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## Network Heterogeneity and a Coordination Game

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## Abstract

We investigate a two-strategy logit choice coordination game on heterogeneous networks. Degree is a number of links a vertex has and heterogeneous network is a network whose variance of degree distribution is large. We obtain mean-field approximate solution. We show that the heterogeneity of a network has an influence on the outcome. The magnitude of heterogeneity determines the number of stable steady states and the characteristics of the stable steady states. The network heterogeneity also determines which of the stable steady states is realized and the probability that a strategy is chosen in a given stable steady state.

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## 1. Introduction

We investigate a two-strategy coordination game on heterogeneous networks. In reality, people interact only with their neighbors, which can be represented by networks. Networks in reality are more heterogeneous than we expect. The heterogeneity of a network is the magnitude of the variance of its degree distribution. A scale-free network is a representative of heterogeneous network. Many real networks are scale-free ones. For instance, Konno (2009) studies the network of inter-firm transactions in Japan, which is a scale-free network with hierarchic. Since many social networks are heterogeneous, the effect of network heterogeneity on the outcome of a model is worth exploring. Konno (2011b), Santos and Pacheco (2005), and Ohtsuki et al. (2006) (also see reviews by Nowak (2006), and Szabo and Fath (2007)) study how underlying networks (in particular heterogeneous networks) determine whether cooperation emerges or not in evolutionary games. Pastor-Satorras and Vespignani (2001) show that an epidemic spread always occurs on scale-free networks, although it occurs under certain conditions on regular networks and random networks. Diffusion processes on social networks and underlying networks are discussed by Vega-Redondo (2007), Goyal (2007), and Jackson (2008). Review on networks are found in Dorogovtesev and Mendes (2003), S.Bornholdt (2003), Albert and Barabási (2002), Newman (2003), and Boccaletti et al. (2006). In logit choice games with local interactions on regular networks, the payoff sensitivity affects the equilibrium configuration, as shown by, e.g., Blume et al. (1993) and Brock and Durlauf (2001). If the payoff sensitivity is large in the games on regular networks, then two stable steady states exist. In this paper, we will study a logit choice coordination game with local interactions on non-regular networks. We will show that the mean degree of nearest neighbors (the heterogeneity of the network) have decisive influence on the equilibrium configuration.

## 2. Model: A Coordination Game on a Heterogeneous Network

Each player is on a vertex and plays games with adjacent players only. The network structure indicates who plays with whom. A player derives a payoff from each game. We study a symmetric-payoff coordination game with two strategies, $R$ and $P$. The payoff matrix is given by

$$\begin{array}{c} \\ R \\ P \end{array} \begin{array}{c} \begin{array}{cc} R & P \end{array} \\ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \end{array}, \tag{1}$$

where we assume that $a+b > d$ and $d > a > b > 0$. Strategy R is risk dominant, whereas the Pareto optimum realizes when all players choose strategy P. A player chooses the strategy by Logit choice with payoff sensitivity $\beta$. After the following transformation (Konno (2011c)) $x = \frac{a-b+d}{4}, y = \frac{a+b-d}{4}, z = \frac{a-b-d}{4}, w = \frac{a+b+d}{4}$ where we let $S_i = +1$ denote strategy R and let $S_i = -1$ denote strategy P for player $i$. The payoff $U_i$ for player $i$ with strategy $S_i$ is expressed in terms of $x$, $y$, $z$, and $w$ as $U_i = \sum_{j\in\partial i}(xS_iS_j + yS_i + zS_j + w)$ where $\partial i$ denotes the set of all players adjacent to player $i$. By substituting $S_i = \pm 1$ and $S_j = \pm 1$ into Eq. (2), we can confirm that the payoff $U_i$ reproduces the payoff matrix in Eq. (1). At each time step, one player is randomly chosen, and this chosen player updates his strategy with logit probability

(Luce (1959); Konno (2011a); Train (2003)). The update processes of the strategies can be viewed as a contagion of strategies through local interactions, which has been studied by Ellison (1993), Morris (2000), López-Pintado (2006), Alós-Ferrer and Weidenholzer (2008), and Galeotti et al. (2010). Unlike these papers, we study logit choice dynamics and stable steady states because they give us a clear picture and we can use mean-field approximation. When a player chooses a strategy, he knows the strategies of all the adjacent players. The idea behind this update rule is that it is not likely that all players change their strategies simultaneously. In order to take into account the fact that players are not perfectly rational in reality, a player updates his strategy with logit probability.The probability that player $i$ chooses strategy $S_i$ is given by (Konno (2011c))

$$\Pr(S_i) = \frac{\exp\left[\beta \sum_{j\in\partial i}(xS_iS_j + yS_i)\right]}{\sum_{S_i'=\pm 1}\exp\left[\beta \sum_{j\in\partial i}(xS_i'S_j + yS_i')\right]}, \tag{2}$$

where $\beta$ is a non-negative constant indicating payoff-sensitiveness. Since a player derives a payoff from each game, his choice is based on the total payoff of each strategy against his neighbors. If a player has more interactions, then he is likely to have a higher payoff, as is likely to happen in reality. What the player actually gains is the total payoff, so he chooses his strategy according to his total payoff. After enough time intervals have passed, the distribution of the strategies becomes stable. We call such a state a stable steady state, and we will focus on these states in this paper. We will study how network heterogeneity affects the fraction of players choosing strategy P. In other words, we will study how network heterogeneity affects cooperation. The logit update rule in Eq. (2) has been used by Blume et al. (1993), Arthur et al. (1997), Young (2001), and Brock and Durlauf (2001). The mathematical structure of this form of logit update probability is equivalent to spin models in statistical physics on a network (Baxter (1982), Kubo (1965), Greiner et al. (1995), and Ising (1925)). We will use the mean-field method from statistical physics. Models on non-regular networks are difficult to solve without approximations expect for few cases. Mean-field approximation has been used.

