Economics Bulletin

Volume 37, Issue 1

Gaussian Quadratures vs. Monte Carlo Experiments for Systematic Sensitivity Analysis of Computable General Equilibrium Model Results

Nelson B Villoria
Department of Agricultural Economics, Kansas State University

Paul V Preckel
Department of Agricultural Economics, Purdue University

Abstract

Third-order Gaussian quadratures (GQ) approximate the mean and variance of model results allowing for computationally inexpensive sensitivity analysis to uncertainty in exogenous parameters. Unfortunately, commonly used GQ approaches restrict the marginal distributions of both parameters and results sacrificing valuable distributional information. Using higher order quadratures, or incorporating more uncertain exogenous parameters, rapidly increases the sample size, undermining the rationale for using GQ. In contrast, Monte Carlo methods directly approximate the distribution of model outcomes without restrictive distributional assumptions on exogenous parameters. We argue that current computing capabilities allow for wider use of Monte Carlo methods for conducting stochastic simulations.

This project was supported by the Agriculture and Food Research Initiative Competitive Grant no. 2015-67023-25258 from the USDA National Institute of Food and Agriculture.

Citation: Nelson B Villoria and Paul V Preckel, (2017) "Gaussian Quadratures vs. Monte Carlo Experiments for Systematic Sensitivity Analysis of Computable General Equilibrium Model Results", *Economics Bulletin*, Volume 37, Issue 1, pages 480-487

Contact: Nelson B Villoria - nvilloria@ksu.edu, Paul V Preckel - preckel@purdue.edu.

Submitted: January 04, 2017. **Published:** March 20, 2017.

1. Introduction

Quantifying the uncertainty introduced in economic models by uncertain exogenous parameters requires stochastic simulations, which may be computationally onerous. One especially economical approach to reduce the time necessary to perform stochastic simulations relies on numerical integration through Gaussian quadratures, or GQs (DeVuyst and Preckel 1997). GQ procedures assess the robustness of model results to assumed parameter values based on explicit definitions of the joint distributions of the model parameters in a procedure known as Systematic Sensitivity Analysis (SSA). The SSA-GQ approach proceeds by making a discrete approximation to the joint parameter distribution, evaluating the model results for each of the discrete mass points in the approximate distribution, and then calculating statistics (e.g. mean and variance) as summary measures of the joint distribution of results.

SSA evaluation of computable general equilibrium (CGE) modeling results is relatively uncommon. However, in those studies that perform SSA, GQ is by far the most common approach. The GQ approach is parsimonious in the number of model evaluations required to perform the SSA, which makes it attractive for CGE models because of their complexity and potential size (e.g., Artavia, Grethe, and Zimmermann 2015). At a time when using Monte Carlo methods was impractical due to computing limitations (Arndt 1996), the development of specialized software to conduct GQ-SSA of CGE models (e.g., Pearson and Arndt 2000) represented a significant step forward in the study of the robustness of CGE models.

Yet, as we argue in this note, the distributional assumptions required to use GQ approaches needed to achieve meaningful computational savings are overly restrictive, leading to potentially inaccurate approximations of the first and second moments of the results of interest and to GQ samples without information about higher distribution moments. In contrast, Monte Carlo experiments produce detailed empirical distributions of model results, and are limited only by our ability to draw random numbers from the distributions of exogenous parameters. With increasingly available hardware and software suitable for parallel execution of large-scale economic modeling exercises, we suggest that Monte Carlo experiments should be the method of choice for conducting SSA of CGE model results. This suggestion apply to other types of models as well. SSA is a useful technique to explore the robustness of any economic model whenever the uncertainty about model parameters or shocks can be formalized. As discussed below, Monte Carlo-SSA provides a much richer perspective on robustness of model results.

2. Background on Gaussian Quadrature for Systematic Sensitivity Analysis

Consider a CGE model of an economy specified as

$$G(x, \beta) = 0, \tag{1}$$

where x denotes the results (e.g., prices), β denotes the exogenous parameters, and $G()$ denotes the structural relationships between the variables and parameters that define equilibrium. Recognizing that solving the system (1) will result in values (ideally unique) for x, we write the results directly as a function of the parameters, $x(\beta)$.

If we denote the joint distribution of the parameters β by the density $F(\beta)$, then we will typically be interested in the mean of one or more components of x, or $\bar{x}_i = E_\beta[x_i(\beta)]$ and perhaps the variance as well, $Var(x_i) = E_\beta[(x_i(\beta) - \bar{x}_i)^2]$. Common practice is to use the estimates of the mean and variance in combination with Chebyshev's inequality to obtain

confidence bounds on x_i (e.g., DeVuyst and Preckel 1997). In this sense the robustness of qualitative model results can be assessed.

