Economics Bulletin

Volume 36, Issue 1

Log-linear demand systems with differentiated products are inconsistent with the representative consumer approach

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Abstract

We argue that log-linear demands with differentiated products, which are viewed as useful modeling from an empirical standpoint, are generated from the representative consumer's utility only in a restrictive form.

We are grateful to Quan Wen (editor-in-charge) and two anonymous referees for providing us with invaluable suggestions. We also thank Michal Fabinger, Makoto Hanazono, Yuji Nakayama, Hikaru Ogawa, Takeshi Ogawa, Masa Tsubuku and Glen Weyl for helpful comments and discussions. Adachi acknowledges a Grant-in-Aid for Young Scientists (B) (24730205) and a Grant-in-Aid for Scientific Research (C) (15K03425) from the Japan Society for the Promotion of Science as well as the Kikawada Foundation for financial supports. Ebina acknowledges a Grant-in-Aid for Scientific Research (C) (24530264) and a Grant-in-Aid for Young Scientists (B) (24730224) from the Society. Any remaining errors are our own.

Citation: Takanori Adachi and Takeshi Ebina, (2016) "Log-linear demand systems with differentiated products are inconsistent with the representative consumer approach", *Economics Bulletin*, Volume 36, Issue 1, pages 260-267

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Submitted: October 19, 2015. **Published:** February 21, 2016.

1. Introduction

Log-linear demands are considered to be useful modeling from an empirical standpoint: in cases of single-product monopolies and homogeneous-product oligopolies, the coefficient for the log of the own price term is interpreted as own price elasticity. However, as products are more or less differentiated in reality, researchers often consider a system of log-linear demand that also includes separate terms of the logs of other products' prices. The coefficient for such a term is then interpreted as the *cross price* elasticity (see, e.g., Baker and Bresnahan (1985) and Hendel and Nevo (2013) for empirical applications). However, log-linear demands with differentiated products are not only intractable as the number of products becomes large[1] but also conceptually flawed even if the number of differentiated products is smaller (i.e., two). Regarding the former point, Jaffe and Weyl (2010) and Jaffe and Kominers (2012) show that *market demand that is additively separable in own price (log-linear demand is one) cannot be generated from discrete choice modeling*, which is ubiquitous in empirical studies of industrial organization.[2] This note argues the latter point: *log-linear demands cannot be generated from the representative consumer's utility, either.*[3] More specifically, we show that *the representative consumer approach can generate log-linear market demands only for the case of complements with the sum of own and cross price elasticities being unity.*

2. Main Argument

We start with a general description of the demand system with constant own

[1] In the empirical industrial organization literature, this is known as the J^2 *problem.* Suppose that consumers face J products. If one starts with (linear or log linear) market demand function for product j as a function of (among others) other rival products' prices as well as j's own price, then the number of parameters to be estimated is at least J^2. Instead, one can think of consumers gaining utility from a product as a bundle of product characteristics, and then each consumer's probability of demanding a particular product is aggregated to construct demand function for the product (this is often called the *product characteristics approach*). See, e.g., Nevo (2000), Davis and Garcés (2010), and Aguirregabiria and Nevo (2013) for excellent surveys on the product characteristics approach.

[2] In relation to these two papers, Armstrong and Vickers (2015) provide a necessary and sufficient condition for a multi-product demand system to be consistent with discrete choice.

[3] See Vives (1999, Chapter 6) for a general exposition on the representative consumer approach. Anderson, de Palma, and Thisse (1992) study the relationships between the two approaches.

and cross price elasticities. In particular, we argue that a parametric restriction is obtained by considering a condition on consumer behavior: whether the total expenditure rises or lowers as a response to an increase in the price of one product. We then argue whether there exists the representative consumer's utility that generates the demand systems with constant own and cross elasticities. We show that it is possible only in a restrictive form.

Suppose first that there are J (≥ 2) products. Each product $j = 1, 2, ..., J$ is identical to brand by Firm j. Let $q = (q_1, q_2, ..., q_J)$ be the representative consumer's commodity bundle, where $q_j \geq 0$ for each j. The representative consumer has the utility function over this bundle and the numéraire. We assume that her utility function is written as $U(q, m) = u(q) + m$, where $u(\cdot)$ satisfies the standard assumptions.[4] She maximizes $U(q, m)$ with respect to q and m, subject to the budget constraint $\sum_{j=1}^{J} p_j q_j + m = I$, where $I > 0$ is her (exogenous) income, and $p_j > 0$ is the price of product j. Let $p = (p_1, p_2, ..., p_J)$. As a result of the utility maximization problem, we derive $q_j = q_j(p)$ as the demand function for product j in this market. The market demand system $(q_j(p))_{j=1}^{J}$ is obtained by solving the inverse demand system: $\partial u(q)/\partial q_j = p_j$ for $j = 1, 2, ..., J$.