## 3. Network Heterogeneity and a Coordination Game

### *3.1. The Network Heterogeneity and the Stable Steady State*

Let $\langle S(k)\rangle$ denote the mean strategy of the players with degree $k$. Let $\langle x\rangle$ denote $E(x)$, where $x$ is some stochastic variable. Mean-field approximation is often used to solve a problem with a network structure, since deriving an exact solution can be difficult, especially when the network consists of many players as in this paper. In the mean-field approximation method, the strategies of adjacent players are replaced by their mean $\langle S_{\rm nn}\rangle$ where "nn" stands for the nearest neighbors. Further, in this approximation, the degree of any adjacent player is replaced by the mean degree of nearest neighbors $\langle k_{\rm nn}\rangle$. Mean-field approximation is an approximation in which the fluctuation of the degree distribution of the nearest neighbors is neglected. A player with arbitrary degree $k$ is surrounded by players with degree $\langle k_{\rm nn}\rangle$, and players with degree $\langle k_{\rm nn}\rangle$ are also surrounded by players with degree $\langle k_{\rm nn}\rangle$ in the mean-field approximation as illustrated in Fig. 1. we have $\langle S_{\rm nn}\rangle = \langle S(\langle k_{\rm nn}\rangle)\rangle$ in the approximation.

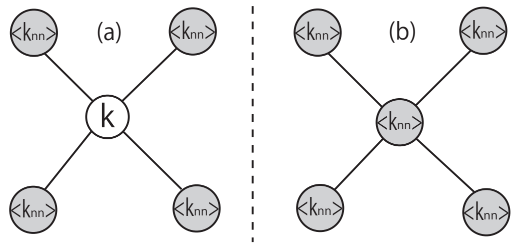



Figure 1: Mean-field approximation scheme: (a) A player is surrounded by players with degree $\langle k_{\rm nn}\rangle$, which is the mean degree of his nearest neighbors; (b) A player with degree $\langle k_{\rm nn}\rangle$ is also surrounded by players with degree $\langle k_{\rm nn}\rangle$.

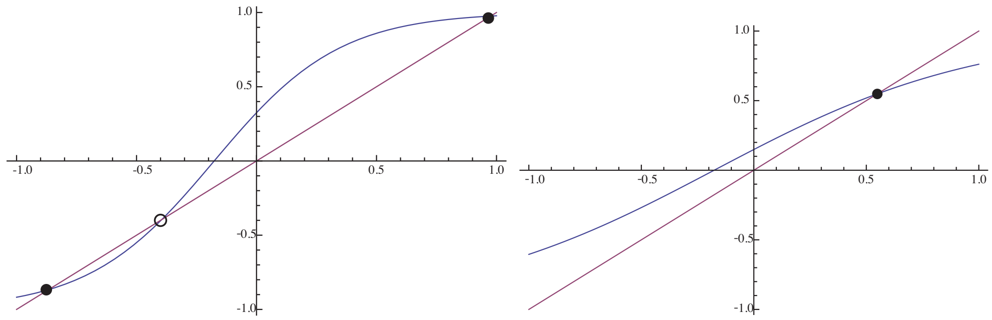
Note that the mean degree of nearest neighbors $\langle k_{\rm nn}\rangle$ is different from the mean degree $\langle k\rangle$. The relation $\langle k_{\rm nn}\rangle > \langle k\rangle$ always holds true in uncorrelated networks unless the network is a regular one. Even though the mean degree $\langle k\rangle$ are the same for two networks, the mean degree of nearest neighbors can be different. This difference has brought a number of interesting phenomena in the study of networks. Suppose we have three networks with the same mean degree, a regular network, a random network, and a scale-free network. The mean degree of nearest neighbors in a regular network is the smallest, and that of a scale-free network is the greatest. For some scale-free networks with infinite network size and exponent $\gamma$ satisfying $2 < \gamma \leq 3$, the mean degree of nearest neighbors is infinite even if the mean degree is the same as that of a regular network and a random network. If the network heterogeneity is large, the mean degree of nearest neighbors is also large. Since the probability that a player with degree $k$ chooses a strategy is given by the logit probability Eq. (2), we have

$$\langle S(k)\rangle = \frac{\sum\limits_{S'(k)=\pm 1} S'(k)\exp\left[\beta k(xS'(k)\langle S_{\rm nn}\rangle + yS'(k))\right]}{\sum\limits_{S'(k)=\pm 1} \exp\left[\beta k(xS'(k)\langle S_{\rm nn}\rangle + yS'(k))\right]} = \tanh\left[\beta k\left(x\langle S_{\rm nn}\rangle + y\right)\right], \qquad (3)$$

in the mean-field approximation. From Eq. (3), we obtain $\langle S(k)\rangle \forall k$ if we know $\langle S_{\rm nn}\rangle$. Let us denote $p_{\rm P}(k)$ and $p_{\rm R}(k)$ the fractions of the players with degree $k$ choosing strategy P and R, respectively. Let us denote $p_{\rm P}$ and $p_{\rm R}$ the fractions of the players choosing the strategy P and R, respectively. The problems of players with degree $\langle k_{\rm nn}\rangle$ are symmetric in the mean-field approximation. Similar to deriving Eq. (3), we have

$$\langle S_{\rm nn}\rangle = \tanh\left[\beta\langle k_{\rm nn}\rangle\left(x\langle S_{\rm nn}\rangle + y\right)\right]. \qquad (4)$$

