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Efficient portfolios and the generalized hyperbolic distribution

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Abstract

This paper proposes a twist to the classical Markowitz approach to build efficient portfolios of risky assets that improves their risk-return performance. The originality of our approach consists in the utilization of a covariance matrix from a member of the Generalized Hyperbolic (GH) Family distribution, instead of the sample covariance matrix described in Markowitz's (1959) seminal contribution. We test the approach with the daily returns of stocks traded in MILA (Mercado Integrado Latino Americano) markets: Chile, Colombia, Mexico and Peru, from January 1st, 2010, to December 31st, 2015. The GH based portfolios are benchmarked with an equally weighted portfolio and a historical covariance based Markowitz portfolio using the coefficient of variation of returns. The results confirm the GH based portfolio's dominance over the other two benchmarks.

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1. Introduction

According to Markowitz (1952, 1959), an efficient portfolio combines market-traded financial assets to maximize returns at a given level of risk, or minimizes risk at a given level of return. Markowitz's optimization method to build efficient portfolios uses the covariance matrix as a key input, and determines the optimal portfolio weights by solving a non-linear programming problem (Alayón, 2014); however, the population covariance matrix is never known in practice, so the best-fit normal multivariate probability density function is generally used, even when there is a consensus about the non-normality of the returns (Hu, 2005).

While many models in finance assume the normality of financial returns, it has been extensively documented that financial data series do not comply with a normal distribution. For that reason, an increasing number of studies are considering alternative methodological approaches (Karoglou 2010; Peiro 1999; Sheikh and Qiao 2010). For example, according to Alayón (2014), "the Generalized Hyperbolic (GH) Distribution has been increasingly used by academicians and practitioners to solve the problem of heavy tails in financial data distributions, and for its usefulness to model asset returns and market risk measures".

The GH family of probability distributions was introduced by Barndorff-Nielsen (1977) and was first used in financial data analysis by Eberlein (1995), who adjusted the univariate hyperbolic distribution to the returns of German stocks. Some years later, Protassov (2004) used the Expectation-Maximization algorithm (EM) to estimate a multivariate GH of dimension five in a sample of OECD countries exchange rates, and is recognized as the first to estimate the GH distribution with more than three dimensions.

This investigation follows Protassov's (2004) algorithm to estimate the parameters of the covariance matrix to solve for the optimal weights of a Markowitz's type portfolio, and takes into consideration some additional numerical aspects mentioned by Breymann (2013). The process also implements McNeil's (2005) specification and does not fix the parameter of the third-order Bessel function. Finally, a hypothesis testing procedure developed by McAssey (2013) to evaluate the goodness-of-fit of the GH distribution corroborates the quality and reliability of the results.

The covariance matrix is estimated with a multivariate GH distribution and its associated parameters, and then used as an input for the Markowitz non-linear programming procedure to obtain efficient-portfolio weights. Tests on the parameters using *t* statistics (from asymptotic theory) confirm the statistical significance of the estimations. Finally, the GH based portfolio is compared to a) an equally weighted and b) a Markowitz portfolio using their coefficient of variation as a measure of performance, confirming the proposed methodology's superiority.

The next section discusses theoretical and applied aspects of the GH family and the hypothesis testing methods used in this work. In the third section, estimates of the portfolios parameters and performance results are explained. Finally, the fourth section discusses the main empirical findings of the study, confirming that the GH-based efficient portfolios' performance clearly dominates that of traditional Markowitz portfolios, and concludes the study.

2. Methodological approach

It is common to observe that financial data returns have heavier tails in comparion with a normal distribution. Generalized Hyperbolic distributions have greater flexibility because their functional form contains more parameters, which allows a better adjustment for the implementation of applications on data with heavy tails. The normal distribution lacks that flexibility because it considers only two parameters, i.e., the mean and the variance (or its square root, the standard deviation). There are very well known examples of members or limiting cases of Generalized Hyperbolic Distributions (e.g. t-Student, Variance-Gamma, Hyperbolic, Normal and Normal Inverse Gaussian), and each incorporate different levels of skewness and kurtosis.

The GH family is established in terms of the λ, χ and ψ parameters of a Generalized Inverse Gaussian (GIG) distribution; μ, and Σ from a normal multivariate distribution, and a vector of bias denoted with γ (Hu, 2005). The last parameter mixes the GIG density function with a normal multivariate in μ, Σ (McNeil, 2005), as in equation (1):

$$f(x;\lambda,\chi,\psi) = \frac{1}{k_\lambda(\chi,\psi)} x^{\lambda-1} \exp\left[-\frac{1}{2}(\chi x^{-1} + \psi x)\right]; \tag{1}$$

and the function $k_\lambda(\chi, \psi)$ is an alternative specification of the third kind Bessel function, (Paolella 2007), given by equation (2), as follows:

$$k_\lambda(\chi,\psi) = \int_0^\infty x^{\lambda-1} \exp\left[-\frac{1}{2}(\chi x^{-1} + \psi x)\right] dx. \tag{2}$$

