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Radner's Theorem on Teams and Games with a Continuum of Players

Takashi Ui
Hitotsubashi University

Yasunori Yoshizawa
Yokohama National University

Abstract

This note considers Bayesian games with a continuum of players, symmetric quadratic payoff functions, and normally distributed signals. It shows that a recent result on the existence and uniqueness of equilibrium is implied by a classical theorem on teams by Radner (1962, Ann. Math. Stat. 33).

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Contact: Takashi Ui - oui@econ.hit-u.ac.jp, Yasunori Yoshizawa - yoshizawa-yasunori-pc@ynu.ac.jp.
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1. Introduction

Since the seminal work of Morris and Shin (2002) on beauty contest games, social value of information in Bayesian games has been experiencing a renewed interest. Angeletos and Pavan (2007) propose a class of Bayesian games which are tractable yet flexible enough to capture a number of applications to study social value of information. In Bayesian games studied by Morris and Shin (2002) and Angeletos and Pavan (2007), there are a continuum of players, payoff functions are symmetric and quadratic, and players' information consists of normally distributed signals. Angeletos and Pavan (2007, 2009) establish the existence and uniqueness of equilibrium.

For the case with a finite number of players, Radner (1962) was the first to study Bayesian games with quadratic payoff functions and normally distributed signals. He assumes that players have identical payoff functions and calls such a game a team (before Harsanyi's Bayesian games). Theorem 5 of Radner (1962) establishes the existence and uniqueness of equilibrium. This result can be directly used in the study of Bayesian games of which best response correspondences are the same as those of teams. Basar and Ho (1974) use Radner's theorem to study Bayesian oligopoly games with linear demand functions, followed by many papers on information sharing in oligopoly (see Raith (1996) and references therein). Ui (2009) studies a class of Bayesian games possessing the same best response correspondences as those of teams, which include Bayesian potential games (Monderer and Shapley, 1996; Heumen et al., 1996) as a special case. Calvó-Armengol and De Martí Beltran (2009) study information gathering in organizations using Bayesian potential games in which players are located on networks through which players can share information.

In both the continuum and finite cases, the unique equilibrium strategy is a linear function of signals. Thus, the difference between them seems not to change the mathematics very much. However, as far as the existence and uniqueness of equilibrium is concerned, the above two strands of literature are completely separated. The purpose of this note is to connect them in terms of Radner's theorem on teams with a finite number of players. Ui (2009) points out that Radner's theorem is useful in the study of Bayesian potential games with a finite number of players. In contrast, this paper demonstrates that Radner's theorem is useful even with a continuum of players.[1]

We show that Radner's theorem implies the existence and uniqueness of equilibrium in a game with a continuum of players established by Angeletos and Pavan (2007, 2009). The intuition is as follows. In the game studied by Angeletos and Pavan (2007, 2009), payoff functions are symmetric, and attention is restricted to a symmetric equilibrium. This implies that the first order condition is reduced to a *single* equation, and there exists an n-player team with the same first order condition, where n is arbitrary. Therefore, the existence and uniqueness of a solution to the first order condition is assured by Radner's theorem.

[1] In fact, games with a continuum of players studied by Angeletos and Pavan (2007, 2009) are bilateral symmetric interaction games (Ui, 2000) with a continuum of players, a special class of potential games in the sense proposed by Ui (2008).

2. Model

There is a continuum of players and an individual player is indexed by $i \in [0,1]$. Player i chooses an action $a_i \in \mathbb{R}$. Given an action profile $\mathbf{a} = (a_i)_{i\in[0,1]}$, the average action is denoted by $A = \int_0^1 a_i di$. Player i's payoff is

$$u_i(a_i, \mathbf{a}, \omega) = -\alpha a_i^2 + 2\beta a_i A + 2\gamma a_i \theta_i + f_i(\mathbf{a}, \omega), \tag{1}$$

where $\omega \equiv (\theta_j)_{j\in[0,1]}$ is a payoff state with $\theta_i \in \mathbb{R}$ for each i, $\alpha, \beta, \gamma \in \mathbb{R}$ are constants with $\alpha > 0$, and f_i is a function with $f_i(\mathbf{a}, \omega) = f_i(\mathbf{a}', \omega)$ if $a_j = a_j'$ for almost all $j \in [0,1]$.

