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Super-majorites and collective surplus in one-dimensional bargaining: Numerical simulations

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Abstract

This note presents numerical simulations computing quota rules that maximize collective surplus for populations choosing a one-dimensional policy through bargaining and voting. These computations are based on the characterization of the unique (asymptotic) equilibrium of Cardona and Ponsatí (2011). We show that under quadratic utility functions, the unique quota rule that maximizes collective surplus ranges from 80% to 95%.

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1 Introduction

This note presents numerical simulations computing quota rules that maximize collective surplus for populations choosing a one-dimensional policy through bargaining and voting. We assume that individuals have single-peaked quadratic preferences and that the different locations of the peaks are the only source of heterogeneity within the population. Bargaining takes place over time. At each period, a randomly selected individual makes a proposal which is approved if it receives the vote of a qualified majority; otherwise, a new proposer is selected next period, and so on. Assuming that individual payoffs are discounted utilities, we have proved in Cardona and Ponsatí (2011) that this game has a unique stationary subgame perfect equilibrium.[1] Thanks to the uniqueness of the equilibrium, and its explicit characterization, the welfare performance of the different majority requirements can be assessed. In Cardona and Ponsatí (2011) we supply sharp results for the special case of symmetric populations: Assuming strict impatience unanimity is the unique (ex-ante) Pareto optimal rule. However, when the population is not symmetric the Pareto criterion is ineffective since any (super)majority may be Pareto optimal.[2] In Cardona and Ponsatí (2012) we present a general discussion of the performance of different quota rules in terms of their delivered collective surplus - i.e., the sum of the individual utilities attained in equilibrium. Here, we present computations of the optimal quotas based on the numerical simulations of the model with quadratic utilities. We carry out this simulation exercise for a rich set of parameterizations and obtain that the optimal quota is always rather demanding: It must be a super-majority between 80 and 95%.

2 The unique asymptotic outcome

The set up is that of Cardona and Ponsatí (2011). A group I of n individuals, n odd, collectively selects an alternative in $[0,1]$. Time is discrete. At each $t = 0,1,2...$ an individual is selected at random (all with equal probability) to make a proposal. Then, she chooses an alternative in $[0,1]$ and all other players, sequentially in any fixed order, reply with acceptance or rejection. If acceptances are at least $nq-1$, $q \in [1/2,1]$, the proposal is implemented and the game ends. Otherwise, the game moves to $t+1$, a new proposer is selected, and so on. Upon a vote that approves $x \in [0,1]$ at t, individual i obtains $\delta^t u(x,i)$ where $\delta \in (0,1)$ and $u(x,i) = 1-(i-x)^2$. Perpetual disagreement yields zero to all agents.

Each $i \in I$ denotes both a generic individual and the location of her peak, so that all the information regarding population heterogeneity is embedded in the cumulative distribution function of peaks, denoted by F. Since we are interested in setups where n

[1] Existence of a Stationary Subgame Perfect Equilibrium follows from the results of Banks and Duggan (2000). See discussion of the literature in Cardona and Ponsatí (2011).

[2] See Examples in Cardona and Ponsatí (2011), p. 72.

is large, we will work with continuous cumulative distribution functions with a positive density f on $(0,1)$.

In Cardona and Ponsatí (2011) we have shown that this setup has a unique stationary subgame equilibrium (SSPE) that depends on the distribution of peaks and, in particular, of two *boundary players*, say $l(q)$ and $r(q)$, satisfying $F(l(q)) = 1-q$ and $F(r(q)) = q$. Moreover, it is immediate that the SSPE yields a unique limit outcome as $\delta \to 1$, which is characterized next.

