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Market size effects on long-run demand of a network good

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Abstract

We consider optimal dynamic pricing under a network externality. We construct the demand dynamic of the network good from the aggregate best response dynamic of agents who have different adoption costs. When the distribution of adoption costs is convex, expansion of potential market inevitably enlarges long-run demand.

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1 Introduction

It is natural to expect expansion of the untapped portion of a market to encourage a supplier to reach new customers. On this note, we consider the effects of such market expansion on long-run demand under a network externality. Dhebar and Oren (1985, henceforth cited as DO85) formulate the optimal dynamic pricing policy for a network good, assuming concavity of the optimal control problem. As we will see in detail, expansion of the untapped portion of the market does not affect long-run demand in the concave case. We adopt their model, while assuming non-concavity in the dynamic governing consumers' demand. In this setting, we show that expansion of the potential market inevitably enlarges long-run demand.

To give an economic interpretation of this nonconcavity, we explicitly derive the demand dynamic as the aggregate best response dynamic of potential customers who have different costs of adopting the good. It follows that nonconcavity of the dynamic is a consequence of non-concavity of the distribution of adoption costs. In particular, convexity of the cumulative distribution function means that the density function is increasing: potential customers are found at an increasing rate as the supplier targets customers with larger adoption costs.

For example, consider the diffusion of a new innovation or a social network on the Internet. People who are less familiar with technology are more reluctant to adopt the new product and thus incur greater costs of doing so. The convex distribution of adoption costs means that there are fewer geeks than laymen, i.e., that most people are unfamiliar with technology. It is commonly observed that diffusion of new technology starts with a few early adopters and then grows exponentially.[1]

Alternatively, we can think of a social club, pub, or nightclub as a platform where people gather and enjoy a positive externality from socializing.[2] A shopping center can also be considered as such a platform because the number of customers may affect prices and the variety of available goods and thus an increase in customers may cause a negative or positive externality.[3] For such a non-virtual platform, transportation costs may be the main source of heterogeneity of adoption costs. If a customer's transportation cost is proportional to the geometric distance between his home and the location of the platform and potential customers are located uniformly around the platform, the distribution of the transportation costs is convex since the area of a circle is proportional to the square of the radius.

Nonconcave dynamic optimization typically results in a bang-bang control. Radner et al. (2014, henceforth cited as RRS) study the dynamic pricing of a network good both in concave and nonconcave (convex) cases, while assuming heterogeneity in customers' sensitivity to the network externality and linearity of individual gross utility in the platform size.[4] (Note that they fix the market size and thus do not study size effects.) They suggest nonexistence of the deterministic optimal control in the nonconcave case and a need to allow randomization in the model. Hence, while we *a priori* restrict the admissible fee levels to the lowest and

[1]See Geroski (2000) for a survey on diffusion of innovation over time.

[2]Becker (1991) and Karni and Levin (1994) explain a restaurant's pricing from a social interactions perspective.

[3]See Beggs (1994) and Smith and Hay (2005) for examples of the theory of shopping centers as platforms with network externalities.

[4]As a result of this linearity, a bang-bang control is the solution in their model in both concave and convex cases, while the optimal control in the concave version of ours, i.e., DO85, is continuous.

highest possible levels, we allow randomization (a mixed strategy) between these. The optimal pricing policy in our model indeed takes the form of a bang-bang control, setting the fee at either bound without randomization until demand reaches a limit state. At the limit state, the fee level is randomized to keep demand unchanged on average.

We find that long-run demand depends on the size of the potential market. Assuming that the aggregate gross utility of the participants grows at a constant elasticity with the number of the participants, there are two distinct cases of the correlation between potential market size and long-run demand. If aggregate gross utility is concave in platform size and consequently participation imposes a negative externality on other individuals, the platform absorbs all agents in the long run when the potential market is small. In the opposite case, where there is a positive externality, the entire potential market is absorbed when the potential market size is large.