Now, let $e_j(p) \equiv p_j q_j(p)$ be the expenditure for product j, and define the expenditure for the numéraire by $m(p, I) \equiv I - \sum_{j=1}^{J} e_j(p)$. Then, the representative consumer increases (reduces) the total expenditure for a small increase in p_j if and only if $\partial \left(\sum_{j'=1}^{J} e_{j'}(p) \right) / \partial p_j > (<)\ 0$, that is,

$$\begin{aligned} & q_j(p) + p_j \frac{\partial q_j(p)}{\partial p_j} + \sum_{j' \neq j} p_{j'} \frac{\partial q_{j'}(p)}{\partial p_j} > (<)\ 0 \\ \Leftrightarrow \quad & \epsilon_{jj}(p) < (>)\ 1 + \frac{\sum_{j' \neq j} p_{j'} q_{j'}(p) \sigma_{j'j}(p)}{p_j q_j(p)}, \end{aligned}$$

where $\epsilon_{jj}(p) \equiv -\left(p_j / q_j(p)\right) \left(\partial q_j(p) / \partial p_j\right) > 0$ is the *own price elasticity* of product j, and $\sigma_{j'j}(p) \equiv \left(p_j / q_{j'}(p)\right) \left(\partial q_{j'}(p) / \partial p_j\right)$ is the *cross price elasticity* of product $j' \neq j$ in response to product j's price. Note that we allow $\sigma_{j'j}(p)$ to be either

[4] In this paper, we assume the quasilinear utility function because we are interested in a partial equilibrium analysis. See Koopman and Uzawa (1990) for a study of constant elasticities with a utility function in a general form.

positive (i.e., $\partial q_{j'}(p)/\partial p_j > 0$; products j and j' are substitutes), negative (products j and j' are complements), or zero. If the own price elasticity is constant and the same for all j's (i.e., $\epsilon_{jj}(p) = \epsilon$) and the cross price elasticity is constant and the same for all j's (i.e., $\sigma_{j'j}(p) = \sigma$), then the inequality above is further rewritten as $\epsilon \lesseqgtr 1 + \sigma \left(\sum_{j' \neq j} p_{j'} q_{j'}(p)\right) / p_j q_j(p)$. If $q_j(p) = q_{j'}(p)$ under $p_j = p_{j'}$ for all $j \neq j'$, then in symmetric price equilibrium, the above inequality is rewritten as

$$\epsilon < (>)\ 1 + (J-1)\sigma, \tag{1}$$

which implies a parametric restriction when one focuses on the symmetric equilibrium. The issue is whether there exists the representative consumer's utility that corresponds one-to-one to the demand system with constant own and cross price elasticities.

To study this problem in a simple manner, consider the system of demand functions for two symmetric firms ($J = 2$), A and B, each of which produces a differentiated product:

$$\begin{cases} q^A = a(p^A)^{-\epsilon}(p^B)^{\sigma} \\ \\ q^B = a(p^B)^{-\epsilon}(p^A)^{\sigma}, \end{cases}$$

where $a > 0$ and $\epsilon > \max\{1, \sigma\}$. This demand system is log-linear in the sense that $\ln q^j = \ln a - \epsilon \ln p^j + \sigma \ln p^{j'}$, $j, j' \in \{A, B\}$, $j \neq j'$. In particular, both own and cross price elasticities are constant because $-(\partial q^A/\partial p^A)\ (p^A/q^A) = \epsilon$ and $(\partial q^A/\partial p^B)(p^B/q^A) = \sigma$. Solving the demand system for p^A and p^B yields the following inverse demand functions:

$$\begin{cases} p^A = a^{\frac{1}{\epsilon-\sigma}}(q^A)^{\frac{-\epsilon}{\epsilon^2-\sigma^2}}(q^B)^{\frac{-\sigma}{\epsilon^2-\sigma^2}} \\ \\ p^B = a^{\frac{1}{\epsilon-\sigma}}(q^B)^{\frac{-\epsilon}{\epsilon^2-\sigma^2}}(q^A)^{\frac{-\sigma}{\epsilon^2-\sigma^2}}. \end{cases}$$