PROPOSITION 1 UNIQUE BARGAINING OUTCOME. *Consider a sequence of environments (q, F, δ_k), where $\delta_k \to 1$. In the limit, as $\delta_k \to 1$, the SSPE approval set converges to a singleton $x(q)$, where $x(q)$ is the unique solution to*

$$K_F(x,q) \equiv F(x)\frac{u_x(x,l)}{u(x,l)} + [1-F(x)]\frac{u_x(x,r)}{u(x,r)} = 0. \tag{1}$$

The unique equilibrium outcome $x(q)$ yields a unique payoff $u(x(q), i)$ for each $i \in I$, which in turn induce collective benefits. Equipped with Eq. (1), we can address the comparative statics for $x(q)$ and its induced individual and collective benefits with respect to q. We turn to numerical simulations to carry out this exercise next.

3 Surplus maximizing quotas: Numerical simulations

When the maximization of surplus is the welfare-maximizing criterion, a *first best policy* x^{fb} is an alternative that maximizes the *collective surplus* $S(x) = \int_0^1 u(x,i)f(i)\,di$. The collective surplus associated to each q, via $x(q)$, is $W(q) = \int_0^1 u(x(q),i)f(i)\,di$. Thus, the best conceivable performance for $W(q)$ would be delivered (if it exists) by a *first best rule* - i.e., a quota rule q^{fb} such that $x(q^{fb}) = x^{fb}$.

It is immediate that when $u(x,i) = 1-(x-i)^2$ the first best policy is the mean of the distribution; *i.e.*, $x^{fb} = \mu = \int_0^1 if(i)\,di$. Hence, our simulations consist on computing q that yield $x(q) = \mu$; that is, solving the system:

$$\begin{aligned} 0 &= F(\mu)\frac{(\mu-l)}{1-(\mu-l)^2} - [1-F(\mu)]\frac{(r-\mu)}{1-(r-\mu)^2}, \\ F(l) &= 1-q, \\ F(r) &= q. \end{aligned} \tag{2}$$

It is trivial to check that in the special case of symmetric populations (*i.e.*, $f(i) = f(1-i)$ for all $i \in [0,1]$), Eq. (1) yields $x(q) = \mu = 1/2$ for all q. For an asymmetric f the bargaining outcome varies with q, and thus the value of q must be fine tuned to attain the first best policy. Our simulations suggest that generally, first best rules can be achieved through bargaining by appropriately selecting the consensus requirement. Moreover, first best rules are large.

We carried out numerical simulations computing the optimal super-majority rule for extensive parameterizations of 4 natural specifications of F: Two-block, Triangular, Beta and Kuramaswamy distributions. Our numerical evaluations require the use of MatLab for the Beta and Kuramaswamy distributions. In these cases, intervals of size $\Delta = 1/10000$ are used.

As shown next, with slight differences between populations, first best rules range from 80% to 95%. Surplus maximization requires a strict super-majority in order to avoid that extreme players (those who suffer high "transportation" costs) are completely excluded from the bargaining. On the other hand, since the mass of extreme agents is relatively low, their influence must be limited. Thus, the optimal rule is lower than unanimity.

We did not find a clear monotonicity relationship between the mass of extreme players, measured in terms of skewness of the distribution, and the optimal rule.[3] This is due to two effects that appear when increasing the skewedness of the population. First, the first best policy moves away from the increased tail. Second, the boundary players also move away from the increased tail, and so does the bargaining outcome. Hence, the total effect on the optimal rule is unclear, as it depends on the relative size of each effect. In particular, when changes in the bargaining outcome due to a change in the skewness of the population exceed (are smaller than) the variation in the optimal policy, then the optimal rule must be increased (reduced) accordingly, in order to weaken (strengthen) the bargaining power of the agents in the largest tail of the distribution.