To simplify the exposition we treat β as a continuous univariate random variable with density $F(\beta)$. However, the theory and procedures are well developed elsewhere for the multivariate case (e.g., Haber 1970; DeVuyst and Preckel 2007). The idea is to make a discrete approximation to the distribution of β that is expressed as a set of J points and associated probabilities $\{[\beta_j,p_j], j=1,\ldots,J\}$. The Gaussian quadrature chooses these points and probabilities so that the moments about zero of the approximating distribution equal the moments of the true distribution from zero through some specified order. That is,

$$\sum_{j=1}^{J} p_j \beta_j^{\,m} = E[\beta^m] \qquad \text{for } m = 0,1,\ldots,M. \tag{2}$$

Because the first M moments of the approximating distribution are equal to the moments of the true distribution, this is called a degree M Gaussian quadrature. In practice, moments calculated as the probability weighted sum of the model results evaluated at the points in the approximate distribution. For example, the mean of the model results is approximated as

$$E[x(\beta)] \approx \sum_{j=1}^{J} p_j x(\beta_j). \tag{3}$$

Bounds on the number of points required to produce a degree M Gaussian quadrature have been developed for the case where β is multivariate. In the case where there are N random variables and an order M quadrature is desired, an upper bound on the number of points required for a quadrature to exist is given by Tchakaloff (1957) as:

$$\bar{J} = \binom{N+M}{M}. \tag{4}$$

This bound rises quickly as a function of N and *very* quickly as a function of M (see Table 1). For the general degree 4 quadrature, the number of points (simulations in the current context) needed is over 10,000 for just 20 variables. For degree 5, the number of variables for which the simulation count exceeds 10,000 drops to 14. The corresponding number of variables for degree 6 and 7 are 11 and 9, respectively. Thus, if higher degree quadratures are needed, as is typically the case when the underlying model is a highly nonlinear function of the model parameters, then the number of required simulations quickly exceeds the 10,000 that is typically used for the Monte Carlo approach (see e.g., Haber 1970), as the number of uncertain parameters increases.

3. Comparison of Gaussian Quadrature and Monte Carlo Stochastic Modeling

For this comparison we revisit the study by Villoria and Mghenyi (2016, henceforth VM). They investigated the extent to which wheat and rice price stabilization policies in India affect price stability in neighboring countries using a GTAP-based equilibrium model (Hertel 1997) to compare the variances of regional prices with and without active stabilization policies. The price variances were obtained by performing a GQ based stochastic simulation of price responses to wheat and rice yield shocks using the SSA tools implemented by Pearson and Arndt (2000). This application is fairly representative of stochastic simulations using general equilibrium models (e.g., Valenzuela et al. 2007).

We focus our discussion on two main model outcomes resulting from stochastic changes in annual rice yields: changes in regional market prices of wheat and rice (processed and paddy) and changes in regional equivalent variation (EV). These changes are expressed as percentage changes from year 2004, the reference year used to calibrate the model. We limit our discussion to the case without stabilization policies in VM. As VM, we assume that the distribution of yield shocks are symmetric about their mean, and that yields vary independently across crops and regions. With nine regions, the GQ proposed by Liu (1997) requires the models to be solved thirty-two times for consistency with an order three Gaussian quadrature approximation to the distribution of yield shocks. In contrast, the Monte Carlo procedure uses 10,000 vectors of regional shocks randomly drawn from a normal distribution based on the historical distribution of the normalized yields.

Table 1. Number of points to guarantee a quadrature exists by number of random variables and degree

