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Modelling under ambiguity with two correlated Choquet-Brownian motions

David Roubaud
Montpellier Business School, Montpellier Research in Management

Alain Lapied
Aix Marseille Univ (Aix-Marseille School of Economics), CNRS & EHESS

Robert Kast
CNRS, LAMETA-INRA

Abstract

Modelling under ambiguity in financial and economic models implies a sound characterisation of ambiguity sources. We expand the seminal work of Kast et al. (2014) who first defined Choquet Random Walks (CRW) and Choquet-Brownian Motions (CBM). Their work allows modelling in the presence of a single source of ambiguity and is used in various contexts, such as investment decisions and portfolio choices. As it is often useful (or even imperative) to introduce multiple sources of ambiguity, we expand the Choquet Brownian model for two correlated sources of ambiguity. Using properties of correlation, we first establish key results for correlated dynamically coherent Choquet Random Walks. We extend it to continuous-time for two correlated sources of ambiguity, each represented by a Choquet-Brownian Motion. Thus, we demonstrate that CBM are sufficiently tractable to adapt to more complex model settings, in the presence of uncertainty represented through two correlated sources of ambiguity. We apply our theoretical model to the optimal portfolio choice of traded assets.

1. Introduction

Interest in correlated random walks (CRW) goes back to Goldstein (1951) and Klein (1952), who considered a symmetric one-dimensional process, which Gillis (1955) expanded to a two-dimensional process. Subsequently, general correlated random walks have proven very useful in physics, biology or chemistry models[1] (Chen and Renshaw 1994). In a random walk, the direction of the next move may depend on time, current position and/or direction of the previous move, or even on the move of another random walk (or several). Gillis (1955), Zhang (1992) and Chen and Renshaw (1994) have explored some specific properties of correlated random walks[2], such as transition probabilities.

Prior literature on correlated random walk analyses directionally reinforced random walks in economic or business situations, such as simple gambler ruin problems. In addition, authors created various financial models in continuous time with correlated Brownian motions, such as in Adkins and Paxson (2011), where both revenues and operating costs follow correlated stochastic processes in the renewal of assets decision. Models often aim at identifying optimal stopping rules under specific assumptions in a random walk or a Brownian motion by extension.

As optimal stopping is often explored under *uncertainty*, which includes *risk* and *ambiguity* concepts (Knight 1921; Ellsberg 1961), building an *ambiguous random walk* introduces an interesting feature to economic modelling. Furthermore, building *ambiguous Brownian motions* is fruitful as they are often used in modelling. Modelling ambiguity is often based on *multiple-priors* utility frameworks (i.e. *maximin* or '*worst case*' criterion, Gilboa and Schmeidler 1989, which show that ambiguity distorts the objective probability distribution by impacting the drift of stochastic processes of various kinds.

However, an alternate approach (Kast and Lapied 2010, Kast *et al.* 2014) not only accounts for the presence of ambiguity but also integrates the decision makers' beliefs in the stochastic processes of the underlying assets; relying on the Choquet ambiguity, a single parameter *c* represents the attitude towards ambiguity (level of ignorance or *c-ignorance*). In contrast to multiple-priors models, both the mean and variance of the Choquet stochastic process are deformed with a lower drift *and* volatility than in the classic probabilistic case; this framework is 'often less trivial and makes the applications to corporate finance more realistic' (Agliardi *et al.* 2016). We illustrate the various applications of the alternate framework in Section 1.3.

Kast and Lapied (2010) created this alternate framework to model ambiguous stochastic process and it was further developed axiomatically by Kast *et al.* (2014). They built an ambiguous random walk through a 'Choquet' version of random walk (CRW) using *capacities*[3] instead of the standard exact probabilities.

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[1] There are various contexts of application, such as dispersal of animals or cells, scattering of waves, etc.

[2] In correlated random walk, the direction chosen at step i+1 may be related to the direction of step i (a phenomenon termed 'persistence', since Patlak 1953).

