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The value of information with neo-additive beliefs

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Abstract

When individual beliefs are not Bayesian, economic agents may refuse further information about the uncertainty they are facing. Choquet decision makers in particular may be information averse. This note shows that, if the capacity is neo-additive, then the information value is necessarily positive.

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1 Introduction

In economics, it is common to assume that economic agents facing uncertainty are prepared to spend scarce resources that will reduce the uncertainty and reveal new opportunities. This automatically implies that information is positively valued. Moreover, when decisions are made under ambiguity, i.e. when decision makers do not perfectly know probability distributions on outcomes, new information changes their initial beliefs. For instance, when beliefs are probabilistic, they are updated according to Bayes' rule. It is well known that Bayesian beliefs allow the information to have a positive value and, from this point of view, are consistent with economic rationality.

Yet when beliefs are not probabilistic, there is no guarantee about the agent's willingness to pay for information. A non-Bayesian decision maker (DM) may refuse freely available information. This seems, if not irrational, then at least counter-intuitive to an economist. Decision models based on non-additive beliefs may therefore be criticized on these grounds, and it becomes necessary to determine in which cases non-Bayesian deciders assign a positive value to information. This note shows that the information value is positive when individual beliefs are represented by neo(non-extreme-outcome)-additive capacities (see Chateauneuf et al., 2007).

2 The model

Let S be a state space endowed with a sigma-algebra Σ. An element A of Σ is called an event. We denote the set of consequences, or outcomes, of the decision problem by X. We assume that there are a worst and a best outcome, respectively denoted by 0 and M, such that M is strictly preferred to 0. The DM's preferences on X are represented by a utility function U. Without any loss of generality, let $U(0) = 0$ and $U(M) = 1$. An act is a function from S to X. A binary act M_A0 is defined by $M_A0(s) = M$ if $s \in A$ and $M_A0(s) = 0$ if $s \in A^c$. Since U is normalized, a *bet* $\tilde{A} := U \circ M_A0$ is simply the indicator function of A, such that $\tilde{A}(s) = 1$ when $s \in A$ and $\tilde{A}(s) = 0$ otherwise. For convenience, we shall assume that the DM's preferences are defined on bets rather than on binary acts. We also assume that they are represented by a Choquet expectation denoted as $V(.)$:

$$V : \tilde{A} \longmapsto \int_S \tilde{A} d\nu$$

In this model, the DM's beliefs are modeled by a Choquet capacity, i.e. a set function $\nu : \Sigma \to \mathbb{R}$ such that (i) ν is normalized, *i.e.* $\nu(S) = 1$ and $\nu(\emptyset) = 0$, and (ii) ν is monotonic with respect to set inclusion, i.e. $A \subset B \Rightarrow \nu(A) \leq \nu(B)$ for all $A, B \in \Sigma$. V is defined by $V(\tilde{A}) := \nu(A)$ for any $A \in \Sigma$.

3 Evaluating information

We consider the following situation, which is quite similar to the one proposed by Vergnaud (2002). Let $\mathcal{A}$ be a finite partition of S, that is $\cup_{A \in \mathcal{A}} A = S$ and for all $A, A' \in \mathcal{A}$, $A \cap A' = \emptyset$. We denote the set of bets by $\tilde{\mathcal{A}} = \{\tilde{A} | A \in \mathcal{A}\}$. The DM initially has to choose a bet in $\tilde{\mathcal{A}}$. This single-stage decision problem may be written as:

$$\max_{\tilde{A}\in\tilde{\mathcal{A}}} V(\tilde{A})$$

As a notational convention, we will denote by $\tilde{A}^*$ a generic element of $\arg\max_{\tilde{A}\in\tilde{\mathcal{A}}} V(\tilde{A})$.

Assume that the DM is given the opportunity to receive information at an intermediate stage. Let $\mathcal{I}$ be a finite partition of S. Before all the uncertainty is resolved, information I from $\mathcal{I}$ is gathered. The DM then has to update her beliefs according to this information. We denote $\nu(.|I)$ as the conditional capacity for $\nu(.)$ given $I \in \mathcal{I}$ such that $\nu(.|I)$ is a Choquet capacity, $\nu(A|I) = \nu(A \cap I|I)$, $\nu(I|I) = 1$ and $\nu(S\backslash I|I) = 0$. Conditionally to each $I \in \mathcal{I}$, the DM has to choose a bet in $\tilde{\mathcal{A}}$. This conditional decision problem is:

$$\max_{\tilde{A}\in\tilde{\mathcal{A}}} V(\tilde{A}|I)$$

where $V(\tilde{A}|I) := \nu(A|I)$. Let

$$B^* \in \bigcup_{I\in\mathcal{I}} \arg\max_{A\cap I} \nu(A|I)$$

Hence the bet $\tilde{B}^*$ is defined recursively and is utility-maximizing conditionally to each I.

Example 1 Let $\mathcal{A} = \{A_1, A_2\}$ and $\mathcal{I} = \{I_1, I_2\}$. Let $\tilde{A}_1 \in \arg\max_{\tilde{A}\in\tilde{\mathcal{A}}} V(\tilde{A}|I_1)$ and $\tilde{A}_2 \in \arg\max_{\tilde{A}\in\tilde{\mathcal{A}}} V(\tilde{A}|I_2)$. Therefore, $B^* = (A_1 \cap I_1) \cup (A_2 \cap I_2)$ maximizes each conditional capacity, $\nu(.|I_1)$ and $\nu(.|I_2)$, and then the bet $\tilde{B}^*$ is conditional utility-maximizing[1].

