1 + x_2)]$ when $(x_1 + x_2) \neq 0$ and $g(x_1, x_2) = 0$ when $(x_1 + x_2) = 0$. As f is strictly increasing, property **P1** must be satisfied. To verify **P2**, for any $r \in \mathbb{R}_{++} = U$ take $x = (r - \varepsilon, r + \varepsilon)$ for some $\varepsilon \in (0, r)$. Then $f(r, r) = f(r - \varepsilon, r + \varepsilon) = 2r$, $g(x) = (r - \varepsilon)(r + \varepsilon)/2r = (r^2 - \varepsilon^2)/2r$ and $g(r, r) = r^2/2r \neq g(x)$ as needed. So we can apply Proposition 1 to show that the resultant order $\succsim$ is not representable.

Example 3: In all the examples presented the set U in condition **P2** turned out to be $\mathbb{R}_{++}$. This example shows that there are cases where the set U in **P2** is a strict subset of $\mathbb{R}_{++}$. Let $f : \mathbb{R}^2_+ \to \mathbb{R}$ and $g : \mathbb{R}^2_+ \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x_1 & \text{if } x_1 \text{ is irrational} \\ x_2 & \text{if } x_1 \text{ is rational} \end{cases}$$

and

$$g(x) = \begin{cases} x_2 & \text{if } x_2 \text{ is irrational} \\ x_1 & \text{if } x_2 \text{ is rational} \end{cases}$$

To verify that **P1** is satisfied, it is sufficient to note that for any $r > 0$ we must have $f(r, r) = r$. Denote the set of positive irrational numbers by $\mathbb{I}_{++}$ and set $U = \mathbb{I}_{++}$, noting that U is uncountable. Consider **P2**, for $r \in U$ we let $x = (r, r+1)$ and note that both r and $r + 1$ are irrational numbers. Our choice of x implies $f(r, r) = f(x) = r$ and $g(x) = r + 1 \neq g(r, r) = r$ as needed for **P2**. Applying Proposition 1 we conclude that $\succsim$ is not representable.

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