There is a notable difference in our model from RRS and their variants.[5] In our model, we add an additively separable idiosyncratic term to the utility function, which better represents heterogeneity in transportation costs or outside options. In the forementioned literature, idiosyncrasy enters the utility function in a multiplicative way, which is interpreted as heterogeneity in sensitivity to network effects. Additionaly separable idiosyncrasy makes our model consistent with econometric models on discrete choices[6] and allow us to justify the dynamic as the aggregate best response dynamic.

In the next section, we set up the model. In Section 3, we formulate the dynamic optimization problem and characterize its solution. We study the comparative statics of long-run demand and clarify the size effects mentioned above in Section 4.

2 The model

There is a mass of agents who decide on whether or not to participate in a platform. Let $\bar{n}$ be the total mass of agents, i.e., the **potential market size**. At each moment in time $t \in \mathbb{R}_+$, an agent gains gross utility $v(n_t)$ from participation in the platform, but incurs adoption cost ϕ and is charged a participation fee P_t by the platform owner. An agent receives zero utility if he does not participate.

Gross utility of participation $v : [0, \bar{n}] \to \mathbb{R}$ is a function of the mass of current participants n_t. We assume that v is C^2 and strictly concave. Adoption cost $\phi \in [0, \bar{\phi}]$ varies among agents, and thus we call it an agent's **type**. Let $F(\phi)$ be the mass of agents whose types are not higher than ϕ. We assume that $F : [0, \bar{\phi}] \to \mathbb{R}_+$ is an increasing and C^2 function with $F(0) = 0$ and $F(\bar{\phi}) = \bar{n}$. We call the type equal to $v(n_t) - P_t$ the **target type** and $F(v(n_t) - P_t)$ the **target demand** at time t. Setting the target type at $\hat{\phi}_t$ means that the fee is set at $P_t = v(n_t) - \hat{\phi}_t$. We restrict the set of feasible target types to $[0, \bar{\phi}]$ and consequently the set of feasible fee levels to $[v(n_t) - \bar{\phi}, v(n_t)]$.[7] The platform owner's instantaneous profit

[5]See Radner (2003) and Radner and Richardson (2003). DO85 proposes a general model without specifying the functional form of heterogeneity in the utility function and argues multiplicative heterogeneity in the canonical example.

[6]See Anderson et al. (1992).

[7]Our assertion that the fee level is bounded above may sound inconsistent with the owner's profit maximization, because the owner may want to exploit the participants as much as possible. We justify our choice of the upper bound $v(n_t)$ by considering that, if $P_t > v(n_t)$, it is obvious for every agent that none of them

is $P_t n_t = \{v(n_t) - \hat{\phi}_t\} n_t$.

In the example of a shopping center, the fee P_t includes not only membership fees or parking fees but also (per-customer) rents paid indirectly from retailers. Incentives to attract shoppers such as coupons and events are considered to be negative components of P_t. The adoption cost ϕ is the transportation cost for an agent to visit the shopping center. Its upper bound $\bar{\phi}$ defines the geographic boundary of the area from which potential customers are drawn, while $\bar{n}$ is its population.

We construct the platform demand dynamic from aggregation of the best response dynamic (BRD), as defined in Gilboa and Matsui (1991); Hofbauer (1995). In the BRD, each agent only occasionally revises his choice of whether or not to participate at a constant frequency, and, upon receiving a revision opportunity at time t, chooses the myopically optimal action based on the current state (P_t, n_t).[8] According to Ely and Sandholm (2005), the aggregate BRD reduces to[9]

$$\dot{n}_t = F(v(n_t) - P_t) - n_t. \tag{1}$$

We make the following assumptions on the functions F and V.

Assumption 1. $F'' > 0$.

Assumption 2. Define the aggregate gross utility of participants $V : \mathbb{R}_+ \to \mathbb{R}$ by $V(n) = v(n)n$. $V'(n)/n$ is decreasing and satisfies

$$\lim_{n\to 0} \frac{V'(n)}{n} > \frac{(2+\rho)\bar{\phi}}{\bar{n}} > \lim_{n\to\infty} \frac{V'(n)}{n}.$$

Assumption 1 means more high types than low types; because of this, the platform demand dynamic exhibits non-concavity and, rather, strict convexity. Assumption 2 is to guarantee the existence and uniqueness of long-run demand under optimal dynamic pricing.