The information value is the difference between the value of the conditional utility-maximizing bet in the dynamic decision problem and the value of the utility-maximizing bet in the static decision problem.

Definition 1 (Information Value)**.** *The information value is given by* $IV[\tilde{A}^*, \tilde{B}^*] := V(\tilde{B}^*) - V(\tilde{A}^*)$.

The value of information is positive if the DM expects a gain from discovering new opportunities, comparatively to her initial situation.

In the additive case, i.e. when ν is a probability, it is easy to check that $IV[\tilde{A}^*, \tilde{B}^*] \geq 0$. However, this property does not hold true, in general, in the non-additive case. Therefore, ambiguity may cause the DM to refuse some freely available information.

4 The result

In order to have a positive information value, we assume that both $\nu(.)$ and $\nu(.|I)$ are neo(non-extreme-outcome)-additive capacities.

[1]An alternative definition could be $\tilde{B}^* = \tilde{A}_1 \otimes \tilde{I}_1 \oplus \tilde{A}_2 \otimes \tilde{I}_2$.

Definition 2 (Neo-additive capacities)**.** *Given any event* $E \in \Sigma$*, the conditional capacity* $\nu(.|E)$ *is neo-additive if there is a conditional probability measure* $\mu(.|E)$: $\Sigma \to [0,1]$ *and* $\delta_E, \lambda_E \in [0,1]$ *such that:*

$$\nu(A|E) = \begin{cases} 0 & \text{if} \quad A \cap E = \emptyset \\ \delta_E \lambda_E + (1-\delta_E)\mu(A|E) & \text{if} \quad A \cap E \neq E, \emptyset \\ 1 & \text{if} \quad E \subseteq A \end{cases}$$

When $E = S$, we simply write the parameters as μ, δ and λ. Observe that the conditional probability $\mu(.|E)$ is given by Bayes' rule: $\mu(.|E) = \mu(. \cap E)/\mu(E)$.

The neo-additive expected utility framework is appealing since it allows one to make a distinction between ambiguity and ambiguity attitude. In particular, δ_E denotes the degree of ambiguity whereas λ_E reflects the degree of optimism (ambiguity loving). When $\delta_E = 0$, there is no ambiguity and beliefs are modeled by means of the probability $\mu(.|E)$. The Choquet expectation then declines to the additive expectation w.r.t. $\mu(.|E)$. Additionally, the DM is said to be extremely optimistic if $\lambda_E = 1$ and extremely pessimistic if $\lambda_E = 0$.

Several updating rules for Choquet beliefs have been proposed in the literature to calculate $\nu(.|E)$: the Dempster-Shafer rule, the "Naive" Bayes rule and the Full-Bayesian updating (FBU) rule. For instance, Eichberger et al. (2007) and Horie (2007) provide axioms for the family of Choquet beliefs that characterize the Full-Bayesian updating rule proposed by Jaffray (1992). Eichberger et al. (2010) show that when the unconditional capacity is neo-additive, any of these three rules yields neo-additive conditional capacities. Yet only the FBU rule implies that the degree of optimism λ is constant, namely $\lambda = \lambda_E$ for any $E \in \Sigma$.

An interesting feature of the neo-additive expected utility model is that it allows $IV(.)$ to be positive.

Proposition 1. *If, for any* $E \in \Sigma$*,* $\nu(.|E)$ *is neo-additive, then the information value is positive.*

Proof By definition, $V(\tilde{B}^*|I) \geq V(\tilde{A}^*|I)$, or, equivalently, $\nu(B^*|I) \geq \nu(A^*|I)$, for any $I \in \mathcal{I}$. Then,

$$\delta_I \lambda_I + (1-\delta_I)\mu(B^*|I) \geq \delta_I \lambda_I + (1-\delta_I)\mu(A^*|I)$$

if and only if $\mu(B^* \cap I) \geq \mu(A^* \cap I)$. Therefore,

$$\sum_{I \in \mathcal{I}} \mu(B^* \cap I) \geq \sum_{I \in \mathcal{I}} \mu(A^* \cap I)$$

which is tantamount to $\mu(B^*) \geq \mu(A^*)$ since μ is additive. Consequently,

$$\delta\lambda + (1-\delta)\mu(B^*) \geq \delta\lambda + (1-\delta)\mu(A^*)$$

hence $IV[\tilde{A}^*, \tilde{B}^*] \geq 0$. □

Vergnaud (2002) showed that if capacity is a possibility measure updated by Bayes' rule or a necessity measure updated by Demspter-Shafer's rule, then the information value is positive. Proposition 1 does not depend on the updating rule used to calculate the conditional capacity $\nu(.|I)$. Therefore, for neo-additive capacities, the information is always positively valued, whatever the updating rule.

References

[1] Chateauneuf, A., Eichberger, J. and Grant, S. (2007) "Choice under uncertainty with the best and worst in mind: Neo-additive capacities" *Journal of Economic Theory* 137, 538-567.

[2] Eichberger, J., Grant, S. and Kelsey, D. (2007) "Updating Choquet beliefs" *Journal of Mathematical Economics* 43, 888-899.

[3] Eichberger, J., Grant, S. and Kelsey, D. (2010) "Comparing three ways to update Choquet beliefs" *Economics Letters* 107, 91-94.

[4] Horie, M. (2007) "Reexamination on updating Choquet beliefs" Kyoto Institute of Economic Research Discussion paper no. 643.

[5] Jaffray, J.-Y. (1992) "Bayesian Updating and Belief Functions" *IEEE Transactions on Systems, Man, and Cybernetics* 22, 1144-1152.

[6] Vergnaud, J.C. (2002) "Information and capacities" *Statistical Papers* 43, 111-125.