Player i observes a vector-valued signal $x_i \in \mathbb{R}^s$ where $s \geq 1$ is an integer. For $i \neq j$, $(x_i, x_j, \theta_i, \theta_j) \in \mathbb{R}^{2s+2}$ is normally distributed with $E[x_i] = E[x_j] = \bar{x}$, $E[\theta_i] = E[\theta_j] = \bar{\theta}$, $\mathrm{var}[x_i] = \mathrm{var}[x_j] = C$, $\mathrm{cov}[x_i, x_j] = \mathrm{cov}[x_j, x_i] = D$, $\mathrm{cov}[x_i, \theta_i] = \mathrm{cov}[x_j, \theta_j] = G$, and $\mathrm{var}(\theta_i) = \mathrm{var}(\theta_j) = H$.

A strategy is a measurable function $a : \mathbb{R}^s \to \mathbb{R}$ which assigns an action $a(x_i) \in \mathbb{R}$ to a signal $x_i \in \mathbb{R}^s$. Since $f_i(\mathbf{a}, \omega)$ in the payoff function does not depend upon player i's action, his best response is determined by the terms other than $f_i(\mathbf{a}, \omega)$, i.e., $-\alpha a_i^2 + 2\beta a_i A + 2\gamma a_i \theta_i$. Hence, a Bayesian Nash equilibrium is defined as a strategy $a : \mathbb{R}^s \to \mathbb{R}$ such that

$$a(x_i) = \arg\max_{a_i} \left(-\alpha a_i^2 + 2\beta a_i A(x_i) + 2\gamma a_i E[\theta_i|x_i]\right)$$

for all $x_i \in \mathbb{R}^s$, where

$$A(x_i) \equiv E\Big[\int_0^1 E[a(x_j)|\omega]dj\Big|x_i\Big] = E[E[a(x_j)|\omega]|x_i] = E[a(x_j)|x_i]$$

for $j \neq i$.[2] The first order condition is

$$\alpha a(x_i) - \beta E[a(x_j)|x_i] = \gamma E[\theta_i|x_i]. \tag{2}$$

Angeletos and Pavan (2007, 2009) show the following result.[3]

Theorem 1. *If $\alpha > 0$ and $\beta/\alpha < 1$, then there exists a unique equilibrium given by*

$$a(x_i) = \gamma G^\top (\alpha C - \beta D)^{-1}(x_i - \bar{x}) + \gamma\bar{\theta}/(\alpha - \beta) \tag{3}$$

for all $x_i \in \mathbb{R}^s$.

3. Radner's theorem

Radner (1962) considers a Bayesian game with a finite number of players who have an identical payoff function. We call this game a team following Radner (1962). There are n players and an individual player is indexed by $i \in \{1, \ldots, n\}$. Player i chooses an action

[2] Since $E[a(x_j)|\omega] = E[a(x_k)|\omega]$ for all $j \neq k$ by the symmetric information structure, it holds that $\int_0^1 E[a(x_j)|\omega]dj = E[a(x_j)|\omega]$.

[3] Angeletos and Pavan (2007) consider the case with $-1 < \beta/\alpha < 1$ and $s = 2$. Angeletos and Pavan (2009) consider the case with $\beta/\alpha < 1$ and $s \geq 2$.

$a_i \in \mathbb{R}$. An action profile is denoted by $\mathbf{a} = (a_i)_{i\in\{1,\ldots,n\}} \in \mathbb{R}^n$, which is understood as a column vector. Each player has the same payoff function

$$v(\mathbf{a}, \omega) = -\mathbf{a}^\top M\mathbf{a} + 2\sum_{i=1}^{n} \gamma_i a_i \theta_i, \tag{4}$$