3 Dynamic optimization

The platform owner chooses the fee schedule $\{P_t\}_{t\in\mathbb{R}_+}$, or equivalently, the target type $\{\hat{\phi}_t\}_{t\in\mathbb{R}_+}$, to maximize the discounted sum of profits under the platform demand dynamic

receives positive net payoff from participation; thus, every agent immediately exits from the platform and n_t jumps to 0. What is crucial for our results is this upper bound assumption; without it, there may not be any optimal price.

[8] Both DO85 and RRS extend the models to allow an agent's choice to be based on a subjective assessment of the platform size, which takes a form of a weighted average between the current platform size and some time-invariant assessment of the would-be platform size: in our term, the latter is $F(\phi)$ for a type-ϕ agent in DO85, and a constant ω commonly held for all agents in RRS. Such extension may be possible for our model but is not discussed in this paper because our focus is on implications of non-concavity.

[9] We should notice aggregability of the dynamic: the evolution of n_t relies only on the aggregate state n_t (and P_t) but not on the composition of participants over different types. For aggregability, DO85 (§2.5) assumes that sequence of participation is always perfectly sorted according to the degree of the net benefit from participation; then, anyone with a lower type than $F^{-1}(n_t)$ participates in the platform and anyone with a greater type does not. But such sorting cannot be expected if individual agents' revision processes are inertial and independent to each other. Zusai (2015) argues that aggregability is not generally guaranteed. However, Ely and Sandholm (2005) verify aggregability of the BRD with additive heterogeneity.

(1). If we had assumed concavity of both V and F, we could find the optimal dynamic pricing scheme as a solution of an optimal control problem that is essentially the same as the problem in DO85:

$$\max_{\{(\hat{\phi}_t, n_t)\}_{t\in\mathbb{R}_+}} \int_0^\infty e^{-\rho t}\{v(n_t) - \hat{\phi}_t\}n_t \quad \text{s.t. (1).}$$

Here $\rho > 0$ is the discount rate. Under the optimal pricing scheme, the target type changes continuously with current demand; see DO85. Demand converges globally and asymptotically to the long-run demand $n^\$ = \min\{\bar{n}, \hat{n}\}$, where $\hat{n}$ is the unique solution to

$$V'(\hat{n}) - F^{-1}(\hat{n}) - (1+\rho)\frac{\hat{n}}{F' \circ F^{-1}(\hat{n})} = 0. \tag{2}$$

Change in the agents' population affects the long-run demand $n^\$$ only if it changes the distribution of the types $[0, F^{-1}(n^\$)]$. That is, a population increase in an area from which the platform has not yet attracted customers does not stimulate the platform to attract them and grow further.[10]

Convexity, as assumed in Assumption 1, implies that the optimal policy is a bang-bang control in which the target type takes either the lowest type 0 or the highest type $\bar{\phi}$. Furthermore, RRS suggests that the optimal control may switch between the two extremes randomly.[11]

Thus, we *a priori* restrict the feasible set of the target types to a binary set $\{0, \bar{\phi}\}$, while allowing the owner to randomize the target type, i.e., to play a mixed strategy. The owner's maximization problem is

$$\max_{\{(\hat{\phi}_t, n_t)\}_{t\in\mathbb{R}_+}} \mathbb{E}\int_0^\infty e^{-\rho t}\{v(n_t) - \hat{\phi}_t\}n_t dt \ \text{ s.t.} \begin{cases} \hat{\phi}_t = \bar{\phi}, & \dot{n}_t = \bar{n} - n_t \quad \text{with prob. } q_t, \\ \hat{\phi}_t = 0, & \dot{n}_t = -n_t \quad \text{with prob. } 1 - q_t. \end{cases} \tag{3}$$