The mean strategy of adjacent players $\langle S_{\rm nn}\rangle$ is the solution of Eq. (4), which is given by the intersection of the 45 degree line and the RHS of Eq. (4). The hyperbolic tangent function $\tanh(s)$ has the following characteristics; $\frac{d}{ds}\tanh(s) > 0$. $\frac{d^2}{ds^2}\tanh(s)$ is less than 0 if $s > 0$, it is 0 if $s = 0$, and it is larger than 0 if $s > 0$. $\frac{d}{ds}\tanh(\alpha s)\big|_{s=0} = \alpha$. $\tanh(s)$ is larger than 0 if $s > 0$, and less than 0 if $s < 0$. Hence, there are two cases depending on the parameters in the RHS of Eq. (4), as illustrated in Fig. 2. Since $y/x < 1$, $x > 0$, and $y > 0$ hold true




(a) Case I (b) Case II

Figure 2: The tangent hyperbolic function and the 45-degree line are illustrated. The intersections are the steady states. An empty circle indicates an unstable steady state, while a filled circle indicates a stable steady state.

from the assumptions, the RHS of Eq. (4) intersects the $\langle S_{\rm nn}\rangle$ axis between $-1$ and 0. In case I, there are two stable steady states and one unstable steady state, whereas there is only one stable steady state in case II. Because of the properties of the hyperbolic tangent function, Eq. (3), and Eq. (4), the signs of $\langle S(k)\rangle$ for all $k$ and $\langle S_{\rm nn}\rangle$ are the same. From $\langle S(k)\rangle$, we get $p_{\rm P}(k)$. Since $p_{\rm R}(k)-p_{\rm P}(k)=\langle S(k)\rangle$ and $p_{\rm R}(k)+p_{\rm P}(k)=1$ both hold, we have $p_{\rm P}(k)=(1-\langle S(k)\rangle)/2$. Therefore, if $\langle S(k)\rangle<0$ holds true for all $k$, then $p_{\rm P}(k)>p_{\rm R}(k)$ holds true as well. Hence, more than half of the players choose strategy P in the state where $\langle S(k)\rangle<0$, and less than half of the players choose the strategy in the state where $\langle S(k)\rangle>0$. The sign of $\langle S(k)\rangle$ is the same for all $k$, as is stated yet. In case I, $p_{\rm R}(k)>p_{\rm P}(k)$ holds for all $k$ in the stable steady state where $\langle S_{\rm nn}\rangle>0$ and $p_{\rm P}(k)>p_{\rm R}(k)$ holds for all $k$ in the other stable steady state where $\langle S_{\rm nn}\rangle<0$. In case II, $p_{\rm R}(k)>p_{\rm P}(k)$ holds for all $k$ in the stable steady state. Hence, the condition when case II occurs and the state converges to the stable steady state where $p_{\rm P}(k)>p_{\rm R}(k)$ for all $k$ are of interest. If $\langle S(k)\rangle<0$ holds which means that more than half of the players choose strategy $P$, a large $P_{\rm P}$ implies the players in the network have a large average utility. We will study how the probability that a player chooses strategy P is affected by network heterogeneity.

We will focus on case I for the time being. Because of the properties of the hyperbolic tangent function, a necessary condition for case I to occur is

$$\left.\frac{\partial\tanh\left[\beta\langle k_{\rm nn}\rangle(x\langle S_{\rm nn}\rangle+y)\right]}{\partial\langle S_{\rm nn}\rangle}\right|_{x\langle S_{\rm nn}\rangle+y=0}>1, \qquad (5)$$

which yields $\beta\langle k_{\rm nn}\rangle>1$. Hence, a sufficient condition for case II to occur is the opposite. This condition is rewritten as $\beta\langle k_{\rm nn}\rangle<1$.

In addition to $\beta\langle k_{\rm nn}\rangle>1$, another necessary condition for case I to occur is the following condition from the properties of the tangent hyperbolic function,

$$\tanh\left[\beta\langle k_{\rm nn}\rangle(x\langle S_{\rm nn}\rangle+y)\right]\big|_{\langle S_{\rm nn}\rangle=S'_{\rm nn}}\le S'_{\rm nn}, \qquad (6)$$

where $S'_{\rm nn}$ is the solution of the following conditions

$$\left.\frac{\partial \tanh\left[\beta\langle k_{\rm nn}\rangle(x\langle S_{\rm nn}\rangle + y)\right]}{\partial\langle S_{\rm nn}\rangle}\right|_{\langle S_{\rm nn}\rangle = S'_{\rm nn}} = 1, \tag{7}$$

$$S'_{\rm nn} < 0. \tag{8}$$

Therefore, we have what follows.

**Result 1.** *The conditions for the two stable steady states illustrated in Fig. 2(a) to exist in the mean-field approximation are given by*

$$\beta\langle k_{nn}\rangle > 1, \tag{9}$$

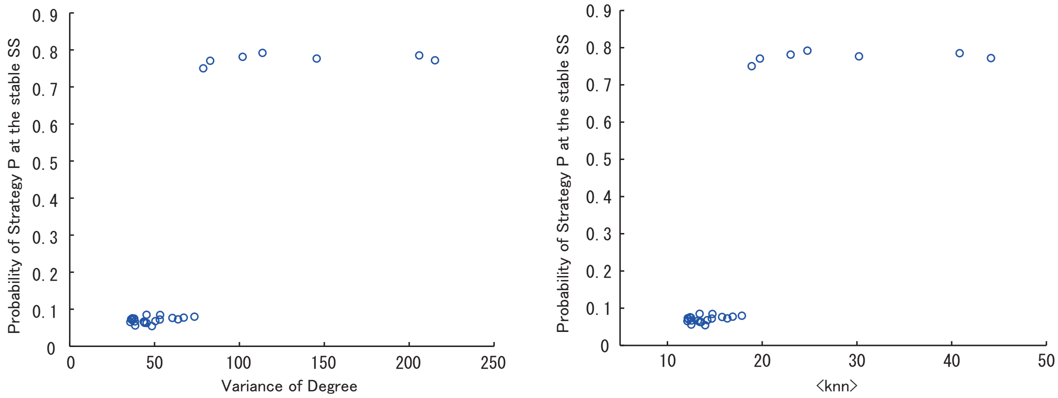
$$\cosh^{-1}\left(\sqrt{x\beta\langle k_{nn}\rangle}\right) + y\beta\langle k_{nn}\rangle \le \sqrt{x\beta\langle k_{nn}\rangle\left(x\beta\langle k_{nn}\rangle - 1\right)}. \tag{10}$$