The parameters satisfy the condition that λ is a real number and $\chi, \psi \geq 0$. Formally, if X is a random vector of dimension $n \times 1$, then, as indicated in equation (3):

$$X|W = w \sim N_n(\mu + w\beta\Delta, w\Delta), \text{ and} \qquad W \sim GIG(\lambda,\chi,\psi), \tag{3}$$

where $N_n(\mu + w\beta\Delta, w\Delta)$ represents a multivariate normal variable with mean $\mu + w\beta\Delta$ and covariance matrix $w\Delta$.
Protassov (2004) rewrites the function of joint probability density – equation (4) – as:

$$f(x;\lambda,\alpha,\beta,\mu,\delta,\Delta) = \frac{\left(\sqrt{\frac{\delta^2+(x-\mu)\Delta^{-1}(x-\mu)\prime}{\alpha^2+\beta'\Delta\beta}}\right)^{\lambda-\frac{n}{2}}}{(2\pi)^{\frac{n}{2}}\left(\frac{\alpha}{\delta}\right)^{-\lambda}} \frac{K_{\lambda-\frac{n}{2}}\left(\sqrt{\delta^2+(x-\mu)\Delta^{-1}(x-\mu)\prime(\alpha^2+\beta\prime\Delta\beta)}\right)}{K_\lambda(\delta\alpha)\exp(-(x-\mu)\beta\prime)} \tag{4}$$

In this function, the parametrization is such that $\chi = \delta^2$, $\psi = \alpha^2 - \beta'\Delta\beta$ and $\alpha, \delta \geq 0$. Also, the modified third order Bessel function for $x > 0$ is, according to equation (5):

$$K_\lambda(x) = \frac{1}{2}\int_0^\infty w^{\lambda-1} \exp\left[-\frac{1}{2}x(w + w^{-1})\right] dw; \tag{5}$$

the expected value and variance of the vector X are calculated, according to the above definitions (Hu, 2005) as:

$$E[X] = \mu + E[W]\beta$$
$$V[X] = E[W]\Delta + V[W]\beta'\beta$$

In the above expressions it can be seen that μ is a transformation of the expected value of X, while β capture part of the bias presented by the random vector around the vector location μ (Alayón, 2014). Also, Δ is a transformation of $V[X]$ and it can be interpreted as a scattering matrix which is weighted by the magnitude of the bias $\beta'\beta$ as well as the expected value and the variance of the random variable in the mixture between GIG and multivariate normal. The λ parameter influences in shape of the tail of the distribution and magnitude of kurtosis (Hu, 2005).

Similarly, α, δ are scaling parameters that influence in the dispersion around the mean (Paolella, 2007).

Finally, the approach proposed uses the covariance matrix $V[X]$ as an input to solve for the optimal weights of an efficient portfolio, following the rest of the steps of the classical Markowitz methodology. The next section briefly discusses the procedure followed to calculate the parameters of the GH family.

2.1. Expectations Maximization Algorithm

The Expectation-Maximization algorithm (EM) consists of two steps. In the first, the expected value of the augmented log likelihood function is obtained (McNeil, 2005) as:

$$\ln\big(L(\Theta;\ x, w)\big) = \sum_{i=1}^{m} \ln(f_{X|W}(x_i|w_i; \mu, \Delta, \beta)) + \sum_{i=1}^{m} \ln(f_W(w_i; \lambda, \alpha, \delta)),$$

with a starting point $\Theta_0 = (\lambda_0, \alpha_0, \delta_0, \beta_0, \mu_0, \Delta_0)$, where $x = (x_1, \dots, x_m)$ comes from a random sample and $w = (\mathrm{w}_1, \mathrm{w}_2, \dots, \mathrm{w}_\mathrm{m})$ comes from a latent variable whose GIG distribution modifies the log-likelihood function (Hu, 2005), as described by equation (7):

$$\ln(L(\Theta; x_1, \dots, x_m) = \textstyle\sum_{i=1}^{m} \ln\big(f(x_i; \Theta)\big). \tag{7}$$

Thus, the kth iteration yields the objective function shown in equation (8):

$$h\big(\Theta; \Theta^{[k]}\big) = E\big[\ln\big(L(\Theta;\ x, w)\big)|x; \Theta^{[k]}\big] \tag{8}$$

Next, the objective function $\mathrm{h}\big(\Theta; \Theta^{[\mathrm{k}]}\big)$ is maximized in step 2, to obtain the parameters $\Theta^{[\mathrm{k}+1]}$ (Breymann, 2013) and, finally, the GH density function is evaluated using a multivariate continuous distributions test, as explained in the following section.

2.2 Hypothesis testing

McAssey (2013) developed a relatively simple procedure for continuous multivariate distributions of any dimension that tests the goodness of fit of the GH family, and reinforces the estimation and inference of the calculations. In this tests (i) The null hypothesis is true; i.e., the random sample follows a GH distribution, and the estimation via the EM algorithm is right; (ii) A random sample is simulated with the parameters obtained from the EM algorithm. This step uses the condition $X|W = w \sim N_n(\mu + w\beta\Delta, w\Delta)$, with $W \sim GIG(\lambda, \chi, \psi)$. (iii)The Mahalanobis distance between series i) and ii) is calculated using the mean $\hat{\mu}$ and the covariance matrix $\hat{\sum}$, as in equation (9):

$$\hat{d}_i = \sqrt{(\hat{u}_i - \hat{\mu})'\hat{\Sigma}^{-1}(\hat{u}_i - \hat{\mu})} \tag{9}$$

(iv) The previous step is repeated until it generates a random sample to calculate:

$$A_T = \sum_{j=1}^{T} \left| 1 - \frac{E_j}{O_j} \right|$$

where E_j is the expected frequency of observations, $\hat{d}_i$ in a fixed interval, and O_j is the observed frequency in that interval. Next, the p-value of the test is calculated to test the null hypothesis. Additionally, the level of significance of each parameter of the GH distribution is calculated. This result is validated by the Asymptotic Theory (Barndorff-Nielsen, 2012), because $\sqrt{n}(\hat{\theta} - \theta) \rightarrow N(0, I^{-1})$, where I is the matrix of information that can be defined as $I = Cov(S_i)$, and:

$$S_i = \frac{\partial \ln[f(x_i;\theta)]}{\partial \theta}.$$

Moreover, the confidence interval $(1-\alpha)\%$ for the parameter $\hat{v}$ is given by:

$$\hat{v} \pm t_{\alpha/2}\sqrt{\frac{1}{n}(I^{-1})_{vv}}\,,$$

and the test statistic to assess the level of significance is:

$$t = \frac{\hat{v} - v_0}{ee(\hat{v})}.$$

The goodness of fit of the adjusted parameters is established through previous estimates, finalizing the process. The next section briefly describes the Markowitz procedure to estimate the weight of risky assets that constitute efficient portfolios.