[3] Capacities are a non-additive unit measure used in Choquet Expected Utility models to represent beliefs. Capacities act as *weighted probability functions*, in which decision weights capture decision makers' subjective degrees of confidence about possible outcomes. The decision weights used in the computation of the Choquet integral overweight high outcomes if the capacity is concave and superadditive ($c > 0.5$), while favoring low outcomes if the capacity is convex and subadditive ($c < 0.5$). The special case $c = 0.5$ corresponds to the traditional probabilistic framework and standard Brownian motion used in risk-neutral option pricing. The capacity variable *c* acts as a proxy for decision-makers' attitudes towards ambiguity; it summarises investors' ambiguity attitudes (aversion or seeking) on future prospects, with $0 < c < 0.5$ ($m < 0$) representing aversion (convex capacities), and $0.5 < c < 1$ *(m > 0)* indicating ambiguity-seeking (concave capacities).

Thus, Choquet Random Walks are defined in discrete time by referring to a binomial tree, in which ambiguity is integrated by assigning capacities to represent the likelihood of the next move at each node. In a dynamic model, consistency is often an issue, Kast *et al.* (2014) show that a *dynamically consistent* Choquet Random Walk may be completely defined by a unique capacity *v*. CRW represents the attitude towards ambiguity and the ambiguity perceived by the decision makers, sometimes referred to as *ambiguity aversion bias*.

In a dynamic setting, CRW were shown to converge in so-called Choquet-Brownian motions (Kast *et al.* 2014), for which an increase in ambiguity decreases both drift and variance (see above); hence, a Choquet-Brownian process is a distorted standard Brownian process due to the ambiguity aversion bias. In a seminal article, the Kast *et al.* (2014) model was applied to a model of intertemporal portfolio choice. In addition, it was used to deal with real investment decisions, as a tool to determine the optimal timing and valuation of real options (Roubaud *et al.* 2010), extending the real options theory and method (Dixit and Pindyck 1994; Trigeorgis 1996) by modelling ambiguous cash flows expected from an investment project.

The model was adapted to determine the optimal timing of environmental policies (Agliardi and Sereno 2011), which uses Choquet-Brownian ambiguity to explore optimal taxes and non-tradable quotas, in the presence of well-known and debated ambiguity on the future costs and benefits of such policies.

The CBM framework was also used in corporate finance valuation, for instance, in Agliardi *et al.* (2015), who model ambiguous earnings before interests and taxes (EBIT) streams into a contingent claim model for convertible debt with CBM. They study how the ambiguity biases of equity holders and debt holders affect convertible debt valuation and conduct a sensitivity analysis of the bond value to changes in attitude toward ambiguity, firm and bond parameters.

CBM are also sufficiently flexible to deal with complex financial products valuation, as in Driouchi *et al.* (2015a) who use it to price European options with stochastic strikes under Choquet uncertainty. They show how swings in investor opinion resulting from ambiguity affect option prices and how the notion of 'fair value' is relative in presence of ambiguity. Furthermore, Driouchi *et al.* (2015b) study the tendency of option investors to deviate from risk-neutrality around extreme financial events, such as the subprime crisis. They reveal that in the context of the subprime crisis (2006-2008), investors' option implied ambiguity moderated the lead–lag relationship between implied and realised volatility.

Recently, Agliardi *et al.* (2016) use the Choquet ambiguity framework for a behavioural perspective of the decisions of equity and debt holders. They describe firm value as a Choquet Brownian process and show that greater ambiguity leads to earlier decrease in equity, increase in debt and financial distress for firms.

Our study aims to further expand the spectrum of situations where the Choquet-Brownian ambiguity may be applied by integrating two correlated sources of ambiguity represented by two correlated CBMs; for instance, modelling irreversible decisions often requires dealing with more than one source of ambiguity, which is tricky and often complex.

Adkins and Paxson (2011)—attempting to solve part of this complexity—observed that it may be possible to treat the option value as a function of homogeneity of degree one, which simplifies the resulting partial differential equation using the ratio of two variables. As discussed by Adkins and Paxson (2011), the main limit of this approach is precisely the strong assumption of a homogeneity of degree one; however, they maintain that it remains a convenient approach in many investment situations. We consider this approach both pragmatic and grounded.