Here, q_t is the probability with which the target type is set to $\bar{\phi}$ at time t. With co-state variable $\mu \in \mathbb{R}$, the corresponding Hamiltonian is

$$H(n, q; \mu) = \{v(n) - q\bar{\phi}\}n + \mu(q\bar{n} - n).$$

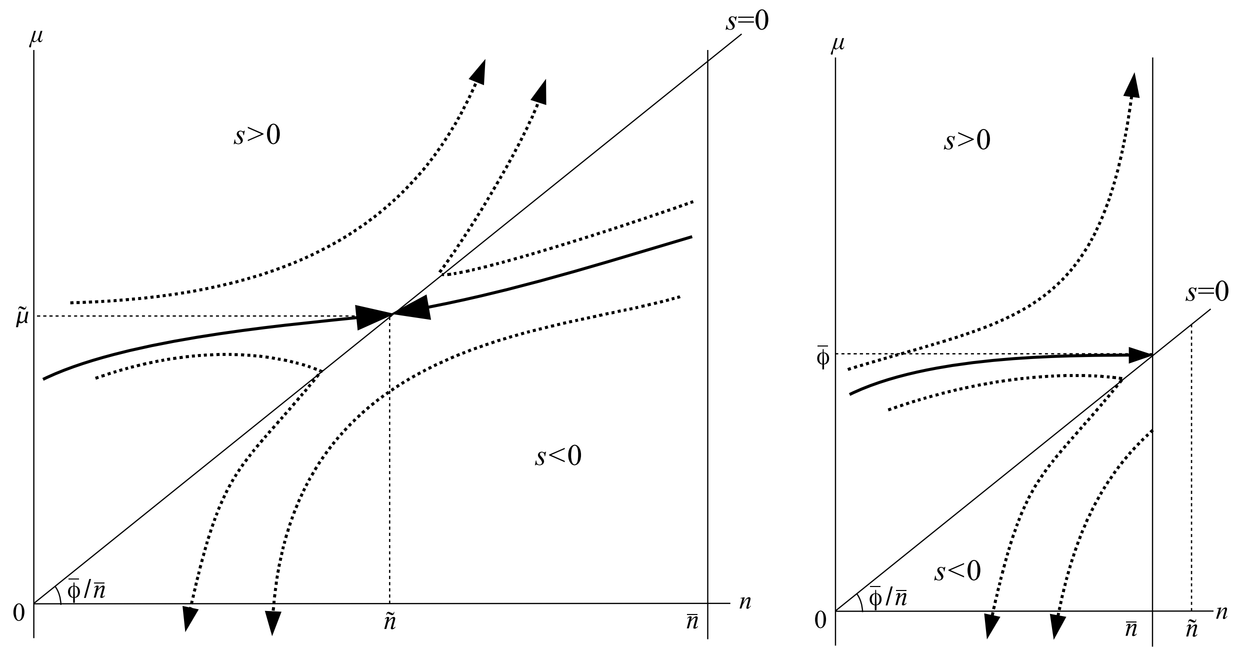
The solution of (3) is thus characterized by the following first-order conditions:

$$q \begin{cases} = 1 & \text{if } \bar{n}\mu > \bar{\phi}n, \\ \in [0,1] & \text{if } \bar{n}\mu = \bar{\phi}n, \\ 0 & \text{if } \bar{n}\mu < \bar{\phi}n, \end{cases} \tag{4a}$$

$$\dot{\mu} = (1+\rho)\mu + q\bar{\phi} - V'(n). \tag{4b}$$

[10] Let $\hat{n}^0$ be the solution of (2) before change in F. Suppose that, with arbitrary small $\varepsilon > 0$, the density of type-ϕ agents changes only if $\phi \in [F^{-1}(\hat{n}^0) + \varepsilon, \bar{\phi}]$. (It may or may not keep $\bar{\phi}$ or $\bar{n}$.) This does not change $F^{-1}(\hat{n}^0)$ or $F' \circ F^{-1}(\hat{n}^0)$ at all and thus $\hat{n}^0$ is still the solution of (2) after this change in F. As long as $\bar{n}$ does not change too much, it does not change which of $\bar{n}$ and $\hat{n}^0$ is smaller; thus, the long-run demand $n^\$$ remains the same.