2.3 Markowitz Portfolio

Markowitz (1952, 1959) assumes that a risk-averse investor has an initial capital endowment and wishes to invest it in risky financial assets. However, the investors' preferences are such that they are willing to accept more risk if adequately compensated with more return. In that sense, investors are maximizers (of return) subject to a certain risk tolerance that is revealed as they seek to determine the optimal proportion of their wealth to invest in each risky asset. The mathematical solution to the investor's problem is a combination of risky asset weights that maximize the portfolio's expected return subject to a given level of risk or, vice versa, that minimizes risk subject to a desired level of expected return[1] (Alaitz, 2002). The portfolio is considered to be efficient provided its expected return is maximized, compared to other possible portfolios with a similar level of risk; or its level of risk is minimized, compared to alternative portfolios with a similar expected return.

In this paper, Markowitz's optimization algorithm is solved using two different covariance matrices: (i) the usual sample covariance matrix (or Markowitz procedure) and (ii) the estimator obtained by adjusting the GH distribution probability. Comparing the performance of both approaches in terms of the coefficient of variation of each portfolio provides valuable information about which methodology is preferable, from an investor's point of view.

The next section reports the empirical results of the two approaches, and adds the performance of an equally weighted portfolio to the comparison.

[1] Markowitz proposed the use of the standard deviation as a measure of risk (Markowitz 1959).

3. Empirical Analysis Results

The data used to build efficient portfolios according to Markowitz (sample covariance matrix) and the GH family estimated covariance matrix, consists of the daily continuous returns of the stocks included in the market indices of Chile, Colombia, Mexico and Peru (the MILA members), for the period 2010-2015.

The MILA is a common trading platform for the stocks listed in its member countries' exchanges, and it is part of the Alianza del Pacífico (AP), a subregional integration project launched to promote economic cooperation among Chile, Colombia, Mexico and Peru. The GDP of the AP is approximately 2 trillion dollars[2], and its total population is 210 million. The MILA allows participants in its member countries to seamlessly trade stocks of the four nations. Small and institutional investors of any of the four MILA member markets gain access to enhanced portfolio diversification opportunities and, in terms of liquidity measured by the volume of trading, the MILA market is significantly more liquid than any of its members.

The empirical analysis first builds and measures the performance of individual country portfolios (that include only domestic listed stocks), and then portfolios that includes all MILA stocks, so the information can reveal the comparative performance of the proposed methodology with respect to the traditional Markowitz approach and the equally weighted portfolio, at different levels of aggregation (see Tables 2 and 3, below).

Daily returns are calculated as:

$$r_{it} = \ln\left(\frac{p_{i,t}}{p_{i,t-1}}\right)$$

where i=1,…, 4 are the individual stock prices, and t denotes time;

$p_{i,t}$= reported daily price of asset i at time t; and

$r_{i,t}$ = daily return of the asset i in time t.

The multivariate returns are adjusted according to the EM algorithm and goodness of fit (McAssey, 2013). At the same time, univariate probability distributions which belong to the GH family are estimated, and their fit is tested using the classical Kolmogorov-Smirnov test for each asset. The tests results tables for each country and for the whole MILA sample are presented in the Appendix as Tables A1 through A5, and they include the estimated parameters, the goodness of fit p-values, and statistical *t* tests of significance for each parameter, according to Asymptotic Theory. Most of the coefficients are statistically relevant at, at least, 90% level. In all cases for the p-value(Normal) we reject the null hypothesis of normality of the data and with the p-values(GH) do not reject the null hypothesis that the data are Generalized Hyperbolic distributed.

Three different portfolios are considered:

a) Portfolio A, with equal weights ($w_i = 1/n$, where n is the number of assets in the sample).
b) Portfolio B, where the weights are obtained following the classical Markowitz methodology (the sample covariance matrix is used).
c) Portfolio C, where the covariance matrix is obtained from the GH family and used as an input to the 'modified" Markowitz method.

[2] If it were a single country, MILA would be the eighth largest economy in the world.

Table 1 reports the performance of each portfolio, represented in terms of the coefficient of variation. As can be seen, when the GH family-based covariance matrix is used, the variation coefficient is smaller, confirming that the GH family covariance matrix can be a useful tool to improve the risk-return performance of portfolios.

Table 1. Mean coefficient of variation for the different portfolios (2010-2015).

Country/MILA	Portfolio A			Portfolio B			Portfolio C		
	Mean	Standard Deviation	Coefficient of Variation	Mean	Standard Deviation	Coefficient of Variation	Mean	Standard Deviation	Coefficient of Variation
Chile	0.0060%	0.8748%	144.74	0.0246%	0.7473%	30.41	0.0251%	0.7139%	28.43
Colombia	0.0018%	0.7772%	426.60	0.0019%	0.5335%	280.80	0.0020%	0.5238%	261.91
Mexico	0.0698%	0.8774%	12.57	0.0778%	0.8120%	10.43	0.0762%	0.7723%	10.13
Peru	0.0187%	1.2998%	69.52	0.0188%	1.1606%	61.75	0.0191%	0.9473%	49.61
MILA	0.0218%	0.5231%	23.97	0.0293%	0.0184%	0.63	0.0297%	0.0140%	0.47

Source: Authors' own compilation based on data from Bloomberg.