It is seen below that σ *cannot be a free parameter if the demand system is generated from the representative consumer's utility*: it must be equal to $1 - \epsilon$. Additionally, it is seen below that $\epsilon > 1$ is necessary (as already assumed) for a positive value for consumer surplus. However, this implies that the two products are *complements* (because $1 - \epsilon < 0 \Leftrightarrow \sigma < 0$). Thus, one cannot deal with substitutes. To verify these claims, notice that the representative consumer's utility $U(q^A, q^B)$ must satisfy

$$\frac{\partial U}{\partial q^A} = a^{\frac{1}{\epsilon-\sigma}} (q^A)^{\frac{-\epsilon}{\epsilon^2-\sigma^2}} (q^B)^{\frac{-\sigma}{\epsilon^2-\sigma^2}} \tag{2}$$

and

$$\frac{\partial U}{\partial q^B} = a^{\frac{1}{\epsilon-\sigma}} (q^B)^{\frac{-\epsilon}{\epsilon^2-\sigma^2}} (q^A)^{\frac{-\sigma}{\epsilon^2-\sigma^2}}. \tag{3}$$

From (2), it is derived that

$$\frac{\partial^2 U}{\partial q^B \partial q^A} = \frac{-\sigma}{\epsilon^2-\sigma^2} a^{\frac{1}{\epsilon-\sigma}} (q^A)^{\frac{-\epsilon}{\epsilon^2-\sigma^2}} \ (q^B)^{\frac{-\sigma}{\epsilon^2-\sigma^2}-1}$$

and from (3),

$$\frac{\partial^2 U}{\partial q^A \partial q^B} = \frac{-\sigma}{\epsilon^2-\sigma^2} a^{\frac{1}{\epsilon-\sigma}} (q^A)^{\frac{-\sigma}{\epsilon^2-\sigma^2}-1} (q^B)^{\frac{-\epsilon}{\epsilon^2-\sigma^2}}.$$

For the utility function to satisfy the symmetry in the cross partial derivatives, it must be that

$$\begin{aligned} \frac{-\sigma}{\epsilon^2-\sigma^2} - 1 &= \frac{-\epsilon}{\epsilon^2-\sigma^2} \\ &\Leftrightarrow \\ \epsilon + \sigma &= 1. \end{aligned} \tag{4}$$

Note that from (1) above, equality (4) is consistent only if the representative consumer *reduces* the total expenditure for a small increase in p_j. Now, if one wishes to base welfare evaluation on the representative consumer's utility, the demand system must be:

$$\begin{cases} q^A = a(p^A)^{-\epsilon}(p^B)^{1-\epsilon} \\ q^B = a(p^B)^{-\epsilon}(p^A)^{1-\epsilon}, \end{cases}$$

which is consistent with the following representative consumer's utility:

$$U(q^A, q^B) = a^{\frac{1}{2\epsilon-1}} \frac{2\epsilon-1}{\epsilon-1} \left(q^A q^B\right)^{\frac{\epsilon-1}{2\epsilon-1}}.$$

and thus, consumer surplus based on the representative consumer utility can be defined by

$$CS(q^A, q^B) = a^{\frac{1}{2\epsilon-1}} \frac{2\epsilon-1}{\epsilon-1} \left(q^A q^B\right)^{\frac{\epsilon-1}{2\epsilon-1}} - p^A q^A - p^B q^B.$$

For $U(q^A, q^B)$ to be positive, it must be that $\epsilon > 1$. Recall that if Firm A raises its price p^A by one percent, then it loses (with everything else held equal) its demand by ϵ percent. Simultaneously, the rival firm gains a $(1-\epsilon)$ percent increase in its demand. Because $1-\epsilon$ is negative, it means that Firm B loses a $|1-\epsilon|$ percent of the consumers in response to the price increase by Firm A. Thus, *the representative consumer approach is consistent with log-linear demands only for the case of complements with the sum of own and cross price elasticities being unity.* Additionally, it is seen from inequality (1) and equality (4) that the representative consumer increases (reduces) the total expenditure for a small increase in p^A or p^B.[5]

Finally, notice that if the market is governed by a monopoly or a homogenous-product oligopoly, then the log-linear market demand is given by $q = a(p)^{-\epsilon}$, where $\epsilon > 1$.[6] The inverse demand is $p = a^{1/\epsilon} q^{-1/\epsilon}$. The representative consumer's utility is simply given by

$$U(Q;p) = \frac{a^{\frac{1}{\epsilon}}\epsilon}{\epsilon - 1} Q^{\frac{\epsilon-1}{\epsilon}} - pQ, \tag{5}$$

where $Q = \sum_{i=1}^{n} q^i$ in the case of homogenous-product oligopoly.