[11] Radner and Richardson (2003) verify that such a random extreme pricing scheme like our solution can be justified as an approximation of a pricing scheme where price oscillates arbitrarily quickly. According to Roth and Sandholm (2013), the BRD in a continuum of agents is also an approximation of the finite population dynamic. If the number of agents was finite, the platform size would change discretely and thus it would oscillate around the limit point in the continuous dynamic. Then, price would also oscillate. We should also note that price in our solution is not randomized until the platform size reaches the limit n_∞.



(a) $\tilde{n} < \bar{n} \;\Rightarrow\; n_\infty = \tilde{n}$ (b) $\tilde{n} > \bar{n} \;\Rightarrow\; n_\infty = \bar{n}$

Figure 1: Phase diagram

The optimal policy of q switches when (n, μ) crosses the line $\mu/n = \bar{\phi}/\bar{n}$ in the phase diagram. The proof of the next lemma shows that it switches only from 1 to 0 if n is below the threshold $\tilde{n}$ and only from 0 to 1 if it is above $\tilde{n}$. This implies the existence of a saddle path to $(\tilde{n}, \tilde{\mu})$ with $\tilde{\mu} = \bar{\phi}\tilde{n}/\bar{n}$.

Lemma 1. *Suppose that assumptions 1 and 2 hold. There exists a unique* $\tilde{n} \in (0, +\infty)$ *s.t.*

$$(2+\rho)\bar{\phi}\tilde{n} = \bar{n}V'(\tilde{n}). \tag{5}$$

There is a saddle path to $(\tilde{n}, \tilde{\mu})$ *if* $\tilde{n} < \bar{n}$*; otherwise, there is a path converging to* $(\bar{n}, \bar{\phi})$*.*

Proof. Define the function $s : \mathbb{R}^2 \to \mathbb{R}$ by $s(n, \mu) := \bar{n}\mu - \bar{\phi}n$. According to (4a), the optimal q is $q = 1$ if $s > 0$ and $q = 0$ if $s < 0$. Note that the derivative of s is $Ds(n, \mu) = (-\bar{\phi}, \bar{n})$. Now we investigate how the value of s changes when $s(n, \mu) = 0$.

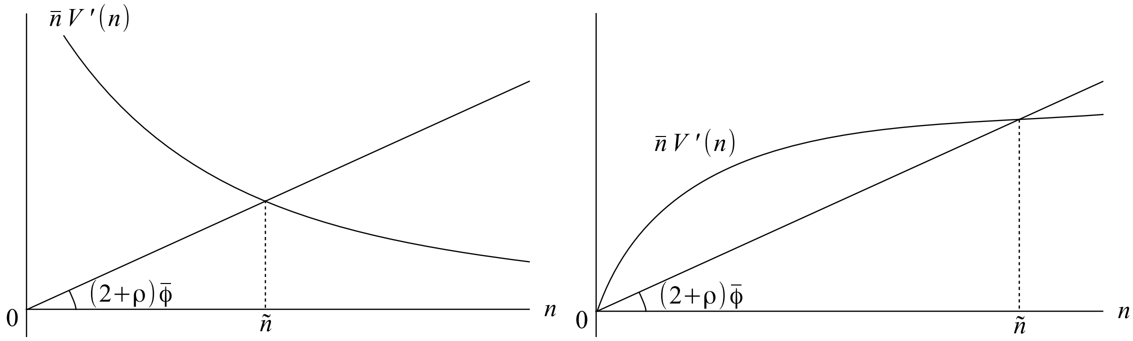
Let $\mathbf{v}_q(n, \mu)$ be the transition vector $(\dot{n}, \dot{\mu})$ at (n, μ) with $q \in [0, 1]$:

$$\mathbf{v}_q(n, \mu) := \begin{pmatrix} q\bar{n} - n \\ (1+\rho)\mu + q\bar{\phi} - V'(n) \end{pmatrix} = \mathbf{v}_0(n, \mu) + q \begin{pmatrix} \bar{n} \\ \bar{\phi} \end{pmatrix}.$$