According to the evidence presented so far, the argument that the classical Markowitz procedure to build efficient risky assets portfolios may be improved with a GH distribution based covariance matrix is clearly validated. This is a result that has importance implications for portfolio managers, ETFs, and other institutional investors, and should be further explored to validate its suitability for other international markets, an endeavour we expect to pursue in the near future.

4. Conclusions

Daily returns for a sample of stocks listed in the MILA (the stock markets of Chile, Colombia, Mexico and Peru) indices are used to analyze the performance of three types of portfolios: a) an equally weighted portfolio; b) a traditional Markowitz portfolio; and, c) a Markowitz portfolio that uses a GH based covariance matrix as an input. The comparison is made using their mean coefficient of variation, for the 2010-2015 period. The stocks included in the sample are those whose returns could be adjusted to a GH multivariate probability distribution. The goodness of fit of the GH distribution is confirmed using McAssey's (2013) test, and the estimated parameters significance is confirmed with an asymptotic test (Barndorff-Nielsen, 2012).

Enough confirmatory evidence that the joint probability distribution for the different subsets and for the whole sample of MILA stocks behaves like a GH multivariate distribution is found. The mean estimated parameters are different from one country to another, and their significance is at least 90%.

The empirical evidence suggests that the GH distribution based covariance matrix can improve the traditional Markowitz portfolio building methodology.

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Appendix

Table A1. GH Distribution Parameters for Mexico 2010-2015

λ	-3.9517 (1.69)*	p-value (GH)	0.3751	p-value (Normal)	0.0001
δ	0.0161 (1.65)*	α	69.7153 (1.89)*	log-likelihood	62963.95

μ		diag(Δ)		β	
0.0000	(1.73)*	0.0002	(1.93)*	0.0003	(1.82)**
-0.0005	(-1.91)*	0.0003	(1.86)*	0.0007	(1.77)**
0.0007	(1.66)*	0.0003	(1.62)	-0.0005	(-1.51)
0.0004	(1.95)*	0.0004	(1.71)*	0.0011	(1.54)
0.0004	(1.64)*	0.0003	(1.78)*	0.0000	(1.69)*
0.0011	(1.67)*	0.0004	(1.67)*	-0.0003	(-1.40)
0.0009	(1.83)*	0.0002	(1.76)*	-0.0001	(1.90)**
-0.0002	(-1.71)*	0.0003	(1.84)*	0.0012	(1.34)
0.0008	(1.89)*	0.0007	(1.98)**	0.0001	(2.34)***
-0.0005	(-2.21)**	0.0002	(1.95)*	0.0008	(1.97)**
0.0004	(1.69)*	0.0003	(1.89)**	-0.0003	(-2.2)***
0.0011	(1.88)*	0.0002	(1.66)*	0.0002	(2.7)***
0.0007	(1.82)*	0.0002	(1.81)*	0.0001	(1.16)
0.0006	(1.78)*	0.0003	(1.67)*	0.0006	(1.2)
0.0010	(1.93)*	0.0002	(1.90)*	-0.0005	(-1.68)**

Source: Authors' own compilation based on data from Bloomberg.

Table A2. GH Distribution Parameters for Chile 2010-2015

λ	2.4982 (1.65)*	p-value (GH)	0.2876	p-value (Normal)	0.00671
δ	0.0129 (1.67)*	α	69.6134 (1.79)*	log-likelihood	100585.4

μ		diag(Δ)		β	
0.0000	(1.84)*	0.0004	(1.71)*	-0.0004	(-1.80)*
0.0000	(1.88)*	0.0005	(1.98)**	-0.0013	(-1.82)*
0.0000	(1.68)*	0.0003	(2.03)**	-0.0001	(-1.91)*
0.0000	(1.75)*	0.0005	(1.97)**	-0.0011	(-1.58)
0.0000	(1.85)*	0.0002	(1.71)*	0.0000	(1.98)**
0.0000	(1.77)*	0.0001	(1.98)**	0.0004	(1.77)*
0.0000	(1.95)*	0.0002	(1.70)*	0.0001	(2.01)**
0.0000	(1.93)*	0.0002	(2.01)**	0.0004	(1.90)*
0.0000	(1.83)*	0.0002	(1.81)*	-0.0001	(1.51)
0.0000	(1.98)**	0.0002	(1.79)*	0.0006	(1.79)*
0.0000	(1.78)*	0.0002	(2.03)**	0.0008	(1.70)*
0.0000	(1.71)*	0.0002	(1.67)*	0.0003	(1.92)*
0.0000	(1.87)*	0.0002	(2.05)**	0.0004	(1.66)*
0.0000	(1.93)*	0.0003	(1.93)*	-0.0003	(-1.77)*
0.0000	(1.65)*	0.0002	(1.69)*	0.0005	(1.71)*
0.0000	(1.99)**	0.0002	(1.74)*	0.0004	(1.64)
0.0000	(1.77)*	0.0001	(1.85)*	0.0004	(1.96)*
0.0000	(1.79)*	0.0001	(1.71)*	0.0003	(1.68)*
0.0000	(1.65)*	0.0003	(1.86)*	0.0000	(1.75)*
0.0000	(1.93)*	0.0002	(1.75)*	0.0000	(1.69)*
0.0000	(1.77)*	0.0002	(1.92)*	0.0004	(2.01)**
0.0000	(1.97)**	0.0002	(1.91)*	-0.0003	(-1.92)*
0.0000	(1.89)*	0.0001	(1.78)*	0.0001	(-1.91)*
0.0000	(2.06)**	0.0002	(1.67)*	0.0000	(-1.72)*
0.0000	(1.68)*	0.0001	(1.94)*	0.0000	(-1.62)*

Source: Authors' own compilation based on data from Bloomberg.