The signs of $\langle S_{\rm nn}\rangle$ and those of $\langle S(k)\rangle$ for all $k$ are opposite in the two stable steady states.
**Interpretation of Result 1** *If the mean degree of nearest neighbors $\langle k_{nn}\rangle$ is large enough, then the stable steady state where $P_P$ is larger than $P_R$, as illustrated in Fig. 2(a), exists. On the other hand, if the mean degree of nearest neighbors is small, then such a stable steady state does not exist and only the stable steady state where $P_R$ is larger than $P_P$, as illustrated in Fig. 2(b), exists.*
This mean-field approximation technique does not capture the effect of the increase in mean degree perfectly. If the mean degree is large enough, case I occurs in numerical simulation. Not only the mean degree of nearest neighbors, but also the mean degree affects. This is shortcoming of the mean-field approximation. Every approximation has its limitation and valid region. However, we can capture how network heterogeneity (and $\langle k_{\rm nn}\rangle$) affects the outcome by the mean-field approximation. To show how the network heterogeneity affects the outcome is why we use the mean-field approximation, and the mean-field indeed plays the role. If the payoff sensitivity $\beta$ is large, the two stable steady states both exist. On the other hand, if the payoff sensitivity $\beta$ is low, only one stable steady state exists. Inequality (10) is satisfied if $\beta\langle k_{\rm nn}\rangle$ is large. Inequality (6) is the condition that the tangent hyperbolic curve intersects the 45 degree line with a point where $S_{\rm nn} < 0$ given that Eq. (5) holds true, and we have inequality (10) from inequality (6).

In case I, the stable steady state depends on the initial value of $P_{\rm P}$, which is denoted by $P^0_{\rm P}$. We will discuss case I in terms of $\langle S_{\rm nn}\rangle$ instead of the initial probability $P^0_{\rm P}$ since we have the relation $\langle S_{\rm nn}\rangle = P_{\rm R} - P_{\rm P}$. We assume that the initial probability that a player chooses a strategy is independent of his degree. Let $S^{\rm T}_{\rm nn}$ denote the *unstable steady state* where the RHS of Eq. (4) intersects 45 degree line in the region where $\langle S_{\rm nn}\rangle$ is negative. If the initial value of $\langle S_{\rm nn}\rangle$ is larger than the threshold $S^{\rm T}_{\rm nn}$, the stable steady state is $\langle S_{\rm nn}\rangle > 0$, and vice versa. Let $\sigma^2$ denote the network heterogeneity, which is the variance of the degree distribution. In uncorrelated networks, we have the following relation

$$\langle k_{\rm nn}\rangle = \frac{\langle k^2\rangle}{\langle k\rangle} = \langle k\rangle + \frac{\sigma^2}{\langle k\rangle}. \tag{11}$$

The reason the above relation holds in uncorrelated networks is discussed in Appendix B. Hence, we have $\frac{\partial\langle k_{\rm nn}\rangle}{\partial\sigma^2} > 0$. Studying the effects of the network heterogeneity $\sigma^2$ is equivalent to studying the effects of the mean degree of nearest neighbors $\langle k_{\rm nn}\rangle$.




(a) The x-axis indicates the mean degree of nearest neighbors, and the y-axis indicates the probability of strategy P in the stable steady state

(b) The x-axis indicates the variance of the degree distribution, and the y-axis indicates the probability of strategy P in the stable steady state

Figure 3: Network heterogeneity and the probability of strategy P in the stable steady state.

### *3.1.1. Numerical Simulation*

We will confirm the interpretation of Result 1 with the following numerical simulation. In this simulation, all the players choose strategy P in the initial state. If there is only one stable steady state and it is the state where the probability of strategy R is larger than that of strategy P, as illustrated in Fig. 2(b), then the state converges to the stable steady state. If two stable steady states exist, as illustrated in Fig. 2(a), then the system converges to the stable steady state where the probability of strategy P is larger than that of R.
In the simulation, we have scale-free networks where the exponent $\gamma$ takes different values ranging from 2.2 to 4 with differences by 0.01, but the mean degree is kept fixed at 8. The payoff parameters are $(a, b, d) = (10, 5, 12)$ and $\beta = 0.05$. The network size is 2000. After the strategies of players are updated $100,000$ times, the simulation finishes for one scale-free network. The scale-free networks are constructed using a preferential attachment mechanism (Barabasi and Albert (1999); Dorogovtsev et al. (2000)). The result is illustrated in Fig. 3. If the mean degree of nearest neighbors $\langle k_{\mathrm{nn}} \rangle$ (the variance of the degree distribution) is small, there is only one stable steady state, as illustrated in Fig. 2(b). The probability of strategy P in this stable steady state is small. On the other hand, if the mean degree of nearest neighbors $\langle k_{\mathrm{nn}} \rangle$ (the variance of the degree distribution) is large, two stable steady states exist, as illustrated in Fig. 2(a), and the probability of strategy P in the stable steady state is large.