Notice that $Ds(n, \mu)\mathbf{v}_q(n, \mu) = Ds(n, \mu)\mathbf{v}_0(n, \mu)$ for any q. Thus we have

$$\dot{s} = Ds(n, \mu)\mathbf{v}_0(n, \mu) \begin{array}{c} > \\ = \\ < \end{array} (2+\rho)\bar{\phi}n - \bar{n}V'(n) \qquad \begin{array}{l} \text{if } s > 0, \\ \text{if } s = 0, \\ \text{if } s < 0. \end{array}$$

(5) implies $\dot{s} = 0$ if $s = 0$ and $n = \tilde{n}$. Assumption 2 guarantees the unique existence of such $\tilde{n}$ in $(0, +\infty)$; in addition, it implies $\dot{s} < 0$ if $s \leq 0$ and $n < \tilde{n}$ and $\dot{s} > 0$ if $s \geq 0$ and $n > \tilde{n}$.



(a) $V'' < 0; b < 1$ in Example 1 (b) $V'' > 0; b \in (1, 2)$ in Example 1

Figure 2: Determination of threshold size $\tilde{n}$ from (5)

If $s(n, \mu) = 0$ and $n < \tilde{n}$, then s changes only from positive to negative when (n, μ) crosses $s = 0$. That is, the optimal q switches from 1 to 0. After this switch, s keeps decreasing and thus (n, μ) never crosses $s = 0$. By the same token, if (n, μ) crosses $s = 0$ at $n > \tilde{n}$, q switches only from 0 to 1 and s changes from negative to positive and keeps increasing.

If $\tilde{n} < \bar{n}$, there is a saddle path converging to $(\tilde{n}, \tilde{\mu})$ in finite time, as shown in Figure 1. Otherwise, there is a path converging to $(\bar{n}, \bar{\phi})$. Notice that, with $q = 1$, n_t converges to $\bar{n}$ only asymptotically and thus the path does not reach $(\bar{n}, \bar{\phi})$ in finite time. □

To select a solution path, we impose the transversality condition

$$\lim_{t \to \infty} e^{-\rho t} n_t \mu_t = 0. \tag{6}$$

If $\tilde{n} < \bar{n}$, the saddle path is the only solution of the FOCs (4) that also satisfies the TVC (6). So is the path to $(\bar{n}, \bar{\phi})$ when $\tilde{n} \geq \bar{n}$. On any other solution paths of (4), μ diverges to ∞ or $-\infty$ faster than ρ after sufficiently long time has passed; these paths violate (6).

$\tilde{n}$ is the threshold between $q = 1$ (pure strategy $\hat{\phi} = \bar{\phi}$) and $q = 0$ (pure strategy $\hat{\phi} = 0$); call it the **threshold demand**, distinguished from the **long-run demand** $n_\infty := \lim_{t \to \infty} n_t$.

Theorem 1. *Suppose assumptions 1 and 2 hold. In the long run, demand converges to* $\tilde{n}$ *defined in* (5) *if* $\tilde{n} < \bar{n}$ *and to* $\bar{n}$ *if* $\tilde{n} \geq \bar{n}$ *. That is, long-run demand is* $n_\infty = \min\{\tilde{n}, \bar{n}\}$.

4 Market size effects

From (5), it is obvious that the threshold demand $\tilde{n}$ depends on the potential market size $\bar{n}$. As is suggested by Figure 2, $\tilde{n}$ increases with $\bar{n}$ (and decreases when $\bar{\phi}$ or ρ increases). In contrast to the concave case, expansion of the potential market triggers larger demand in the long run no matter where it occurs.

Corollary 1. *Suppose assumptions 1 and 2 hold. As potential market size* $\bar{n}$ *increases, so does long-run demand* n_∞.