Table A3. GH Distribution Parameters for Colombia 2010-2015

λ	-1.0837 (1.71)*	p-value (GH)	0.1874	p-value (Normal)	0.00393
δ	0.0120 (1.69)*	α	53.2674 (1.66)*	log-likelihood	49825.59

μ		diag(Δ)		β	
-0.0001	(-1.97)**	0.0002	(2.02)**	0.0001	(1.84)*
-0.0016	(-1.76)*	0.0003	(1.99)**	0.0019	(1.65)*
-0.0030	(-1.98)**	0.0008	(1.67)*	0.0043	(2.09)**
0.0002	(1.72)*	0.0002	(1.71)*	-0.0002	(-1.84)*
-0.0005	(-1.97)**	0.0002	(1.79)*	0.0002	(1.89)*
-0.0003	(-1.89)*	0.0001	(1.91)*	-0.0001	(-1.73)*
-0.0001	(-1.74)*	0.0002	(1.86)*	-0.0002	(-1.88)*
-0.0008	(-2.07)**	0.0001	(1.91)*	0.0002	(1.94)*
-0.0009	(-1.61)	0.0002	(1.76)*	0.0012	(1.66)*
0.0000	(1.68)*	0.0002	(1.70)*	-0.0002	(-1.88)*
-0.0011	(-1.75)*	0.0001	(1.81)*	0.0010	(1.71)*
-0.0018	(-1.83)*	0.0002	(1.69)*	0.0020	(2.01)**
0.0007	(1.86)*	0.0002	(1.84)*	-0.0005	(-1.76)*

Source: Authors' own compilation based on data from Bloomberg.

Table A4. GH Distribution Parameters for Peru 2010-2015

λ	-0.9894 (1.71)*	p-value (GH)	0.1346	p-value (Normal)	0.0098
δ	0.0040 (1.65)*	α	9.4562 (1.68)*	log-likelihood	78541.75

μ		diag(Δ)		β	
-0.0010	(-1.77)*	0.0002	(1.88)*	0.0002	(1.70)*
-0.0014	(-1.69)*	0.0002	(1.96)**	0.0009	(1.80)*
0.0015	(1.78)*	0.0005	(1.67)*	-0.0007	(-1.67)*
0.0015	(1.91)*	0.0013	(1.97)**	0.0000	(1.95)*
0.0019	(1.86)*	0.0007	(1.94)*	-0.0009	(-1.76)*
0.0028	(1.72)*	0.0008	(1.83)*	-0.0012	(-1.95)*
-0.0013	(-1.88)*	0.0001	(1.67)*	0.0008	(1.74)*
0.0008	(1.75)*	0.0005	(1.81)*	0.0001	(1.88)*
0.0002	(1.65)*	0.0005	(1.84)*	-0.0001	(-1.87)*
-0.0013	(-1.74)*	0.0001	(1.91)*	0.0006	(1.69)*
-0.0006	(-1.97)**	0.0002	(1.74)*	0.0005	(1.75)*
-0.0003	(-1.69)*	0.0003	(1.85)*	0.0000	(1.72)*
0.0008	(1.84)*	0.0003	(1.66)*	0.0000	(1.69)*
-0.0011	(-1.95)*	0.0002	(1.96)**	0.0011	(1.85)*
0.0001	(1.76)*	0.0003	(1.94)*	-0.0001	(-1.86)*
0.0000	(1.65)*	0.0004	(1.78)*	0.0005	(2.05)**
-0.0011	(-1.93)*	0.0003	(1.92)*	0.0006	(1.78)*
0.0002	(1.85)*	0.0004	(1.65)*	-0.0002	(-1.91)*
-0.0005	(-1.52)	0.0007	(1.81)*	0.0013	(1.81)*
-0.0012	(-1.94)*	0.0003	(1.89)*	0.0006	(1.93)*
-0.0015	(-1.89)*	0.0002	(1.77)*	0.0012	(1.94)*

Source: Authors' own compilation based on data from Bloomberg.