Consequently, in our model with more than one source of ambiguity, we suggest relating the two simultaneous sources of ambiguity through a *correlation coefficient*; thereby, avoiding the intricacy of multidimensional partial differential equations. Note that the adoption of this correlation factor is often economic: multiple sources of ambiguity may be subjected to the influence of the same « market forces ». Hence, it is possible that their evolution is at least partially linked. Many examples have been given to justify adopting correlation in models, such as when economic outlook, regulations or technological breakthroughs may impact not only the expected cash flows of an asset but also the price that can be obtained from selling it.

2. Model

A Choquet-Brownian motion (CBM) is a distorted Wiener process where the distortion may derive from individual preferences towards ambiguity (see Section 1). CBM were shown to be the continuous time limit of a specific type of random walk: the Choquet Random Walk (CRW). The CRWs are binomial lattices with equal capacities c (instead of additive probabilities) on the two states at each node. The constant conditional capacity c plays a key role in such a setting; it summarises the decision makers' attitude towards ambiguity. Indeed, in a symmetrical CRW, dynamics is described by a discrete time motion, in which probability ½ is replaced by a constant c (See note 3) that represents the ambiguous weight exerted by the decision maker on the event « up » and the event « down » instead of the unambiguous ½.

To characterise a Choquet Random Walk, Kast and Lapied (2010) impose that for any node s_t at date t ($0 \leq t < T$), if s_{t+1}^u and s_{t+1}^d are the two possible successors of s_t at date $t+1$ (for, respectively, an 'up' or a 'down' movement in the binomial tree), the conditional capacity is a constant, such as $v(s_{t+1}^u/s_t) = v(s_{t+1}^d/s_t) = c$, with $0 < c < 1$.

Convergence to a specific kind of Brownian motion in continuous time has been established, which may be termed Choquet-Brownian Motions (CBM). Let us recall that in continuous time, symmetric random walks—when the up and down movements are of the same magnitude—converge to general Wiener processes.

In the case of CBM, a distorted Brownian is obtained, with mean $m = 2c{-}1$ and variance $s^2 = 4c(1{-}c)$.

This convergence towards a CBM in continuous time allows applications of Choquet-Brownian in various settings, such as real option models. We extend this to two correlated risks, each represented by a Choquet-Brownian process.

2.1 Correlated Dynamically Consistent Choquet Random Walks

Let Y_1 and Y_2 be two symmetric Dynamically Consistent Choquet Random Walks. At each date, the two processes can take two possible values, $+1$ and -1, and thus, determine four elementary states.

$\omega_1 = \{Y_1 = +1, Y_2 = +1\}$, $\omega_2 = \{Y_1 = +1, Y_2 = -1\}$,
$\omega_3 = \{Y_1 = -1, Y_2 = +1\}$, $\omega_4 = \{Y_1 = -1, Y_2 = -1\}$,
and let
$u_1 = \{\omega_1, \omega_2\} = \{Y_1 = +1\}$, $d_1 = \{\omega_3, \omega_4\} = \{Y_1 = -1\}$,
$u_2 = \{\omega_1, \omega_3\} = \{Y_2 = +1\}$, $d_2 = \{\omega_2, \omega_4\} = \{Y_2 = -1\}$.
The states are measured by a sub-linear non-additive capacity v such that
$v(u_1) = v(d_1) = v(u_2) = v(d_2) = c$, with $0 < c < ½$.
The Choquet expected[4] values are then

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[4] Let f be a real function such that $f = (x_1, E_1; \ldots; x_m, E_m)$, where $x_1 \leq x_2 \leq \ldots \leq x_m$.