Vars.\Degree	3*	3	4	5	6	7	8	9
1	2	4	5	6	7	8	9	10
2	4	10	15	21	28	36	45	55
3	6	20	35	56	84	120	165	220
4	8	35	70	126	210	330	495	715
5	10	56	126	252	462	792	1,287	2,002
6	12	84	210	462	924	1,716	3,003	5,005
7	14	120	330	792	1,716	3,432	6,435	11,440
8	16	165	495	1,287	3,003	6,435	12,870	24,310
9	18	220	715	2,002	5,005	11,440	24,310	48,620
10	20	286	1,001	3,003	8,008	19,448	43,758	92,378
11	22	364	1,365	4,368	12,376	31,824	75,582	167,960
12	24	455	1,820	6,188	18,564	50,388	125,970	293,930
13	26	560	2,380	8,568	27,132	77,520	203,490	497,420
14	28	680	3,060	11,628	38,760	116,280	319,770	817,190
15	30	816	3,876	15,504	54,264	170,544	490,314	1,307,504
16	32	969	4,845	20,349	74,613	245,157	735,471	2,042,975
17	34	1,140	5,985	26,334	100,947	346,104	1,081,575	3,124,550
18	36	1,330	7,315	33,649	134,596	480,700	1,562,275	4,686,825
19	38	1,540	8,855	42,504	177,100	657,800	2,220,075	6,906,900
20	40	1,771	10,626	53,130	230,230	888,030	3,108,105	10,015,005
* Order three symmetric case with Stroud or Liu quadrature.								

Figure 1 describes the quartiles of the ratios of the GQ based summary statistics to their MC counterparts, pooled across countries in the case of the EV, and across countries and commodities in the case of the market price. If the two techniques yielded similar results, one would expect most ratios to be close to unity. However as is evident in the boxplots in figure 1, the GQ means and standard deviations differ without any apparent systematic pattern from the MC summary statistics.

Turning our attention to the third central moment, or skewness, our simulations reveal a much more troublesome pattern. Each panel in figure 2 displays the distribution of the (10,000) Monte Carlo outcomes for the percentage change in the price of a single commodity and region superimposed on a histogram plot of the distribution of (32) Gaussian quadrature outcomes for the same variable. We have also included the estimated skewness coefficients based on each sample. For most regions the results of the GQ outcomes are divided in two clusters to either side of the distribution peak and appear to be perfectly symmetric (skewness near zero). Meanwhile, the MC results reveal important degrees of positive skewness in all the countries (except for the EU25, where the MC results are symmetric while the GQ results display positive skewness). In sum, it appears that by taking the GQ approach, considerable information is lost regarding the shape of the distribution, its higher order moments, and its range.

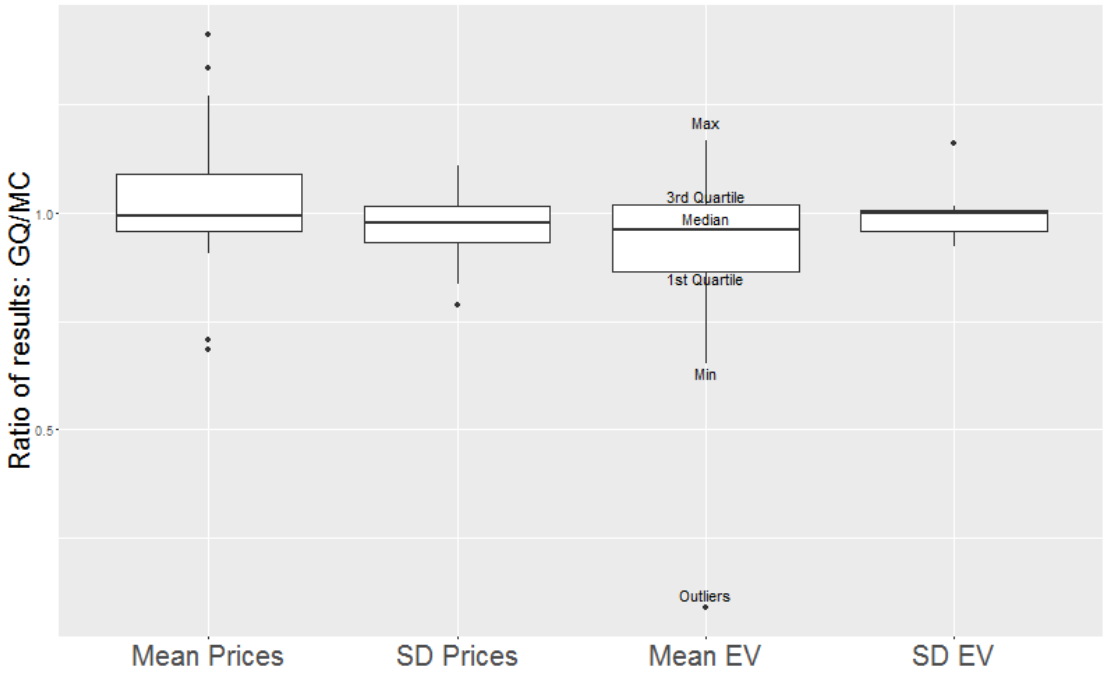

Figure 1. Gaussian Quadrature vs. Monte Carlo: If the GQ and MC techniques produced similar estimates, most result ratios would cluster around a median value of unity.