Table A5. GH Distribution Parameters for MILA 2010-2015

λ	-7.1328 (1.74)*	p-value (GH)	0.1003	p-value (Normal)	0.0003
δ	0.0012 (1.78)*	α	76.3264 (1.76)*	log-likelihood	276326.6
μ		diag(Δ)		β	
-0.0019 (1.85)*	-0.0002 (1.95)*	0.0002 (1.91)*	0.0006 (1.67)*	0.0011 (1.77)*	-0.0009 (1.8)*
0.0041 (1.82)*	-0.0006 (1.83)*	0.0002 (1.65)*	0.0002 (1.92)*	-0.0024 (1.91)*	0.0004 (1.89)*
-0.0022 (1.92)*	0.0007 (1.75)*	0.0005 (1.77)*	0.0002 (2.04)**	0.0021 (1.78)*	-0.0011 (1.68)*
-0.0007 (1.77)*	-0.0014 (1.91)*	0.0013 (1.74)*	0.0002 (1.77)*	0.0011 (1.67)*	0.0012 (1.79)*
-0.0018 (1.83)*	-0.0015 (1.71)*	0.0007 (2.04)**	0.0002 (1.86)*	0.0016 (1.82)*	0.0014 (1.71)*
-0.0007 (1.76)*	0.0008 (1.69)*	0.0008 (2.05)**	0.0002 (1.82)*	0.0004 (1.85)*	0.0009 (1.83)*
-0.0011 (1.81)*	-0.0005 (1.93)*	0.0002 (1.78)*	0.0002 (1.79)*	0.0009 (1.75)*	0.0014 (1.88)*
-0.0021 (1.67)*	0.002 (1.68)*	0.0006 (1.86)*	0.0002 (1.83)*	0.0007 (1.67)*	-0.0008 (1.85)*
-0.0001 (1.78)*	-0.0006 (1.96)*	0.0005 (1.94)*	0.0002 (1.76)*	0.0002 (1.91)*	0.0024 (1.91)*
-0.0001 (1.65)*	-0.0013 (1.93)*	0.0001 (1.74)*	0.0002 (1.73)*	0.0004 (1.83)*	0.0013 (1.87)*
0.0017 (1.77)*	-0.0032 (1.91)*	0.0002 (2.04)**	0.0003 (1.86)*	-0.0013 (1.72)*	0.0031 (1.84)*
0.0005 (1.88)*	-0.0023 (1.79)*	0.0003 (1.78)*	0.0002 (1.86)*	-0.0009 (1.81)*	0.0031 (1.81)*
0.0009 (1.72)*	-0.0014 (1.72)*	0.0003 (1.81)*	0.0002 (1.83)*	-0.0006 (1.86)*	0.0026 (1.83)*
0.0019 (1.73)*	-0.0003 (1.66)*	0.0002 (1.99)*	0.0001 (1.95)*	-0.0012 (1.84)*	-0.0003 (1.95)*
0.0011 (1.93)*	0.0001 (1.89)*	0.0003 (1.71)*	0.0002 (2.01)**	0.0002 (1.82)*	0.0001 (1.87)*
-0.0026 (1.66)*	0.0001 (1.73)*	0.0004 (2.02)**	0.0003 (2.05)**	0.0021 (1.77)*	-0.0002 (1.87)*
-0.0024 (1.82)*	-0.0003 (1.82)*	0.0003 (1.98)**	0.0002 (2.05)**	0.0018 (1.82)*	0.0009 (1.68)*
-0.0018 (1.69)*	0.0002 (1.81)*	0.0004 (1.92)*	0.0002 (1.71)*	0.0015 (1.76)*	0.0002 (1.67)*
-0.0035 (1.70)*	-0.0001 (1.81)*	0.0007 (1.95)*	0.0002 (1.84)*	0.0029 (1.86)*	0.0001 (1.78)*
-0.0004 (1.68)*	0.0005 (1.94)*	0.0003 (1.81)*	0.0001 (1.88)*	0.0004 (1.83)*	-0.0006 (1.69)*
-0.0008 (1.85)*	0.0019 (1.69)*	0.0002 (1.87)*	0.0002 (1.75)*	0.0003 (1.71)*	-0.0014 (1.67)*
-0.0017 (1.67)*	-0.0013 (1.79)*	0.0003 (1.71)*	0.0002 (1.7)*	0.0016 (1.81)*	0.0018 (1.94)*
-0.001 (1.68)*	0.0032 (1.69)*	0.0003 (1.82)*	0.0002 (1.77)*	0.0007 (1.91)*	-0.0026 (1.69)*
0.0002 (1.66)*	-0.0002 (1.72)*	0.0009 (2.05)**	0.0003 (1.68)*	0.0003 (1.66)*	0.0013 (1.87)*
0.0006 (1.79)*	-0.0014 (1.95)*	0.0003 (1.91)*	0.0004 (2.04)**	-0.0003 (1.95)*	0.0007 (1.73)*
-0.0005 (1.94)*	-0.0017 (1.72)*	0.0002 (1.97)*	0.0004 (2.03)**	0.0007 (1.91)*	0.0016 (1.95)*
0.0001 (1.80)*	-0.0032 (1.86)*	0.0001 (1.74)*	0.0003 (2.03)**	0.0001 (1.67)*	0.0026 (1.91)*
0.0015 (1.87)*	-0.0017 (1.93)*	0.0002 (1.97)**	0.0004 (1.84)*	-0.0013 (1.83)*	0.0016 (1.78)*
0.0011 (1.90)*	-0.001 (1.87)*	0.0001 (1.76)*	0.0002 (1.77)*	0.0003 (1.87)*	0.0013 (1.86)*
0.0011 (1.83)*	-0.0003 (1.91)*	0.0003 (1.98)**	0.0004 (1.94)*	-0.0003 (1.68)*	-0.0001 (1.72)*
-0.0003 (1.93)*	0.0002 (1.82)*	0.0002 (2.01)**	0.0004 (1.88)*	0.0011 (1.83)*	0.0002 (1.76)*
-0.0013 (1.84)*	0.0018 (1.83)*	0.0001 (1.75)*	0.0002 (1.92)*	0.0022 (1.68)*	-0.0014 (1.82)*
0.0002 (1.72)*	0.0012 (1.85)*	0.0003 (1.79)*	0.0003 (1.71)*	0.0006 (1.74)*	-0.0008 (1.85)*
-0.0015 (1.78)*	-0.0009 (1.96)*	0.0002 (1.85)*	0.0002 (1.74)*	0.0016 (1.85)*	0.0012 (1.67)*
-0.0012 (1.88)*	0.0017 (1.65)*	0.0004 (1.94)*	0.0003 (2.01)**	0.0015 (1.78)*	-0.0014 (1.67)*
-0.0005 (1.88)*	0.0019 (1.74)*	0.0005 (1.95)*	0.0003 (2.05)**	0.0002 (1.66)*	-0.0011 (1.71)*
-0.0015 (1.94)*	0.0017 (1.67)*	0.0003 (2.02)**	0.0002 (1.93)*	0.0016 (1.82)*	-0.0016 (1.73)*

Source: Authors' own compilation based on data from Bloomberg.