The Choquet's expectation of f with respect to v is:

$$E_v(Y_1) = \sum_{s\in\{u_1,d_1\}} Y_1(s)\Delta v(s) = 2c-1, \quad E_v(Y_2) = \sum_{s\in\{u_2,d_2\}} Y_2(s)\Delta v(s) = 2c-1,$$

and the values that correspond to variances in probability theory are defined by the following relations:

$$Var_v(Y_1) = \sum_{s\in\{u_1,d_1\}} [Y_1(s) - E_v(Y_1)]^2 \Delta v(s) = 4c(1-c),$$

$$Var_v(Y_2) = \sum_{s\in\{u_2,d_2\}} [Y_2(s) - E_v(Y_2)]^2 \Delta v(s) = 4c(1-c).$$

However, a first difficulty appears; we have two candidates for the definition of the covariance between Y_1 and Y_2 in a Choquet framework.

$$I = \sum_{i=1}^{4} [Y_1(\omega_i) - E_v(Y_1)][Y_2(\omega_i) - E_v(Y_2)]\Delta\{v[Y_1 = Y_1(\omega_i)/Y_2 = Y_2(\omega_i)]v[Y_2 = Y_2(\omega_i)]\},$$

and

$$J = \sum_{i=1}^{4} [Y_1(\omega_i) - E_v(Y_1)][Y_2(\omega_i) - E_v(Y_2)]\Delta\{v[Y_2 = Y_2(\omega_i)/Y_1 = Y_1(\omega_i)]v[Y_1 = Y_1(\omega_i)]\}.$$

If the decision maker is dynamically consistent, Kast *et al.* (2014) state[5] that for a random variable X

$$\sum_{i=D,D^C} [\sum_{s\in S} X(s)\Delta v^i(s)]\Delta v(i) = \sum_{s\in S} X(s)\Delta v(s) \tag{1}$$

where S is the set of states, D and D^C the possible information sets
and the following normalisation of the conditional capacities holds:
$\forall\ D \subset S,\ v(\varnothing/D) = 0,\ v(S/D) = 1,\ \forall\ B \subset S,\ v(B/D) = v(B\cap D/D).$

Proposition 1: Under relation (1), $\forall\ i = 1, \ldots, 4,$
$v[Y_1 = Y_1(\omega_i)/Y_2 = Y_2(\omega_i)]v[Y_2 = Y_2(\omega_i)] = v[Y_2 = Y_2(\omega_i)/Y_1 = Y_1(\omega_i)]v[Y_1 = Y_1(\omega_i)]$
$$= v\{[Y_1 = Y_1(\omega_i)]\cap[Y_2 = Y_2(\omega_i)]\}.$$

Proof: We use the following simplified notations.
For some i, $[Y_1 = Y_1(\omega_i)] = B$ and $[Y_2 = Y_2(\omega_i)] = D$.[6]
Then, for $X = 1_{B\cap D}$, relation (1) becomes

$$\sum_{i=D,D^C} [\sum_{s\in S} 1_{B\cap D}(s)\Delta v^i(s)]\Delta v(i) = \sum_{s\in S} 1_{B\cap D}(s)\Delta v(s) \tag{2}$$

The left hand of equation (2) is
$$[\sum_{s\in S} 1_{B\cap D}(s)\Delta v^D(s)]v(D) = v(B/D).v(D),$$

because
$$\sum_{s\in S} 1_{B\cap D}(s)\Delta v^{D^C}(s) = v(B\cap D/D^C) = v(\varnothing/D^C) = 0,$$

and

$$\int_S f\, dv = \sum_{j=1}^{m} x_j \left[v\left(\bigcup_{i=j}^{m} E_i\right) - v\left(\bigcup_{i=j+1}^{m} E_i\right) \right], \text{ where, by convention, } E_{m+1} = \varnothing.$$

[5] In relation (1).
[6] Notice that $B = u_1$ or $B = d_1$ and $D = u_2$ or $D = d_2$.

$$\sum_{s \in S} 1_{B \cap D}(s) \Delta v^D(s) = v(B \cap D / D) = v(B / D).$$

The right hand of equation (2) is

$$\sum_{s \in S} 1_{B \cap D}(s) \Delta v(s) = v(B \cap D).$$

Then, $v(B/D).v(D) = v(B \cap D)$.