3.1 Two-block distributions

We first examine the optimal rules in the simple one-parameter (the median) specification of the population, where densities are constant over *two-blocks* with the same mass of players each. The *two-block* density and cumulative distribution functions, defined over $[0, 1]$, are given by

$$f_b(x;\theta) = \begin{cases} \frac{1}{2\theta} & 0 \leq x \leq \theta \\ \frac{1}{2(1-\theta)} & \theta < x \leq 1 \end{cases} \text{ and } F_b(x;\theta) = \begin{cases} \frac{x}{2\theta} & 0 \leq x \leq \theta \\ 1 - \frac{1-x}{2(1-\theta)} & \theta < x \leq 1 \end{cases},$$

where the median is $x^m = \theta \in [0, 1]$, $\mu = (1 + 2\theta)/4$, $r(q) = \theta + 2\left(q - \frac{1}{2}\right)(1 - \theta)$ and

[3] The skewness of a real-valued random variable - aiming to estimate probability distribution asymmetry - is usually measured by the third standardized moment, defined as $E\left[\left(\frac{x-\mu}{\sigma}\right)^3\right]$, where σ is the standard deviation.

$l(q) = 2\theta(1-q)$. Two examples are depicted in the following figures.

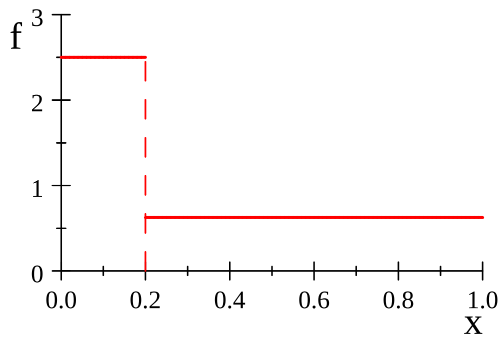

Figure 1: $f_b(x; 0.2)$

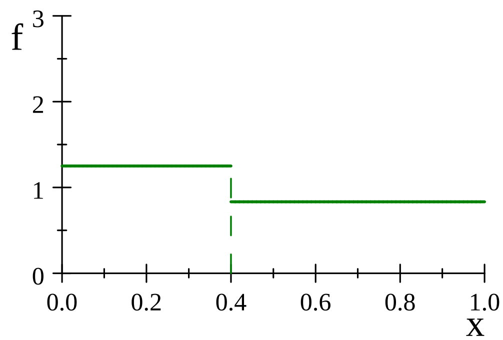

Figure 2: $f_b(x; 0.4)$

The first best rules, for different values of $\theta \in (0, 1/2)$, are presented in Table 1.

$\theta = x^m$	$\mu = x^{fb}$	q^{fb}	l	r	$Skew.$
0.1	0.3	0.82332	0.035336	0.68198	0.83181
0.2	0.35	0.83383	0.066468	0.73413	0.68491
0.3	0.4	0.84769	0.091386	0.78677	0.48993
0.35	0.425	0.85640	0.10052	0.813 32	0.37692
0.4	0.45	0.8667	0.10664	0.840 04	0.25596
0.45	0.475	0.87905	0.10886	0.866 96	0.12913
0.49	0.495	0.89081	0.10701	0.888 63	0.02598

Table 1

We observe that $q^{fb} \in [0.8, 0.9]$ for any distribution. Moreover, for this specification increasing the skewness of the population, which is equivalent to decreasing the median, reduces the optimal super-majority: Starting from the optimal rule in any given population, the change in the optimal policy due to an increase in the skewness always exceeds the corresponding change in the bargaining outcome. Hence, the required super-majority must be weakened in order to decrease the bargaining power of agents in the largest tail.

3.2 Triangular distributions

The Triangular density and cumulative distribution functions, defined over $[0, 1]$, are given by

$$f_T(x;d) = \begin{cases} 2\frac{x}{\theta} & 0 \leq x \leq \theta \\ 2\frac{(1-x)}{1-\theta} & \theta < x \leq 1. \end{cases} \text{ and } F_T(x;\theta) = \begin{cases} \frac{x^2}{\theta} & 0 \leq x \leq \theta \\ 1 - \frac{(1-x)^2}{1-\theta} & \theta < x \leq 1 \end{cases}.$$

We restrict, w.l.o.g., to the case where $\theta \in (1/2, 1]$. In these cases, the median is $x^m = (\theta/2)^{1/2}$ and $\mu = (1+\theta)/3$. Moreover, $l(q) = \sqrt{(1-q)\theta}$ for all q, $r(q) = \sqrt{q\theta}$ if $q \leq \theta$ and $r(q) = 1 - \sqrt{(1-q)(1-\theta)}$ if $q \geq \theta$. Next figures illustrate two triangular density functions.