where $M = [m_{ij}]_{n\times n}$ is a symmetric matrix and $\omega \equiv (\theta_1, \ldots, \theta_n) \in \mathbb{R}^n$ is a payoff state. Player i observes a vector-valued signal $x_i \in \mathbb{R}^s$, and $(x_1, \ldots, x_n, \theta_1, \ldots, \theta_n)$ is normally distributed. We denote player i's strategy by $\sigma_i : \mathbb{R}^s \to \mathbb{R}$. A strategy profile $(\sigma_i)_{i\in\{1,\ldots,n\}}$ is a Bayesian Nash equilibrium if

$$\sigma_i(x_i) = \arg\max_{a_i} E[v((a_i, \sigma_{-i}), \omega)|x_i]$$

for all $x_i \in \mathbb{R}^s$ and $i \in \{1, \ldots, n\}$, where $\sigma_{-i} = (\sigma_j(x_j))_{j\neq i}$.

If $v(\mathbf{a}, \omega)$ is strictly concave in $\mathbf{a}$ and ω is commonly known, then the first order condition for an equilibrium coincides with that for maximizing $v(\mathbf{a}, \omega)$ with respect to $\mathbf{a}$, and an equilibrium is unique. This holds even under incomplete information as shown by Theorem 5 of Radner (1962).

Theorem 2 (Radner, 1962)**.** *If M is positive definite, then there exists a unique equilibrium $(\sigma_i)_{i\in\{1,\ldots,n\}}$. It is a unique solution of the first order condition*

$$\sum_{j=1}^{n} m_{ij} E[\sigma_j(x_j)|x_i] = \gamma_i E[\theta_i|x_i]$$

for all $x_i \in \mathbb{R}^s$ and $i \in \{1, \ldots, n\}$, and σ_i is linear.

We show that Theorem 1 is implied by Theorem 2. Consider a team with the same parameters as those in a Bayesian game in Section 2. Let $m_{ii} = \alpha$, $m_{ij} = -\beta/(n-1)$, and $\gamma_i = \gamma$ for all $i \neq j$, and assume that M is positive definite. This is true if and only if $\alpha > 0$ and $-(n-1) < \beta/\alpha < 1$ because the leading principal minors of M are $(\alpha+\beta/(n-1))^{k-1}(\alpha-(k-1)\beta/(n-1))$ for $k = 1, \ldots, n$. The information structure is also the same as that in Section 2: for $i \neq j$, $(x_i, x_j, \theta_i, \theta_j) \in \mathbb{R}^{2s+2}$ is normally distributed with $E[x_i] = E[x_j] = \bar{x}$, $E[\theta_i] = E[\theta_j] = \bar{\theta}$, $\mathrm{var}[x_i] = \mathrm{var}[x_j] = C$, $\mathrm{cov}[x_i, x_j] = \mathrm{cov}[x_j, x_i] = D$, $\mathrm{cov}[x_i, \theta_i] = \mathrm{cov}[x_j, \theta_j] = G$, and $\mathrm{var}(\theta_i) = \mathrm{var}(\theta_j) = H$.

The first order condition is

$$\alpha\sigma_i(x_i) - \frac{\beta}{n-1}\sum_{j\neq i} E[\sigma_j(x_j)|x_i] = \gamma E[\theta_i|x_i]$$

for all i. By Theorem 2, this equation has a unique solution $(\sigma_i)_{i\in\{1,\ldots,n\}}$. Since the joint probability distribution of $(x_1, \ldots, x_n)$ is symmetric and that of (x_i, θ_i) is the same for all i, for any permutation $\pi : \{1, \ldots, n\} \to \{1, \ldots, n\}$, we have

$$\alpha\sigma_i(x_{\pi(i)}) - \frac{\beta}{n-1}\sum_{j\neq i} E[\sigma_j(x_{\pi(j)})|x_{\pi(i)}] = \gamma E[\theta_{\pi(i)}|x_{\pi(i)}]$$

for all i. This implies that a strategy profile $(\sigma'_i)_{i\in\{1,\ldots,n\}}$ with $\sigma'_{\pi(i)} = \sigma_i$ for all i is also a unique solution. Hence, we must have $\sigma_i = \sigma_j$ for all i, j, and let $a \equiv \sigma_i$. Then, the first order condition is reduced to

$$\alpha a(x_i) - \frac{\beta}{n-1}\sum_{j\neq i} E[a(x_j)|x_i] = \alpha a(x_i) - \beta E[a(x_j)|x_i] = \gamma E[\theta_i|x_i] \tag{5}$$

since $E[a(x_j)|x_i] = E[a(x_k)|x_i]$ for all $j, k \neq i$. Theorem 2 implies that (5) has a unique solution if $\alpha > 0$ and $-(n-1) < \beta/\alpha < 1$.