QED

With Proposition 1, the covariance between Y_1 and Y_2 is uniquely given by

$$Cov_v(Y_1, Y_2) = I = J = \sum_{i=1}^{4} [Y_1(\omega_i) - E_v(Y_1)][Y_2(\omega_i) - E_v(Y_2)] \Delta v\{[Y_1 = Y_1(\omega_i)] \cap [Y_2 = Y_2(\omega_i)]\}$$

$$= 4[-c(1-c) + c\,b + (1-2c)\,a],$$

with the following notations: $a = v(\omega_1)$, $b = v(\omega_1, \omega_4)$.

The correlation between Y_1 and Y_2 is then defined by

$$Cor_v(Y_1, Y_2) = \frac{-c(1-c) + cb + (1-2c)a}{c(1-c)}.$$

Remark 1: With $a \le c < \frac{1}{2}$, it is easy to check that $-1 \le Cor_v(Y_1, Y_2) \le 1$.

Remark 2: As special cases, we have

- Perfect positive correlation, when $a = c$ and $b = 1$, $Cor_v(Y_1, Y_2) = 1$,
- Perfect negative correlation, when $a = b = 0$, $Cor_v(Y_1, Y_2) = -1$,
- Independence corresponds to

$$v(Y_1 = y_1 / Y_2 = y_2) = v(Y_1 = y_1), \; v(Y_2 = y_2 / Y_1 = y_1) = v(Y_2 = y_2),$$

and, with proposition 1

$$v(Y_1 = y_1 / Y_2 = y_2) v(Y_2 = y_2) = v(Y_1 = y_1) v(Y_2 = y_2) = v[(Y_1 = y_1) \cap (Y_2 = y_2)].$$

This leads to $a = c^2$, and in this case, $Cor_v(Y_1, Y_2) = 0$, if and only if $b = c^2 + (1-c)^2$.

If b takes this value, then $Cor_v(Y_1, Y_2) = \dfrac{1-2c}{c(1-c)}(a - c^2)$.

2.2 Continuous-time limit Choquet-Brownian processes

Take a time interval $[0, T]$, where the number of periods in the interval is N, and the length of each period is $h = T/N$, and define two processes as $W_i(n) = W_i(n-1) + X_i(n)$ with $W_i(0) = 0$, and $X_i(n) = m\,h + s\,h^{1/2}\,U_i(n)$, $n = 1, \ldots, N$, for $i = 1, 2$, where m and s are two parameters, and $U_i(n)$ are two processes with the following properties.

For any $n = 1, \ldots, N$, $U_i(n)$, $i = 1, 2$, can take two possible values, $+1$ and -1, with probabilities $\frac{1}{2}$, $U_i(n)$ is independent from $U_i(n')$, and $U_1(n)$ is independent from $U_2(n')$, for $n' = 1, \ldots, N$, $n' \ne n$.

For any $n = 1, \ldots, N$,

$Pr\{U_1(n) = +1, U_2(n) = +1\} = Pr\{U_1(n) = -1, U_2(n) = -1\} = p$, $0 \le p \le \frac{1}{2}$,

$Pr\{U_1(n) = +1, U_2(n) = -1\} = Pr\{U_1(n) = -1, U_2(n) = +1\} = \frac{1}{2} - p$.

We have, $\forall i = 1, 2, \forall n = 1, \ldots, N$:

$E[U_i(n)] = 0$, $Var[U_i(n)] = 1$, $Cov[U_1(n), U_2(n)] = E[U_1(n)U_2(n)] = 4p - 1$,

and

$$W_i(N) = \sum_{n=1}^{N} X_i(n) = Nmh + sh^{1/2} \sum_{n=1}^{N} U_i(n),$$

$E[W_i(N)] = m\,N\,h$, $Var[W_i(N)] = s^2\,N\,h$,

$$Cov[W_1(N), W_2(N)] = s^2 h E\{[\sum_{n=1}^{N} U_1(n)][\sum_{n=1}^{N} U_2(n)]\} = s^2 Nh E[U_1(n)U_2(n)], \forall\, n = 1,\ldots, N$$

$$= s^2\, N\, h\, (4p - 1).$$

$\forall\, i = 1, 2,\ \underset{N \to +\infty}{Lim}\ W_i(N) = W_i(t) = m\, t + s\, B_i(t)$, where $B_i(t)$ are Brownian motions such that

$Cor[B_1(t), B_2(t)] = \rho = (4\, p - 1)$.