3. Consumer Surplus with Log-Linear Market Demands

In the case of log-linear demands, if one wants to let σ be a free parameter, a natural definition of consumer surplus for product $j = A, B$, given $q^{j'}$, $j' = A, B$, $j' \neq j$, would be

$$CS^j(q^j, q^{j'}; p^j) \equiv \int_0^{q^j} \left[a^{\frac{1}{\epsilon-\sigma}} (\tilde{q}^j)^{\frac{-\epsilon}{\epsilon^2-\sigma^2}} (q^{j'})^{\frac{-\sigma}{\epsilon^2-\sigma^2}} - p^j \right] d\tilde{q}^j$$

[5] In contrast, linear market demands can be generated from the representative consumer's utility,

$$U(q^A, q^B) = \alpha \cdot (q^A + q^B) - \frac{1}{2}\left(\beta [q^A]^2 + 2\gamma q^A q^B + \beta [q^B]^2\right),$$

where $|\gamma| < \beta$ denotes the degree of horizontal product differentiation: the two products are substitutes (complements) if $\gamma > 0$ ($\gamma < 0$).

[6] Aguirre and Cowan (2015) use this market demand to study monopolistic third-degree price discrimination with constant elasticity. Galera and Zaratiegui (2006) also consider constant elasticity market demands to study price-discriminating duopolists. However, in each market m, two firms compete in the Cournot manner (i.e., quantity competition), facing the *common* market demand, $Q = a_m p^{-\epsilon_m}$, where $a_m > 0$ and $\epsilon_m > 1$ capture heterogeneity across markets.

$$= \frac{(\epsilon^2 - \sigma^2) a^{\frac{1}{\epsilon - \sigma}}}{\epsilon^2 - \epsilon - \sigma^2} \left(q^j\right)^{\frac{\epsilon^2 - \epsilon - \sigma^2}{\epsilon^2 - \sigma^2}} \left(q^{j'}\right)^{\frac{-\sigma}{\epsilon^2 - \sigma^2}} - p^j q^j. \tag{6}$$

Thus, if both firms are symmetric, then the symmetric equilibrium realizes ($q^A = q^B \equiv q$ and $p^A = p^B \equiv p$), and the consumer surplus for each product j defined as above becomes

$$CS^j(q, p) \equiv CS^j(q, q; p) = \frac{(\epsilon^2 - \sigma^2) a^{\frac{1}{\epsilon - \sigma}}}{\epsilon^2 - \epsilon - \sigma^2} (q)^{\frac{\epsilon - \sigma - 1}{\epsilon - \sigma}} - pq.$$

Then, letting $Q = 2q$ and the aggregate consumer surplus be defined by $CS(Q; p) \equiv CS^A(Q/2, p) + CS^B(Q/2, p)$ yields

$$CS(Q; p) \to 2^{\frac{1}{\epsilon}} \frac{a^{\frac{1}{\epsilon}} \epsilon}{\epsilon - 1} (Q)^{\frac{\epsilon - 1}{\epsilon}} - pQ$$

as $\sigma \to 0$. That is, when the total output converges to the case of monopoly and homogenous oligopoly (and the equilibrium price always coincides with the monopolist's optimal price), $\lim_{\sigma \to 0} CS(Q, p)$ is greater than $U(Q; p)$ for any $\epsilon > 1$ because $2^{1/\epsilon} > 1$ (see equation (5) above). Thus, even in the limit of σ close to zero (i.e., two firms act as an independent monopolist), $CS(Q; p)$ is *over*valued. This is caused by the 'duplication' of integrals: consumer surplus (6) is defined for each j, whereas $U(Q; p)$ is defined over (q^A, q^B) 'at once.' Note that as the own price elasticity, ϵ, goes to infinity, $\lim_{\sigma \to 0} CS(Q, p)$ approaches to $U(Q; p)$. In our symmetric case, this means that a firm would lose almost all of its own demand if it raises its price even by a small amount; firms produce almost perfect substitutes. As long as the absolute value of consumer surplus itself is not discussed (as in the case of, e.g., comparison of consumer surplus under uniform pricing and under price discrimination), our definition (6) above would seem innocuous.[7]

[7] In relation to this point, Adachi (2004), in the case of monopolistic third-degree price discrimination, argues that, in response to Bertoletti (2004), a representative consumer modeling and a discrete choice modeling can generate different conclusions regarding the welfare effects of monopolistic third-degree price discrimination. Adachi and Ebina (2016) employ the definition of consumer surplus (6) above to study the welfare effects of oligopolistic third-degree price discrimination when own and cross price elasticities are constants.

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