The use of MC simulations has an additional advantage in communicating the uncertainty surrounding model results. In particular, the limited knowledge of the distribution of the GQ samples makes confidence intervals based on Chebyshev's inequality much larger than confidence intervals based on known distributions (e.g., a 95% confidence interval based on Chebyshev's inequality extends 4.4 times the standard deviation of the GQ samples in either direction; a Normal-based CI extends 1.96 times the standard deviation.) In the case of the MC simulations, it is straightforward to construct empirical confidence intervals without recourse to distributional assumptions by using data quantiles; e.g., a 95% CI is obtained by sorting the 10,000 observations in increasing order and using the 250^{th} and $9{,}750^{th}$ values as the bounds of the interval (see figure 3).

For perspective on wall clock time for serial versus parallel calculation, we again consider our example. With the GQ approach, 32 model solves would be required. Using T to

denote the time required for a single model to solve, with serial computation the GQ would solve in 32×T. With the Monte Carlo approach and serial computation, 10,000 solves would be needed for a time cost of 10,000×T – a factor of over 300 more than for the GQ. As the number of stochastic variables increases or the quadrature degree rises, the number of solves required to execute the GQ analysis rises rapidly, and the time required for Monte Carlo with parallel execution rapidly becomes competitive with GQ (see Table 1). This is in part because the recommendation of roughly 10,000 simulations for Monte Carlo is independent on the number of stochastic variables (e.g., Haber 1970). If parallel computation is feasible, and n processors are available, then the time for the 10,000 solves drops to roughly 10,000×T/n.

Hypothetically, the availability of massively parallel computing hardware and software may eventually level the playing field between the GQ and MC approaches in terms of time to complete the sensitivity analysis. In both cases, if the available number of processors is as large as the number of points in the GQ and the number of draws for MC, then the wall clock time to do the analysis falls to the time required for a single model to solve (plus some overhead to manage spawning the jobs and collecting the results). In practice, the availability of cores, RAM memory, and competing projects in shared clusters are likely to constraint the number of feasible parallel simulations within reasonable limits so more accurate GQ simulations become competitive with Monte Carlo experiments. With current parallel hardware and software resources, the differences between computation time for GQ and MC are not nearly as great as they once were, and given the richness of the MC results, the extra expense of time and resources may be justified.

4. Conclusions

Efforts to reduce the computational time to perform SSA of model results come at a cost of restrictions on the joint distributions of the model parameters and the information captured in the approximations to those distributions. Even when those restrictions are satisfied, the required computational effort rises rapidly with the degree of the quadrature, and the number of uncertain parameters. With recent advances in computing software and hardware, the computations required for performing stochastic modeling via Monte Carlo methods no longer requires a prohibitive amount of wall clock time. Given the significant increase in flexibility in specifying the joint distributions of model stochastic variables and the progress on hardware and software, the time may have come for Monte Carlo approaches to become the clear choice for implementing stochastic models.