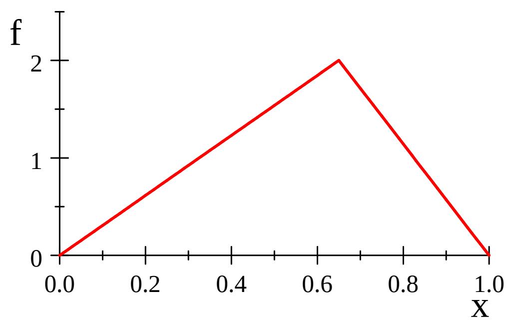

Figure 3: $f_T(x; 0.65)$

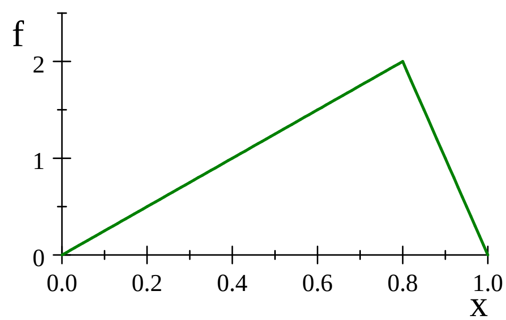

Figure 4: $f_T(x; 0.8)$

The first best rules, for different values of $\theta \in (0, 1/2)$, are presented in Table 2.

θ	x^m	$\mu = x^{fb}$	q^{fb}	l	r	$Skew.$
0.6	0.547 72	0.533 33	0.917 65	0.222 28	0.818 51	−0.1912
0.7	0.591 61	0.566 67	0.893 25	0.273 36	0.821 04	−0.3561
0.8	0.632 46	0.6	0.882 53	0.306 56	0.846 72	−0.4761
0.9	0.670 82	0.633 33	0.885 33	0.320 97	0.892 63	−0.5447
0.95	0.689 2	0.65	0.88758	0.326 82	0.893 76	−0.5612

Table 2

Again, optimal quotas are large. However, and in contrast to the two-block case, under *Triangular distributions*, the difference between the change of the optimal policy and the change in the bargaining outcome due to an increase in the skewness of the population depends on the initial distribution. Thus, the optimal quota is not monotone in the skewness.

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3.3 Beta distributions

The Beta distributions are characterized by two parameters $\alpha, \beta > 0$, and offer a very flexible family of specifications. Specifically, the density and cumulative distribution functions, defined over $[0,1]$, are given by[4]

$$f_B(x;\alpha,\beta) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)} \text{ and } F_B\left(x;\alpha,\beta\right) = \frac{1}{B\left(\alpha,\beta\right)}\int_0^x z^{\alpha-1}\left(1-z\right)^{\beta-1}dz,$$

with

$$B(\alpha,\beta) = \int_0^1 z^{\alpha-1}(1-z)^{\beta-1}dz.$$

While the median has no explicit form, the mean is given by $\mu = \alpha/(\alpha+\beta)$, and the mode is $(\alpha-1)/(\alpha+\beta-2)$. Two particular specifications are depicted next.