### *3.2. Network Heterogeneity and the Probability of Strategy P in the Stable Steady State*

We study case I, where two stable steady states exist, as illustrated in Fig. 2(a). First, we focus on the case where $\langle S_{\mathrm{nn}} \rangle$ converges to a positive value at the stable steady state, which is denoted by $\langle S_{\mathrm{nn}}^{+} \rangle > 0$. This stable steady state is where $P_{\mathrm{R}} > P_{\mathrm{P}}$ holds. Thus, we have

$$\frac{d\langle S_{\mathrm{nn}}^{+} \rangle}{d\langle k_{\mathrm{nn}} \rangle} = \frac{\partial \tanh\left[\beta \langle k_{\mathrm{nn}} \rangle \left(x \langle S_{\mathrm{nn}}^{+} \rangle + y\right)\right]}{\partial \langle k_{\mathrm{nn}} \rangle} \left(1 - \frac{\partial \tanh\left[\beta \langle k_{\mathrm{nn}} \rangle \left(x \langle S_{\mathrm{nn}} \rangle + y\right)\right]}{\partial \langle S_{\mathrm{nn}}^{+} \rangle}\right)^{-1}. \quad (12)$$

We have

$$\frac{\partial \tanh\left[\beta\langle k_{\mathrm{nn}}\rangle\left(x\langle S_{\mathrm{nn}}^{+}\rangle+y\right)\right]}{\partial\langle S_{\mathrm{nn}}^{+}\rangle}<1, \tag{13}$$

because the slope of the RHS of Eq. (4) at the intersection with the 45 degree line is less than 1. We have

$$\frac{d\langle S_{\mathrm{nn}}^{+}\rangle}{d\langle k_{\mathrm{nn}}\rangle}>0. \tag{14}$$

Let $\langle S(k)^{+}\rangle$ denote the value of $\langle S(k)\rangle$ at stable steady state, which is derived from $\langle S_{\mathrm{nn}}^{+}\rangle>0$. The inequality $\langle S(k)^{+}\rangle>0$ holds for all $k$, because $\langle S_{\mathrm{nn}}^{+}\rangle>0$ holds true. We have

$$\frac{d\langle S(k)^{+}\rangle}{d\langle k_{\mathrm{nn}}\rangle}=\frac{\partial \tanh\left[\beta k\left(x\langle S_{\mathrm{nn}}^{+}\rangle+y\right)\right]}{\partial\langle S_{\mathrm{nn}}^{+}\rangle}\frac{d\langle S_{\mathrm{nn}}^{+}\rangle}{d\langle k_{\mathrm{nn}}\rangle}>0. \tag{15}$$

We also have

$$\frac{d\langle S(k)^{+}\rangle}{dk}>0. \tag{16}$$

These properties also hold in case II, where only one stable steady state exists. At the stable steady state, $\langle S_{\mathrm{nn}}\rangle$ is positive and $\langle S(k)\rangle>0$ holds for all $k$.

Second, we focus on the case where $\langle S_{\mathrm{nn}}\rangle$ converges to a negative value in the stable steady state, which is denoted by $\langle S_{\mathrm{nn}}^{-}\rangle<0$. At the stable steady state $P_{\mathrm{P}}>P_{\mathrm{R}}$ holds true. Let $\langle S(k)^{-}\rangle$ denote the stable steady state of $\langle S(k)\rangle$ that is derived from $\langle S_{\mathrm{nn}}^{-}\rangle<0$. $\langle S(k)^{-}\rangle<0$ holds for all $k$, because $\langle S_{\mathrm{nn}}^{-}\rangle<0$ holds. Similar to the previous case where $\langle S_{\mathrm{nn}}\rangle$ converges to $\langle S_{\mathrm{nn}}^{+}\rangle$, we have

$$\frac{d\langle S_{\mathrm{nn}}^{-}\rangle}{d\langle k_{\mathrm{nn}}\rangle}<0, \tag{17}$$

$$\frac{d\langle S(k)^{-}\rangle}{d\langle k_{\mathrm{nn}}\rangle}<0, \tag{18}$$

$$\frac{d\langle S(k)^{-}\rangle}{dk}<0. \tag{19}$$

In case I, the signs of $\langle S_{\mathrm{nn}}\rangle$ and $\langle S(k)\rangle$ are determined depending on the initial probability of strategy P $P_{\mathrm{P}}^{0}$. If $S_{\mathrm{nn}}$ is positive in the stable steady state, we have the following result from the relations in Eqs. (14), (15), and (16). In case I, if the initial probability for strategy P $P_{\mathrm{P}}^{0}$ is below a certain threshold, then $S_{\mathrm{nn}}$ at the stable steady state is positive. In case II, it is always positive.

**Result 2.** *If the mean degree of nearest neighbors $\langle k_{nn}\rangle$ is greater, $P_P(k)$ at the stable steady state is small in the R-type stable steady state in the mean-field approximation. If the degree $k$ is greater, $P_P(k)$ at the stable steady state is smaller in the R-type stable steady state in the mean-field approximation.*

If $S_{\mathrm{nn}}$ at the stable steady state is negative, we have the following Result from the relations in Eqs. (17), (18), and (19). This is the case where the initial probability of strategy P exceeds a certain threshold in case I.