Note that (5) coincides with the first order condition (2). Thus, a Bayesian game in Section 2 has a unique equilibrium if $\alpha > 0$ and $-(n-1) < \beta/\alpha < 1$ by Theorem 2. Since we can choose arbitrary n, it has a unique equilibrium if $\alpha > 0$ and $\beta/\alpha < 1$.

Finally, we solve (5) for completeness. By Theorem 2, the solution must be a linear function, and let $a(x_i) = b^\top(x_i - \bar{x}) + c$ where $b \in \mathbb{R}^s$ and $c \in \mathbb{R}$. Plugging this into (5),

$$\alpha(b^\top(x_i - \bar{x}) + c) - \beta(b^\top(E[x_j|x_i] - \bar{x}) + c) = \gamma E[\theta_i|x_i]. \tag{6}$$

By the property of multivariate normal distributions,[4]

$$E[x_j|x_i] = \bar{x} + DC^{-1}(x_i - \bar{x}),\ E[\theta_i|x_i] = \bar{\theta} + G^\top C^{-1}(x_i - \bar{x}).$$

Plugging this into (6),

$$b^\top(\alpha I - \beta DC^{-1})(x_i - \bar{x}) + (\alpha - \beta)c = \gamma G^\top C^{-1}(x_i - \bar{x}) + \gamma\bar{\theta}$$

for all $x_i \in \mathbb{R}^s$. This implies that $b^\top = \gamma G^\top(\alpha C - \beta D)^{-1}$ and $c = \gamma\bar{\theta}/(\alpha - \beta)$.

References

Angeletos, G.-M. and A. Pavan (2007) "Efficient Use of Information and Social Value of Information" *Econometrica* **75**, 1103–1142.

Angeletos, G.-M. and A. Pavan (2009) "Policy with Dispersed Information" *Journal of the European Economic Association* **7**, 11–60.

Basar, T. and Y. Ho (1974) "Informational Properties of the Nash Solutions of Two Stochastic Nonzero-sum Games" *Journal of Economic Theory* **7**, 370–384.

Calvó-Armengol, A. and J. De Martí Beltran (2009) "Information Gathering in Organizations: Equilibrium, Welfare and Optimal Network Structure" *Journal of the European Economic Association* **7**, 116–161.

van Heumen, R., B. Peleg, S. Tijs and P. Borm (1996) "Axiomatic Characterizations of Solutions for Bayesian Games" *Theory and Decision* **40**, 103–129.

Monderer, D. and L. S. Shapley (1996) "Potential Games" *Games and Economic Behavior* **14**, 124–143.

Morris, S. and H. S. Shin (2002) "Social Value of Public Information" *American Economic Review* **92**, 1521–1534.

Radner, R. (1962) "Team Decision Problems" *Annals of Mathematical Statistics* **33**, 857–881.

Raith, M. (1996) "A General Model of Information Sharing in Oligopoly" *Journal of Economic Theory* **71**, 260–288.

[4]Let $X = (X_1, X_2)$ be a random vector whose distribution is multivariate normal. Let $\mu_i = EX_i$ and $C_{ij} = \mathrm{cov}(X_i, X_j)$ for $i, j = 1, 2$. Then, $E[X_2|X_1] = \mu_2 + C_{21}C_{11}^{-1}(X_1 - \mu_1)$.

Ui, T. (2000) “A Shapley Value Representation of Potential Games” *Games and Economic Behavior* **31**, 121–135.

Ui, T. (2008) “Non-atomic Potential Games and the Value of Vector Measure Games” Working paper.

Ui, T. (2009) “Bayesian Potentials and Information Structures: Team Decision Problems Revisited” *International Journal of Economic Theory* **5**, 271–291.