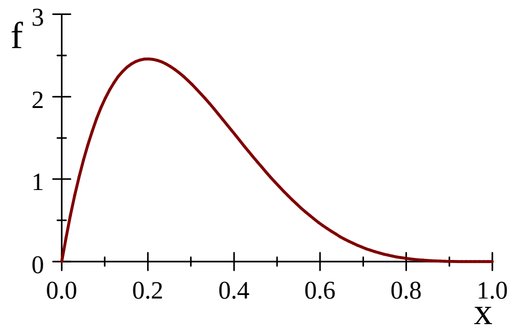

Figure 5: $f_B(x;2,5)$

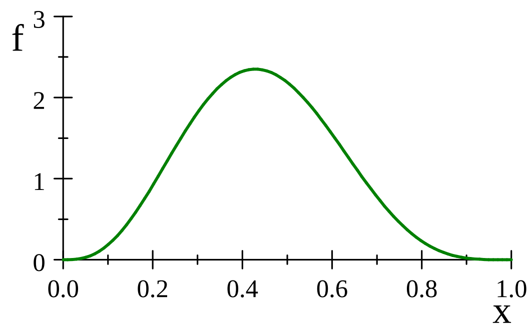

Figure 6: $f_B(x;4,5)$

The first best rules, for different values of (α,β), are presented in Table 3. Now, optimal quotas stand around 0.9. Moreover, they are not directly related to the skewness of the distribution. Roughly speaking, the absolute value of the difference between the parameters is positively related to the skewness of the density function. Now, when $\alpha < \beta$, the skewness might be increased either (i) by decreasing α while maintaining β fixed or (ii) by fixing α and increasing β. Moreover, our computations show that the optimal super-majority rule decreases in case (i) and it increases in case (ii). Thus, while large super-majorities are obtained in all cases, they are not monotone in the skewness. As an illustration, consider the beta distribution with parameters $(\alpha,\beta) = (3,6)$. The optimal rule is 0.9146. If the skewness is increased by reducing α to 2 then the optimal

[4] $a > 1$ and $b > 1$ guarantee the single-peakedness of the density function.

rule decreases to 0.9134. Instead, when the skewness increases by changing β to 7, the optimal rules goes up to 0.9163.

α	β	x^m	$\mu = x^{fb}$	q^{fb}	l	r	$Skew.$
1	3	0.2063	0.25	0.8989	0.0349	0.5341	0.8607
2	3	0.3857	0.4	0.9028	0.1403	0.6829	0.2857
2.5	3	0.4487	0.4545	0.9048	0.1922	0.7266	0.1241
1	4	0.1591	0.2	0.9053	0.0246	0.4453	1.0498
2	4	0.3138	0.3333	0.9075	0.1074	0.5930	0.4677
3	4	0.4214	0.4286	0.9100	0.1926	0.6773	0.1814
3.5	4	0.4636	0.4667	0.9111	0.2315	0.7074	0.0820
1	5	0.1294	0.1667	0.9093	0.0188	0.3812	1.1832
2	5	0.2644	0.2857	0.9109	0.0867	0.5232	0.5963
3	5	0.3641	0.375	0.9126	0.1606	0.6094	0.3098
4	5	0.4401	0.4444	0.9142	0.2281	0.6688	0.1286
4.5	5	0.4718	0.4737	0.9149	0.2589	0.6921	0.0594
1	6	0.1091	0.1429	0.9120	0.0152	0.3331	1.2830
2	6	0.2285	0.25	0.9134	0.0726	0.4676	0.6928
3	6	0.3205	0.3333	0.9146	0.1376	0.5533	0.4066
5	6	0.4517	0.4545	0.9169	0.2541	0.6605	0.0973
5.5	6	0.4769	0.4783	0.9174	0.2795	0.6796	0.0456
1	7	0.0943	0.125	0.914	0.0128	0.2957	1.3607
3	7	0.2862	0.3	0.9163	0.1203	0.5064	0.4825
4	7	0.3551	0.3636	0.9171	0.1763	0.5677	0.3021
6	7	0.4595	0.4615	0.9188	0.2742	0.6529	0.0770
6.5	7	0.4805	0.4815	0.9192	0.2956	0.6692	0.0364

Table 3

3.4 Kuramaswamy distributions

The Kumaraswamy density and cumulative distribution functions, defined over $[0, 1]$, are given by

$$f_K(x; \alpha, \beta) = \alpha\beta x^{\alpha-1} (1 - x^{\alpha})^{\beta-1} \text{ and } F_K(x; \alpha, \beta) = 1 - (1 - x^{\alpha})^{\beta} \text{ where } \alpha, \beta > 0.$$