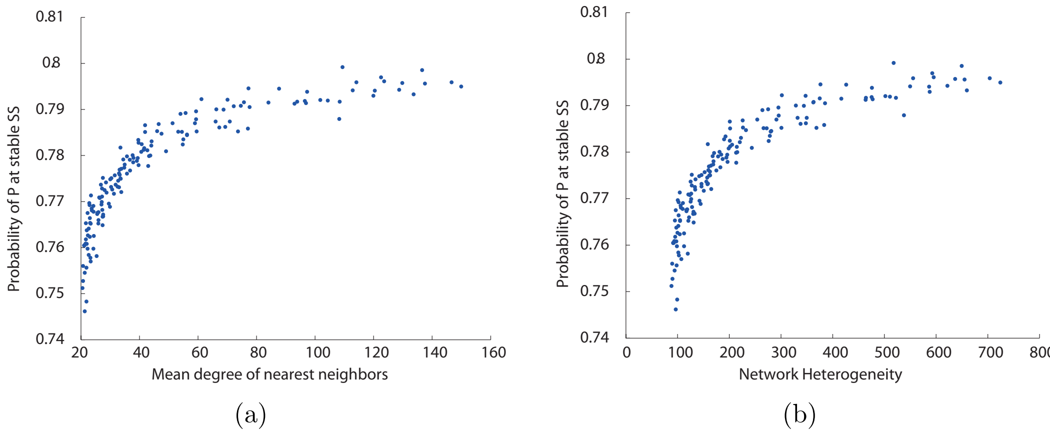



Figure 4: (a) The x-axis indicates the mean degree of nearest neighbors, and the y-axis indicates the probability of strategy P in the stable steady state. The mean degree is the same. (b) The x-axis indicates the variance of the degree distribution and the y-axis indicates the probability of strategy P in the stable steady state. The mean degree is the same.

**Result 3.** *If the mean degree of nearest neighbors $\langle k_{nn} \rangle$ is great, $P_P(k)$ and the average utility of the players at the stable steady state are both great in the P-type stable steady state in the mean-field approximation. If the degree $k$ is great, $P_P(k)$ and the average utility of a player with degree $k$ are both great in the P-type stable steady state in the mean-field approximation.*

The effect of network heterogeneity on $P_{\mathrm{P}}$ in the stable steady state is opposite depending on whether the initial probability for the strategy P $P_{\mathrm{P}}^0$ exceeds the threshold or not. If the initial probability is below the threshold, then $P_{\mathrm{P}}(k)$ in the stable steady state decreases as the network heterogeneity increases. If the initial probability is above the threshold, then $P_{\mathrm{P}}(k)$ increases as the network heterogeneity increases. The network heterogeneity magnifies the absolute value of $S_{\mathrm{nn}}$ at the stable steady state in both cases. If $P_{\mathrm{P}}^0$ exceeds the threshold and, thus, the probability of strategy P is larger than that of R at the stable steady state, then the greater the network heterogeneity $\sigma^2$ (the mean degree of the nearest neighbors $\langle k_{\mathrm{nn}} \rangle$ ) is, the greater the average utility of the players on the network is.

*3.2.1. Numerical Simulation*

We confirm that the probability of strategy P at the stable steady state increases as the network heterogeneity increases by a numerical simulation. The numerical simulation is performed in the case where two stable steady states exist, as illustrated in Fig. 2(a), and the probability of strategy P is larger than that of R in the stable steady state. All the players choose strategy P in the initial state. Scale-free networks with exponent $\gamma$ ranging from 2.2 to 3 with difference 0.005 are constructed. But the mean degree is fixed to 8. The network size is $2,000$. The payoff parameters are $(a, b, d) = (10, 5, 12)$ and $\beta = 0.05$. We assume that the state converges to the stable steady state after the strategies of the players have been updated $100,000$ times. The result of the simulation is illustrated in Fig. 4. The simulation shows that the greater the mean degree of nearest neighbors (the network heterogeneity) is, the greater the probability of strategy P in the stable steady state is.

*3.3. Network Heterogeneity and the Threshold Initial Probability*

It has been shown that if the initial probability of strategy P is beyond the threshold, then the probability of strategy P is larger than that of R in stable steady state, and vice versa, in case I where two stable steady states exist. We see that network heterogeneity lowers the threshold. Network heterogeneity widens the range of initial probabilities of strategy P that result in the stable steady state where the probability of strategy P is larger than that of R. We now consider only case I, where there are two stable steady states. We study the problem in terms of the threshold initial value of $\langle S_{\rm nn}\rangle$, which is denoted by $S^T_{\rm nn}$. An initial probability of strategy P that is above the threshold probability corresponds to an initial value of $\langle S_{\rm nn}\rangle$ that is below the threshold $\langle S_{\rm nn}\rangle^T$, and vice versa. If the initial value of $\langle S_{\rm nn}\rangle$ is below $S^T_{\rm nn}$, then $P_{\rm P}$ is larger than $P_{\rm R}$ in the stable steady state. The threshold initial value $S^T_{\rm nn}$ satisfies

$$\tanh\left[\lambda\langle k_{\rm nn}\rangle\left(xS^T_{\rm nn}+y\right)\right]=S^T_{\rm nn}, \tag{20}$$

$$\frac{\partial\tanh\left[\lambda\langle k_{\rm nn}\rangle\left(xS^T_{\rm nn}+y\right)\right]}{\partial S^T_{\rm nn}}>1, \tag{21}$$

because the tangent hyperbolic curve intersects the 45 degree line at three points and $S^T_{\rm nn}$ is the point where the slope of the tangent hyperbolic curve is larger than 1. We have

$$\frac{dS^T_{\rm nn}}{d\langle k_{\rm nn}\rangle}=\frac{\partial\tanh\left[\lambda\langle k_{\rm nn}\rangle\left(xS^T_{\rm nn}+y\right)\right]}{\partial\langle k_{\rm nn}\rangle}\left(1-\frac{\partial\tanh\left[\lambda\langle k_{\rm nn}\rangle\left(xS^T_{\rm nn}+y\right)\right]}{\partial S^T_{\rm nn}}\right). \tag{22}$$

Therefore, we obtain

$$\frac{dS^T_{\rm nn}}{d\langle k_{\rm nn}\rangle}>0. \tag{23}$$

We have the following Result.