The correspondence with the processes Y_i are obtained, if and only if

$$m = 2c - 1,\ s^2 = 4c\,(1 - c),\ \rho = \frac{-c(1 - c) + cb + (1 - 2c)a}{c(1 - c)}.$$

3. Application Results

We apply our theoretical model to the optimal portfolio choice of traded assets using a framework from the stationary version of the Intertemporal Capital Asset Pricing Model (Merton 1969, 1971, 1973).

Suppose the wealth of an investor $(w(t))_{0 \le t \le T}$, is to be allocated between a riskless asset with constant instantaneous rate of return r, $r > 0$, and two risky assets, the prices of which are driven by two (possibly) partially correlated Choquet-Brownian motions.[7]

$$\frac{dP_i(t)}{P_i(t)} = \mu_i dt + \sigma_{i1} dB_1^C(t) + \sigma_{i2} dB_2^C(t) \tag{3}$$

with $P_i(0) > 0, \mu_i > 0, \sigma_{i1} > 0, \sigma_{i2} > 0$, for $i = 1, 2$.

Simultaneously, we express relation (3) for standard Brownian motions and deal with the correlation of the two Choquet-Brownian motions. The second point introduces a third Brownian motion $B_3(t)$, independent of $B_1(t)$, for which we have

$B_2(t) = \rho B_1(t) + \sqrt{1 - \rho^2} B_3(t)$, where ρ is the correlation coefficient between $B_1(t)$ and $B_2(t)$.

Then,

$$\frac{dP_i(t)}{P_i(t)} = \mu_i dt + \sigma_{i1}\{mdt + sdB_1(t)\} + \sigma_{i2}\{mdt + s[\rho dB_1(t) + \sqrt{1 - \rho^2} dB_3(t)]\},$$

with $m = 2c - 1, s^2 = 4c(1 - c), \rho = \dfrac{-c(1-c)+cb+(1-2c)a}{c(1-c)}, 0 \le a \le b \le 1$, and $a \le c \le \frac{1}{2}$, for $i = 1, 2$,

and finally

$$\frac{dP_i(t)}{P_i(t)} = [\mu_i + (\sigma_{i1} + \sigma_{i2})m]dt + (\sigma_{i1} + \sigma_{i2}\rho)sdB_1(t) + \sigma_{i2}s\sqrt{1 - \rho^2}dB_3(t) \tag{4}$$

for $i = 1, 2$.

With the following notations:

$m_i = \mu_i + (\sigma_{i1} + \sigma_{i2})m,\ s_{i1} = (\sigma_{i1} + \sigma_{i2}\rho)s,\ s_{i3} = \sigma_{i2}\sqrt{1 - \rho^2}s$, for $i = 1, 2$, we turn back to the standard case

$$\frac{dP_i(t)}{P_i(t)} = m_i dt + s_{i1}dB_1(t) + s_{i3}dB_3(t) \tag{5}$$

for $i = 1, 2$.

With $\rho^2 \ne 1$ and $c > 0$, the matrix

$\Sigma = \begin{pmatrix} s_{11} & s_{13} \\ s_{21} & s_{23} \end{pmatrix}$ is non-singular, if and only if, the standard matrix $\begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$ is non-singular.

We suppose this condition satisfied in the sequel.

If $x_1(t)$ and $x_2(t)$ are parts of the capital invested in risky assets at date t, the following stochastic differential equation characterises the agent wealth.