Proof. Notice that, if $\bar{n} \leq \tilde{n}$, $n_\infty = \bar{n}$ and thus the claim is immediate. We show $\tilde{n}$ increases with $\bar{n}$. Assumption 2 and (5) imply

$$0 > \left(\frac{V'(n)}{n}\right)'\Big|_{n=\tilde{n}} = \frac{V''(\tilde{n})\tilde{n} - V'(\tilde{n})}{\tilde{n}^2} = \frac{\bar{n}V''(\tilde{n}) - (2+\rho)\bar{\phi}}{\bar{n}\tilde{n}},$$

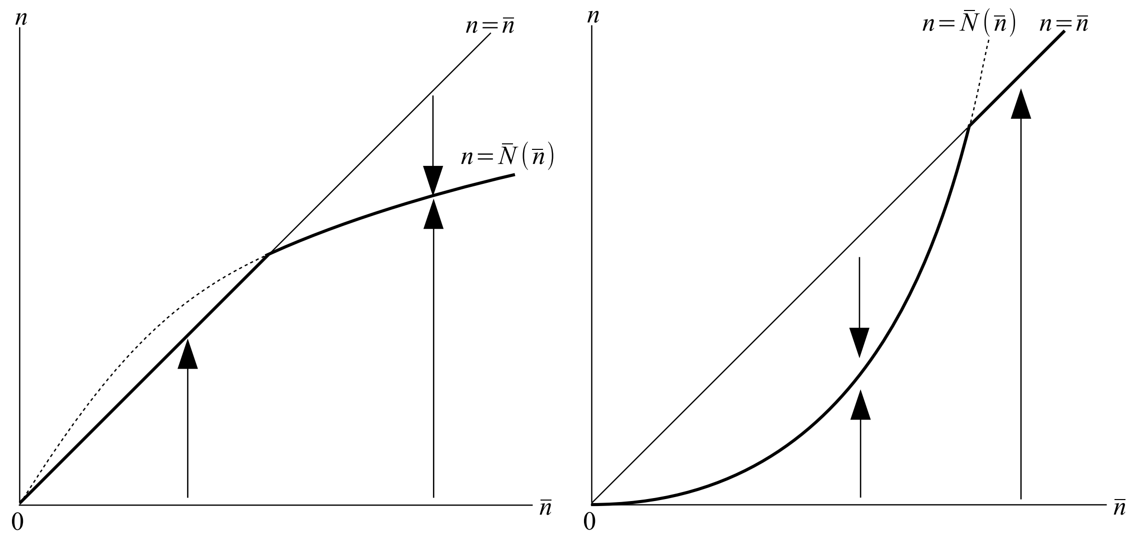


Figure 3: The relationship between long-run demand n_∞ (the bold line) and potential market size $\bar{n}$.

and thus $\bar{n}V''(\tilde{n}) - (2+\rho)\bar{\phi} < 0$. From this, implicit differentiation of (5) yields

$$\frac{d\tilde{n}}{d\bar{n}} = \frac{V'(\tilde{n})}{(2+\rho)\bar{\phi} - \bar{n}V''(\tilde{n})} > 0.$$

□

In the following example, we specify the aggregate gross utility function V and calculate long-run demand numerically from (5).

Example 1 (Constant elasticity). Let V be $V(n) = An^b$ with $A, b > 0$. This functional form implies that the aggregate gross utility of the participants grows at a constant elasticity b with the platform size n. Assumption 2 reduces to $b \leq 2$. (5) is solved by

$$\tilde{n} = \tilde{N}(\bar{n}) := \left(\frac{Ab\bar{n}}{(2+\rho)\bar{\phi}}\right)^{1/(2-b)}.$$

$\tilde{N}$ is convex in $\bar{n}$ if $b \in (1,2)$ and concave if $b < 1$. The relationship between potential market size $\bar{n}$ and long-run demand n_∞ is significantly different in these two cases, as seen in Figure 3.

Corollary 2. *Assume constant elasticity of V, i.e., $V(n) = An^b$ with $A > 0$ and $b \in (0,2)$.*

Consider the case $b < 1$. Regardless of initial demand n_0, demand n_t converges to $\bar{n}$, and thus the platform eventually absorbs all agents in the economy, when potential market size $\bar{n}$ is sufficiently small.

If $b \in (1,2)$, this happens when potential market size $\bar{n}$ is sufficiently large.

Notice that individual participant's gross utility from participation is $v(n) = V(n)/n = An^{b-1}$. Thus, additional participation causes a negative externality if $b < 1$; otherwise, it causes a positive externality.

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