**Result 4.** *The larger the mean degree of the nearest neighbors (the network heterogeneity $\sigma^2$), the wider the range of initial probabilities $P^0_P$ from which the state converges to the stable steady state where the probability of strategy P is larger than that of R in mean-field approximation.*

*3.3.1. Numerical Simulation*

We confirm Result 4 by following numerical simulation. We have a scale-free network with $\gamma$ ranging from 2.2 to 3 with difference 0.005. But the mean degree is kept fixed at 8. The network size is 2000. The payoff parameters are $(a, b, d) = (10, 5, 12)$ and $\beta = 0.05$. We assume that the state converges to the stable steady state after the strategies of the players are updated $100,000$ times. In Fig. 5, the x-axis indicates the mean degree of the nearest neighbors, and the y-axis is the threshold probability of strategy P. If the initial probability of strategy P is less than the threshold probability, the state converges to the stable steady state where the probability of strategy P is less than that of R. If the initial probability of strategy P is larger than the threshold probability, the state converges to the stable steady state where the probability of strategy P is larger than that of R. In the simulation, the larger the mean degree of the nearest neighbors $\langle k_{\rm nn}\rangle$ is, the smaller the threshold initial probability of strategy P is. The greater the network heterogeneity is, the wider the range of initial probabilities of strategy P from which the state converges to the stable steady state where the probability of strategy P is larger than that of strategy R is.

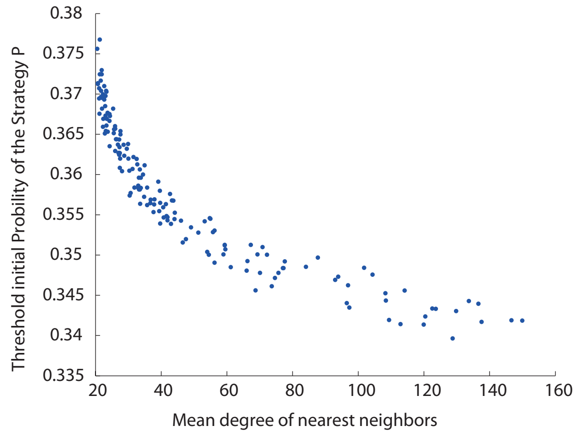



Figure 5: Threshold: The x-axis indicates the mean degree of nearest neighbors $\langle k_{\mathrm{nn}} \rangle$ and the y-axis indicates the threshold initial probability of strategy P.

## 4. Concluding Remarks

We studied a coordination game with logit choice in heterogeneous networks. There are two strategies, P and R, where strategy R is risk dominant and the Pareto optimal state is when all players chose strategy P. Real networks are heterogeneous and network heterogeneity affects the outcomes of models. Hence, games on networks have been studied by a number of papers. We studied how the probability that a player chooses strategy P in a stable steady state is affected by network heterogeneity. The case where all players chose strategy P is studied first. We showed there are two cases. In case I, two stable steady states exist. The probability of strategy P is larger than that of R in one stable steady state, whereas the probability of strategy R is larger than that of P in the other stable steady state. In case II, only one stable steady state exists, in which the probability of strategy R is larger than that of P. We gave the mean-field conditions where case I occurs and the stable steady state where the probability of strategy P is larger than that of R exists. Next, we studied the case initial probability for strategies vary. We showed that if the initial probability of strategy P is beyond a certain threshold, the probability of strategy P in the stable steady state is larger than that of R, and vice versa, in case I. We also showed that if the probability of strategy P is less than that of R at the stable steady state, then what follows holds. The greater the mean degree of nearest neighbors $\langle k_{\mathrm{nn}} \rangle$ is, the smaller $P_{\mathrm{P}}(k)$ is in the stable steady state. The greater the degree $k$ is, the smaller $P_{\mathrm{P}}(k)$ is in the stable steady state. On the other hand, if the probability of strategy P is larger than that of R in the stable steady state, then what follows holds. The greater the mean degree of nearest neighbors $\langle k_{\mathrm{nn}} \rangle$ is, the greater $P_{\mathrm{P}}(k)$ and the average utility of the players are in the stable steady state. The greater the degree $k$ is, the greater $P_{\mathrm{P}}(k)$ and the average utility of the players with degree $k$ are. We showed that the larger the network heterogeneity is, the wider the range of $P_{\mathrm{P}}^{0}$ from which the state converges to the stable steady state where the probability of strategy P is larger than that of R is. In other words, the larger the network heterogeneity is, the lower the threshold initial probability of strategy P is. We investigated how network heterogeneity has influence on the outcome and that which of the stable steady states is realized is determined by magnitude of the network heterogeneity in the coordination game on a network. If the network heterogeneity is large, the stable steady state where the probability of strategy P is larger than that of R emerges.

## Appendix A. The Difference between the mean degree and the mean degree of nearest neighbors

In this appendix, We will demonstrate that $\langle k \rangle$ and $\langle k_{\rm nn} \rangle$ are indeed different with the example illustrated in Fig. A.6.

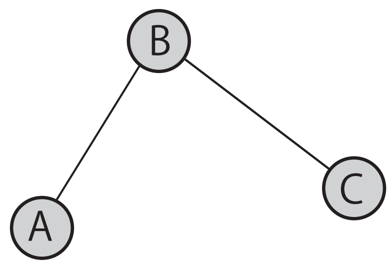



Figure A.6: In this network $\langle k \rangle \neq \langle k_{\rm nn} \rangle$.