Table A.6 Weights for the Mexican Stocks Portfolios (2010-2015)

Ticker	Portfolio A	Portfolio B	Portfolio C
ELEKTRA	6.6667%	10.0695%	9.0539%
GFNORTE	6.6667%	0.0837%	0.0000%
BIMBO	6.6667%	0.9017%	0.0000%
LIVEPOL	6.6667%	6.7855%	5.3852%
FEMSA	6.6667%	2.7502%	2.2844%
GCARSO	6.6667%	0.1311%	0.0633%
GRUMA	6.6667%	13.3490%	9.3128%
GFINBUR	6.6667%	4.4102%	6.4810%
WALMEX	6.6667%	3.3977%	6.3483%
AMXL	6.6667%	11.0806%	9.7214%
KIMBER	6.6667%	3.4697%	3.4653%
TLEVISA	6.6667%	16.3785%	15.1527%
CEMEX	6.6667%	6.1579%	4.6996%
ALSEA	6.6667%	4.1264%	5.0887%
ALFA	6.6667%	16.9081%	22.9433%
Mean	0.0698%	0.0778%	0.0762%
Standard Deviation	0.8774%	0.8120%	0.7723%
Coefficient of Variation	12.5664	10.4343	10.1349

Source: Authors' own compilation based on data from Bloomberg.

Table A.7 Weights for the Chilean Stocks Portfolios (2010-2015)

Ticker	Portfolio A	Portfolio B	Portfolio C
ENTEL	4.0000%	0.0000%	0.0000%
ANDINA	4.0000%	0.0000%	0.0000%
BEVIDE	4.0000%	1.3083%	2.5862%
CHILE	4.0000%	0.0000%	0.0000%
ECL	4.0000%	7.1308%	7.7288%
QUINENCO	4.0000%	21.4085%	19.6793%
SONDA	4.0000%	4.7823%	4.7669%
RIPLEY	4.0000%	4.7753%	5.3051%
EMBONOR	4.0000%	10.0811%	9.3591%
FALABELLA	4.0000%	0.3873%	1.8313%
COLBUN	4.0000%	1.1713%	2.5958%
BANMEDICA	4.0000%	7.8968%	5.9815%
SALFACORP	4.0000%	10.3014%	8.2800%
AESGENER	4.0000%	0.6084%	0.9663%
AGUAS	4.0000%	0.0000%	0.0000%
BCI	4.0000%	2.2200%	3.4282%
SECURITY	4.0000%	8.9782%	7.9627%
CMPC	4.0000%	8.1246%	9.4201%
LAN	4.0000%	0.0000%	0.0000%
COPEC	4.0000%	0.0000%	0.0000%
SK	4.0000%	0.0000%	0.0000%
BUPACL	4.0000%	0.0000%	0.0000%
SQM	4.0000%	8.5142%	8.6241%
CAP	4.0000%	0.0000%	0.0000%
ENDESA	4.0000%	2.3113%	1.4846%
Mean	0.0060%	0.0246%	0.0251%
Standard Deviation	0.8748%	0.7473%	0.7139%
Coefficient of Variation	144.7368	30.4098	28.4306

Source: Authors' own compilation based on data from Bloomberg.

Table A.8 Weights for the Colombian Stocks Portfolios (2010-2015)

Ticker	Portfolio A	Portfolio B	Portfolio C
CLH	7.6923%	4.4391%	3.9997%
ECOPETROL	7.6923%	7.7404%	6.9360%
BVC	7.6923%	1.4725%	2.8459%
PFDAVVNDA	7.6923%	4.3629%	3.1616%
PFAVH	7.6923%	4.7031%	5.7665%
CELSIA	7.6923%	9.3440%	9.1549%
CNEC	7.6923%	5.0268%	4.6245%
PFAVAL	7.6923%	2.8744%	4.9164%
EEB	7.6923%	6.4155%	7.0820%
PREC	7.6923%	4.1315%	4.6216%
GRUPOSURA	7.6923%	8.9062%	7.8902%
BOGOTA	7.6923%	6.4573%	7.3758%
ISAGEN	7.6923%	34.1263%	31.6251%
Mean	0.0018%	0.0019%	0.0020%
Standard Deviation	0.7772%	0.5335%	0.5238%
Coefficient of Variation	426.5961	280.8001	261.9085

Source: Authors' own compilation based on data from Bloomberg.