These distributions resemble to the Beta distributions. In particular, if $x_{\alpha,\beta}$ is a Kumaraswamy distributed random variable with parameters α and β, and $y_{1,\beta}$ denotes a Beta distributed random variable with parameters 1 and β, then one has that $x_{\alpha,\beta} = y_{1,\beta}^{1/\alpha}$. As a comparison, Figure 7 displays a population distributed according to $f_K(x; 2, 8)$ (solid line) and a population distributed according to $f_B(x; 2.7, 6)$ (dashed line).

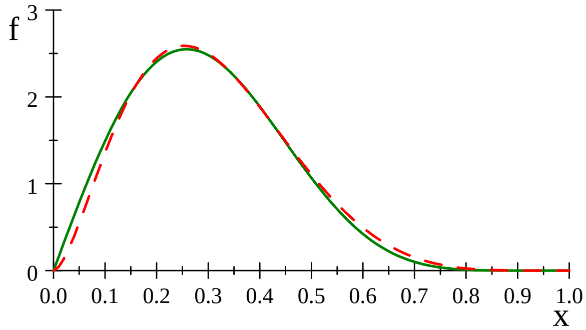

Figure 7

The first best rules, for different values of (α, β), are presented in Table 4. As observed, these distributions involve large optimal quotas that stand around 90%. Again, they are not directly related to the skewness of the distribution. For instance, when $\alpha = 5$ and β increases, so that the skewness is reduced, the optimal quota increases. However, when $\alpha = 3$ and the skewness is reduced by increasing β, we observe a reduction in the size of the optimal quota.

α	β	$mode$	x^m	$\mu = x^{fb}$	q^{fb}	l	r	$Skew.$
2	5	0.3333	0.3598	0.3694	0.9132	0.1341	0.6219	0.2560
2	6	0.3015	0.3303	0.341	0.9146	0.1215	0.5800	0.3132
2	7	0.2773	0.307	0.3183	0.9158	0.1117	0.5457	0.3529
2	8	0.2582	0.2881	0.2995	0.9168	0.1039	0.5168	0.3838
3	5	0.5228	0.5059	0.5007	0.9026	0.2727	0.7194	−0.1212
3	6	0.4900	0.4778	0.4743	0.8973	0.2616	0.6810	−0.0778
3	7	0.4641	0.4551	0.4528	0.8871	0.2570	0.6445	−0.0459
3	8	0.4430	0.4362	0.4347	0.8669	0.2605	0.6063	−0.0212
4	5	0.6304	0.5998	0.5884	0.9127	0.3668	0.7882	−0.3449
4	6	0.6100	0.5747	0.5648	0.91282	0.3505	0.7603	−0.3055
4	7	0.5773	0.5541	0.5454	0.91285	0.3373	0.7365	−0.2766
4	8	0.5577	0.5367	0.5288	0.9128	0.3263	0.7160	−0.2544
5	5	0.6988	0.6644	0.6504	0.9164	0.4442	0.8289	−0.4956
5	6	0.6729	0.6420	0.6294	0.9167	0.4281	0.8056	−0.4583
5	7	0.6518	0.6236	0.6119	0.9170	0.4149	0.7856	−0.4209
5	8	0.6342	0.6079	0.5970	0.9172	0.4039	0.7682	−0.4101

Table 4

4 Conclusions

In the context of a multilateral one-dimensional bargaining game, we used numerical simulations to analyze the performance of alternative (super)majority rules in achieving outcomes that maximize collective surplus. Our exercise is based on the uniqueness of equilibrium (see Cardona and Ponsatí 2011, 2012) when negotiations take place over time through a process of alternating proposals, where the proposer is randomly selected. For the specific case of uniform recognition probabilities and quadratic preferences, our computations show that, for standard specifications of the population, the optimal rule is always higher than $q = 0.8$ and smaller that $q = 0.95$.

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