In the example, the degrees of the vertices are $k_A = 1$, $k_B = 2$, and $k_C = 1$, respectively. The mean degree is

$$\langle k \rangle = \frac{1}{3}(1+2+1) = \frac{4}{3}. \tag{A.1}$$

On the other hand, the mean degree of the nearest neighbors is given by

$$\begin{aligned} \langle k_{\rm nn} \rangle =& \frac{1}{3}\left( k_B + \frac{k_A + k_C}{2} + k_B \right) \\ =& \frac{1}{3}\left( 2 + \frac{1+1}{2} + 2 \right) = \frac{5}{3}. \end{aligned} \tag{A.2}$$

In fact, $\langle k \rangle \neq \langle k_{\rm nn} \rangle$ even in the simple network.

## Appendix B. The mean degree of nearest neighbors

In uncorrelated networks, the probability $P_{\rm nn}(k)$ that the end of a link is attached to a vertex with degree $k$ is given by

$$\begin{aligned} P_{\rm nn}(k) =& \frac{\#\ \text{Ends attached to vertices with degree } k}{\#\ \text{All ends of links in the network}} \\ =& \frac{NkP(k)}{N\sum_k kP(k)}, \end{aligned} \tag{B.1}$$

where $P(k)$ is the degree distribution and $N$ is the network size. Therefore, we have

$$\begin{aligned}\langle k_{\rm nn}\rangle &\equiv \sum_k k P_{\rm nn}(k) \\ &= \frac{\sum_k k^2 P(k)}{\sum_{k'} k' P(k')} = \frac{\langle k^2\rangle}{\langle k\rangle}.\end{aligned} \tag{B.2}$$

### References


Albert, R., Barabási, A.-L., Jan 2002. Statistical mechanics of complex networks. Review of Modern Physics 74 (1), 47–97.

Alós-Ferrer, C., Weidenholzer, S., 2008. Contagion and efficiency. Journal of Economic Theory 143 (1), 251–274.

Arthur, W., Durlauf, S., Lane, D., 1997. The economy as an evolving complex system II. Addison-Wesley Reading, MA.

Barabasi, A., Albert, R., 1999. Emergence of scaling in random networks. Science 286 (5439), 509.

Baxter, R., 1982. Exactly solved models in statistical mechanics. Academic press London.

Blume, L., et al., 1993. The statistical mechanics of strategic interaction. Games and economic behavior 5 (3), 387–424.

Boccaletti, S., Latora, V., Moreno, Y., Chavez, M., Hwang, D. U., 2006. Complex networks: Structure and dynamics. Physics Reports 424 (4-5), 175–308.

Brock, W., Durlauf, S., 2001. Discrete choice with social interactions. The Review of Economic Studies 68 (2), 235–260.

Dorogovtesev, Mendes, 2003. Evolution of Networks. Oxford University Press.

Dorogovtsev, S., Mendes, J., Samukhin, A., 2000. Structure of growing networks with preferential linking. Physical Review Letters 85 (21), 4633–4636.

Ellison, G., 1993. Learning, local interaction, and coordination. ECONOMETRICA-EVANSTON ILL- 61, 1047–1047.

Galeotti, A., Goyal, S., Jackson, M., Vega-Redondo, F., Yariv, L., 2010. Network games. Review of Economic Studies 77 (1), 218–244.

Goyal, S., 2007. Connections: An Introduction to the Economics of Networks. Princeton University Press.

Greiner, W., Neise, L., Stocker, H., 1995. Thermodynamics and statistical mechanics. Springer.

Ising, E., 1925. Beitrag zur theorie des ferromagnetismus. Zeitschrift fur Physik A Hadrons and Nuclei 31 (1), 253–258.

Jackson, M., 2008. Social and economic networks. Princeton University Press.

Konno, T., 2009. Network structure of japanese firms. scale-free, hierarchy, and degree correlation: Analysis from 800,000 firms. Economics: The Open-Access, Open-Assessment E-Journal 3 (2009-31).

Konno, T., 2011a. An alternative explanation for the logit form probabilistic choice model from the equal likelihood hypothesis. Economics Letters.

Konno, T., 2011b. A condition for cooperation in a game on complex networks. Journal of Theoretical Biology 269 (1), 224 – 233.

Konno, T., 2011c. "the exact solution of spatial logit response games". Unpublished.

Kubo, R., 1965. Statistical mechanics: an advanced course with problems and solutions. North-Holland Pubishing Cooporation.

López-Pintado, D., 2006. Contagion and coordination in random networks. International Journal of Game Theory 34 (3), 371–381.

Luce, R., 1959. Individual choice behavior. Wiley New York.

Morris, S., 2000. Contagion. The Review of Economic Studies 67 (1), 57.

Newman, M. E. J., 2003. The structure and function of complex networks. SIAM Review 45 (2), 167–256.

Nowak, M., 2006. Five rules for the evolution of cooperation. Science 314 (5805), 1560–1563.

Ohtsuki, H., Hauert, C., Lieberman, E., Nowak, M. A., 2006. A simple rule for the evolution of cooperation on graphs and social networks. Nature 441 (7092), 502–505.

Pastor-Satorras, R., Vespignani, A., 2001. Epidemic spreading in scale-free networks. Physical Review Letters 86 (14), 3200.

Santos, F., Pacheco, J., 2005. Scale-free networks provide a unifying framework for the emergence of cooperation. Physical Review Letters 95 (9), 98104.

S.Bornholdt, 2003. Handbook of Graphs and Networks: From the Genome to the Internet. Wiley-VCH.

Szabo, G., Fath, G., 2007. Evolutionary games on graphs. Physics Reports 446 (4-6), 97–216.

Train, K., 2003. Discrete choice methods with simulation. Cambridge Univ Press.

Vega-Redondo, F., 2007. Complex Social Networks. Cambridge Univerity Press.

Young, H., 2001. Individual strategy and social structure: An evolutionary theory of institutions. Princeton Univ Press.