Table A.9 Weights for the Peruvian Stocks Portfolios (2010-2015)

Ticker	Portfolio A	Portfolio B	Portfolio C
IFS	4.7619%	0.0000%	0.0000%
GRA	4.7619%	0.0000%	0.0000%
CON	4.7619%	1.2094%	3.7295%
MIN	4.7619%	0.9097%	1.2723%
TV	4.7619%	1.8191%	3.6887%
LGC	4.7619%	6.1827%	6.6827%
UNA	4.7619%	19.8465%	14.1749%
LUS	4.7619%	0.0000%	0.0000%
MIL	4.7619%	0.0000%	0.0000%
ATA	4.7619%	0.6366%	15.8101%
COR	4.7619%	1.2083%	5.8560%
REL	4.7619%	5.9012%	3.4088%
EDE	4.7619%	17.2455%	17.1097%
FER	4.7619%	1.0347%	10.0382%
BVN	4.7619%	0.0000%	0.0000%
BAP	4.7619%	0.5551%	0.0000%
CPA	4.7619%	0.0000%	0.0000%
ENE	4.7619%	0.0000%	0.0000%
SID	4.7619%	15.2648%	10.4565%
CAS	4.7619%	5.7364%	0.0000%
ALI	4.7619%	22.4500%	7.7725%
Mean	0.0187%	0.0188%	0.0191%
Standard Deviation	1.2998%	1.1606%	0.9473%
Coefficient of Variation	69.5185	61.7456	49.6056

Source: Authors' own compilation based on data from Bloomberg.

Table A.10 Weights for the MILA Stocks Portfolios (2010-2015)

Ticker	Portfolio A	Portfolio B	Portfolio C	Ticker	Portfolio A	Portfolio B	Portfolio C
IFS	1.3514%	0.0000%	0.0000%	ECOPETROL	1.3514%	1.3598%	1.2185%
GRA	1.3514%	0.0000%	0.0000%	BVC	1.3514%	0.2587%	0.5000%
CON	1.3514%	0.3432%	1.0584%	PFDAVVNDA	1.3514%	0.7665%	0.5554%
MIN	1.3514%	0.2582%	0.3611%	PFAVH	1.3514%	0.8262%	1.0130%
TV	1.3514%	0.5162%	1.0468%	CELSIA	1.3514%	1.6415%	1.6083%
LGC	1.3514%	1.7545%	1.8964%	CNEC	1.3514%	0.8831%	0.8124%
UNA	1.3514%	5.6321%	4.0226%	PFAVAL	1.3514%	0.5050%	0.8637%
LUS	1.3514%	0.0000%	0.0000%	EEB	1.3514%	1.1270%	1.2441%
MIL	1.3514%	0.0000%	0.0000%	PREC	1.3514%	0.7258%	0.8119%
ATA	1.3514%	0.1807%	4.4867%	GRUPOSURA	1.3514%	1.5646%	1.3861%
COR	1.3514%	0.3429%	1.6618%	BOGOTA	1.3514%	1.1344%	1.2958%
REL	1.3514%	1.6747%	0.9674%	ISAGEN	1.3514%	5.9952%	5.5558%
EDE	1.3514%	4.8940%	4.8555%	ENTEL	1.3514%	0.0000%	0.0000%
FER	1.3514%	0.2936%	2.8487%	ANDINA	1.3514%	0.0000%	0.0000%
BVN	1.3514%	0.0000%	0.0000%	BEVIDE	1.3514%	0.4420%	0.8737%
BAP	1.3514%	0.1575%	0.0000%	CHILE	1.3514%	0.0000%	0.0000%
CPA	1.3514%	0.0000%	0.0000%	ECL	1.3514%	2.4090%	2.6111%
ENE	1.3514%	0.0000%	0.0000%	QUINENCO	1.3514%	7.2326%	6.6484%
SID	1.3514%	4.3319%	2.9674%	SONDA	1.3514%	1.6157%	1.6104%
CAS	1.3514%	1.6279%	0.0000%	RIPLEY	1.3514%	1.6133%	1.7923%
ALI	1.3514%	6.3709%	2.2057%	EMBONOR	1.3514%	3.4058%	3.1618%
ELEKTRA	1.3514%	2.0411%	1.8352%	FALABELLA	1.3514%	0.1309%	0.6187%
GFNORTE	1.3514%	0.0170%	0.0000%	COLBUN	1.3514%	0.3957%	0.8770%
BIMBO	1.3514%	0.1828%	0.0000%	BANMEDICA	1.3514%	2.6679%	2.0208%
LIVEPOL	1.3514%	1.3754%	1.0916%	SALFACORP	1.3514%	3.4802%	2.7973%
FEMSA	1.3514%	0.5575%	0.4631%	AESGENER	1.3514%	0.2056%	0.3264%
GCARSO	1.3514%	0.0266%	0.0128%	AGUAS	1.3514%	0.0000%	0.0000%
GRUMA	1.3514%	2.7059%	1.8877%	BCI	1.3514%	0.7500%	1.1582%
GFINBUR	1.3514%	0.8940%	1.3137%	SECURITY	1.3514%	3.0332%	2.6901%
WALMEX	1.3514%	0.6887%	1.2868%	CMPC	1.3514%	2.7448%	3.1825%
AMXL	1.3514%	2.2461%	1.9706%	LAN	1.3514%	0.0000%	0.0000%
KIMBER	1.3514%	0.7033%	0.7024%	COPEC	1.3514%	0.0000%	0.0000%
TLEVISA	1.3514%	3.3200%	3.0715%	SK	1.3514%	0.0000%	0.0000%
CEMEX	1.3514%	1.2482%	0.9526%	BUPACL	1.3514%	0.0000%	0.0000%
ALSEA	1.3514%	0.8364%	1.0315%	SQM	1.3514%	2.8764%	2.9136%
ALFA	1.3514%	3.4273%	4.6507%	CAP	1.3514%	0.0000%	0.0000%
CLH	1.3514%	0.7798%	0.7026%	ENDESA	1.3514%	0.7808%	0.5016%
Mean	0.0218%	0.0293%	0.0297%				
Standard Deviation	0.5231%	0.0184%	0.0140%				
Coefficient of Variation	23.9736	0.6267	0.4719				

Source: Authors' own compilation based on data from Bloomberg.