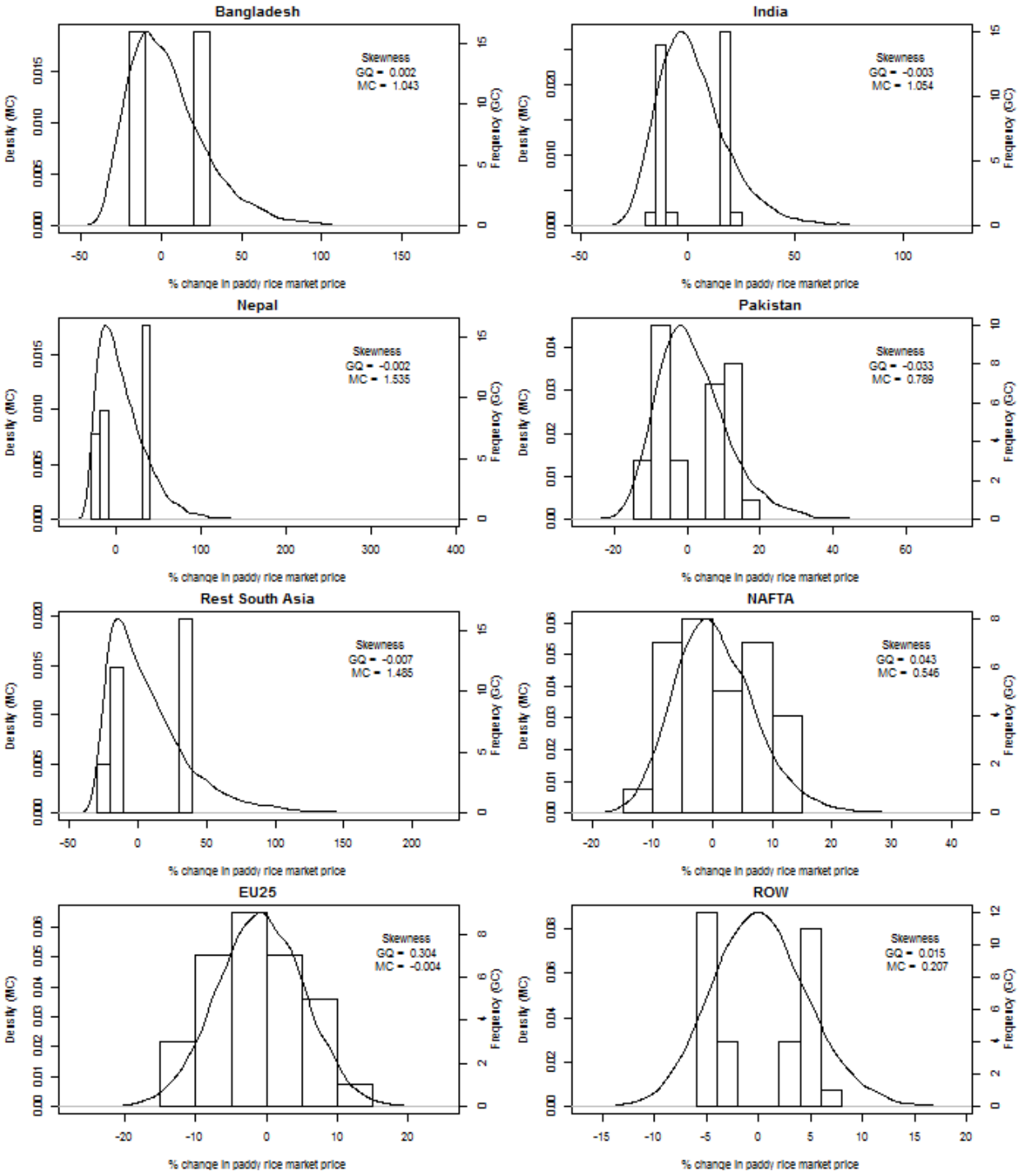

Figure 2. Monte Carlo (density of 10,000 observations, frequency in left axis) and GQ (histogram of 32 observations, counts in right axis) empirical distributions of prices for paddy rice.