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[7] We propose two generalisations of the standard model: the introduction of Choquet-Brownian motions and the possibility of partial correlation between them.

$$\frac{dw(t)}{w(t)} = x_1(t)\frac{dP_1(t)}{P_1(t)} + x_2(t)\frac{dP_2(t)}{P_2(t)} + [1 - x_1(t) - x_2(t)]rdt,$$

and with relation (5)

$$\frac{dw(t)}{w(t)} = [r + (m_1 - r)x_1(t) + (m_2 - r)x_2(t)]dt + [s_{11}x_1(t) + s_{21}x_2(t)]dB_1(t) +$$

$$[s_{13}x_1(t) + s_{23}x_2(t)]dB_3(t) \tag{6}$$

The program of the agent for a time horizon T is to maximise the expected utility of its final wealth with respect to relation (6).

$$\max\nolimits_{x(t)_{0 \le t \le T}} E_t\{u[w(T)]\}, \ w(0) = w_0 > 0, \tag{7}$$

where $u(.)$ is an increasing and concave utility function and $x(t) = (x_1(t), x_2(t))$.

The well-known solution of this stationary problem, in which $w(t)$ is the one-dimensional state variable is given by

That is $\quad x^*(t, w) = \dfrac{1}{I(t,w)}\Gamma^{-1}(m - r) \tag{8}$

where $I(t,w) = -w\dfrac{\frac{\partial^2 J}{\partial w^2}(t,w)}{\frac{\partial J}{\partial w}(t,w)}$,

$J(t, w)$ is the value function of the Bellman's dynamic programming[8], and $J(T, w) = u(w)$, $\Gamma = \Sigma^t\Sigma$, and $m - r = (m_1 - \mathrm{r}, m_2 - \mathrm{r})$.

We now consider the special case of an iso-elastic utility function

$u(w) = \dfrac{w^{1-\alpha}}{1-\alpha}$, $\alpha > 0$, $\alpha \ne 1$. The relative risk aversion coefficient $-w\dfrac{u''(w)}{u'(w)}$ is the constant α, and the value function consistent with this utility function is

$J(t,w) = \dfrac{\exp \delta(T-t)}{1-\alpha}w^{1-\alpha}$, with $\delta = (1 - \alpha)[r + \frac{1}{2\alpha}\langle m - r, \Gamma^{-1}(m - r)\rangle$ and $I(t, w) = \alpha$.

The solution for the optimal control is a constant

$$x^* = \frac{1}{\alpha}\Gamma^{-1}(m - r). \tag{9}$$

The effect of ambiguity on m and s^2 is clear because $\dfrac{\partial m}{\partial c} > 0$, $\dfrac{\partial s^2}{\partial c} \ge 0$ and in the standard case (where $c = \frac{1}{2}$), $m = 0$ and $s^2 = 1$. However, even when the Choquet-Brownian motions are independent ($\rho = 0$), the effect of ambiguity on $x_1(t)$ and $x_2(t)$ cannot be decided, as it depends on the links between the prices of the assets and the standard Brownian motions (σ_{ij}, $i, j = 1, 2$).

4. Conclusion

Identifying and dealing with sources of uncertainty remains a major challenge for value creation in many sectors. Although there is more to financial decision-making than just sophisticated mathematical models, integrating flexibility and ambiguity remains intellectually challenging and promising in potential implications. Although ambiguity often results in aversion in preferences, one cannot reject ambiguity seeking if one is to account for decisions taken in various ambiguous situations. The use of Choquet-Brownian ambiguity prevents from radical assumptions *a priori*. Our model of correlated Choquet-Brownian deals with two ambiguity sources and can enrich some real option models. There is a rich literature examining the *interdependence* of different options in one project and accounting for the existence of *interrelated* projects. However, in contrast, there are few proposals to apply multiple risk factors on one real option. Finally, expanding the Choquet ambiguity applications presented in Section 1.3 to multiple correlated sources of ambiguity may be useful in complementing other models in continuous-time, in very ambiguous environments.

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[8] $I(t, w)$ is usually interpreted as the relative risk aversion coefficient.

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