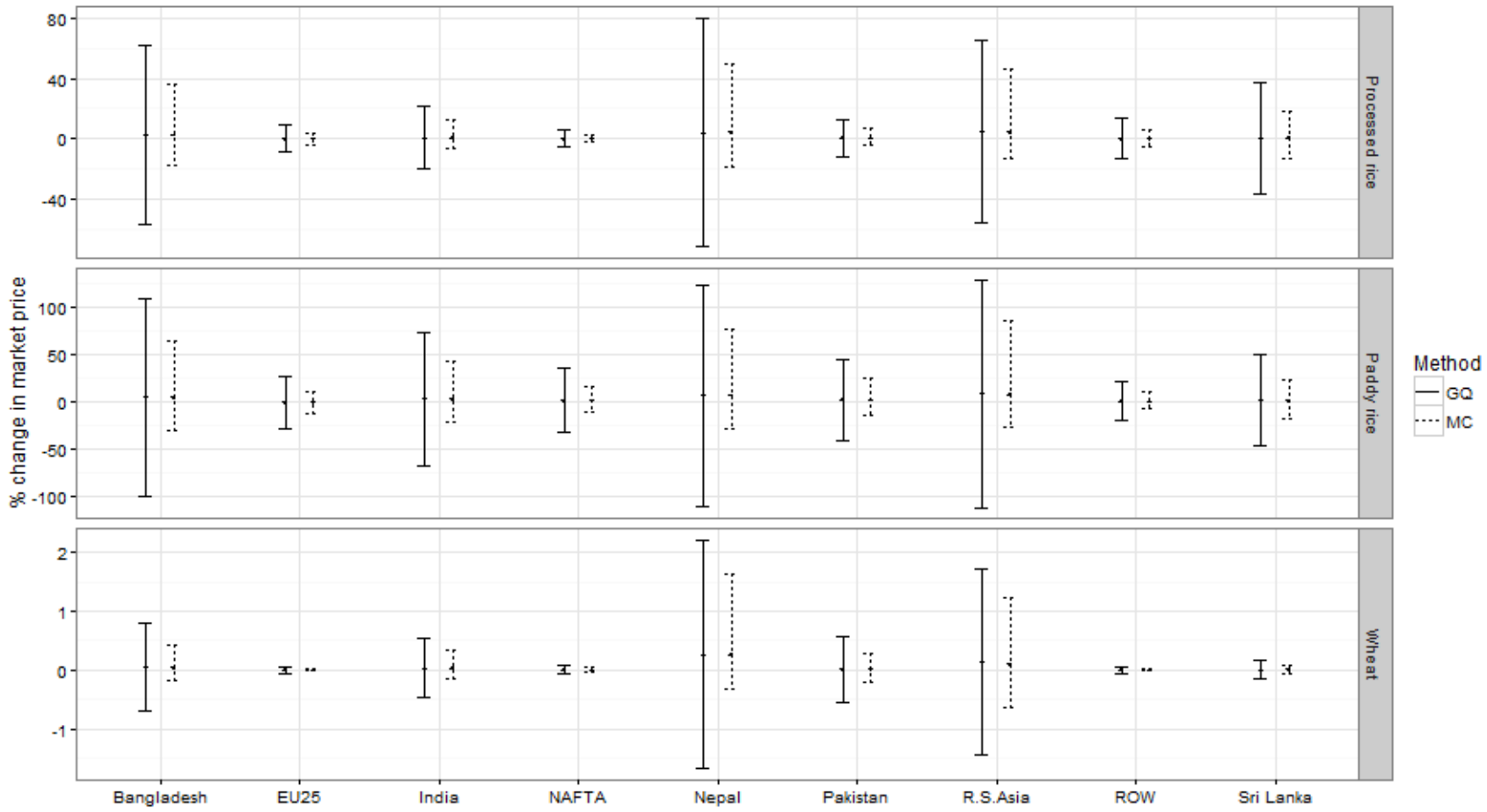

Figure 3. Confidence intervals (95%) for model-generated percentage changes in market prices based on GQ (using Chebyshev's inequality) and Monte Carlo (using data quantiles) results.

References

Arndt, Channing. 1996. “An Introduction to Systematic Sensitivity Analysis via Gaussian Quadrature.” GTAP Technical Paper 02. GTAP Technical Paper. GTAP Center, Department of Agricultural Economics, Purdue University.

Artavia, Marco, Harald Grethe, and Georg Zimmermann. 2015. “Stochastic Market Modeling with Gaussian Quadratures: Do Rotations of Stroud’s Octahedron Matter?” *Economic Modelling* 45 (February): 155–68. doi:10.1016/j.econmod.2014.10.017.

DeVuyst, Eric A., and Paul V. Preckel. 1997. “Sensitivity Analysis Revisited: A Quadrature-Based Approach.” *Journal of Policy Modeling* 19 (2): 175–85.

———. 2007. “Gaussian Cubature: A Practitioner’s Guide.” *Mathematical and Computer Modelling* 45 (7–8): 787–94. doi:10.1016/j.mcm.2006.07.021.

Haber, Seymour. 1970. “Numerical Evaluation of Multiple Integrals.” *SIAM Review* 12 (4): 481–526.

Hertel, Thomas W. 1997. *Global Trade Analysis: Modeling and Applications*. Cambridge, MA: Cambridge University Press.

Pearson, Ken, and Channing Arndt. 2000. “Implementing Systematic Sensitivity Analysis Using GEMPACK.” *GTAP Technical Paper No. 03*. https://goo.gl/sOFm5L.

Tchakaloff, Vladimir. 1957. “Formules de Cubatures Mécaniques À Coefficients Non Négatifs.” *Bull. Sci. Math* 81 (2): 123–134.

Valenzuela, Ernesto, Thomas W. Hertel, Roman Keeney, and Jeffrey J. Reimer. 2007. “Assessing Global Computable General Equilibrium Model Validity Using Agricultural Price Volatility.” *American Journal of Agricultural Economics* 89 (2): 383–97.

Villoria, Nelson Benjamin, and Elliot Wamboka Mghenyi. 2016. “The Impacts of India’s Food Security Policies on South Asian Wheat and Rice Markets.” *The World Bank Economic Review*, February. doi:10.1093/wber/lhw002.