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First-price vs second-price auctions under risk aversion and private affiliated values

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Abstract

Under a specific informational framework, we compare the seller's expected revenue from a first-price auction and a second-price auction when bidders are risk averse and have private affiliated values.

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1 Introduction

The pioneer article of Vickrey (1961) has established that the four main auction mechanisms[1] yield the same expected revenue for the seller. However, the Vickrey theorem only holds under the specific assumptions of the independent private value (hereafter IPV) paradigm. Namely, bidders are assumed to adopt symmetric strategies, are risk neutral and the value of the auctioned object for each bidder is known to himself but not to other bidders whose valuations are independent. The IPV model can be restrictive in practice as bidders may have private values related to each other. Besides, when bidders are risk averse *e.g.*, the Vickrey theorem no longer holds since the first-price sealed-bid auction (hereafter FPA) or the dutch auction yields a higher expected revenue for the seller than the second-price sealed-bid auction (hereafter SPA) or the english auction.[2]

Another polar case mainly analyzed in the literature is the common value (hereafter CV) model. This model assumes that the value of the auctioned object is the same but unknown to all bidders who have different (private) estimates of this common value. Although the IPV model recognizes differences in individuals' preferences, the CV model does not. Thus, as noticed by Campo et al. (2003) and Li et al. (2002) many auction situations may not correspond to either one of these two polar cases. Your value for a painting *e.g.* may depend on both a private information (how much you like it) and on others private valuations because this affects the resale value.[3]

Milgrom and Weber (1982) provide a more realistic and general model of auctions that allows for affiliated values. Positive affiliation *e.g.* means that higher values of the item for one bidder make higher values for other bidders more likely. Namely, they show that when bidders are risk neutral and have affiliated values, the SPA (or the english auction) yields a higher expected revenue than the FPA (or the dutch auction). However, in their general model, no clear qualitative comparison can be made between the FPA and the SPA in the presence of risk aversion and affiliated values. *The comparison between the FPA and the SPA in terms of expected revenue is the main goal of this paper.*

We analyze this comparison in a setting where bidders have affiliated valuations. However, we consider that these valuations are private. This means that each bidder has perfect information regarding his own value for the object at auction, but higher values of the item for one bidder make higher values for other bidders more likely. As noted by Kagel et al. (1987) "a simple example of an auction with affiliated private values would be a charity fundraiser of consumer perishables, where an item unusually appealing to you is tipically more appealing to other bidders as well". The affiliated private values model[4] seems to be quite realistic since it allows for dependence among bidders private values while retaining a bidders private valuation for the auctioned object. We also consider that bidders are risk averse. This assumption seems to be reasonable since bidders face many uncertainties. It has been tested extensively on experimental data. Bajari and Hortacsu (2005) *e.g.* show that the risk aversion model provides the best fit for experimental data. Perrigne and Vuong (2007) provide an estimation of bidders' risk aversion in first-price auctions. Namely, they find significant risk aversion in the case of timber auction data.

Following Von Ungern-Sternberg (1991) (hereafter VUS), we adopt a simple infor-

[1] The first-price sealed-bid auction, the dutch auction, the second-price sealed-bid auction and the english auction.

[2] see *e.g.* Milgrom and Weber (1982), Maskin and Riley (1984) or Matthews (1983).

[3] See *e.g.* Klemperer (1999) for a survey on auction theory.

[4] See. *e.g.* Li et al. (2002) for a structural estimation of the symmetric affiliated private value model.

mational paradigm. VUS's model actually combines the simplifying properties of the standard independent private values model and the common value model. Indeed, each bidder learns his own valuation with certainty and, as in the simplest version of the common value model, each bidder has no grounds for believing his own valuation estimate to be higher or lower on average than his competitor's valuations. Formally, VUS models this by assuming that the different bidders' valuations are independent drawings from a *known* distribution with an *unknown* mean.[5] Then, since the bidders know the functional form of the distribution but are uncertain about the mean, they will infer this mean from their own private valuation.

VUS's model considers symmetric risk neutral bidders. In this paper, we extend the analysis of VUS to the case of risk averse buyers. Then we compare the seller's expected revenue from an FPA and an SPA. The paper is organized as follows. In the next section, we detail the model assumptions. In section 3, we analyze the FPA. In section 4, we compute the expected revenue from an SPA. Section 5 yields a comparison between both auction mechanisms. Section 6 offers concluding remarks.

2 The assumptions of the model

We consider a model with $n \geq 2$ bidders. Each bidder has a private valuation v. All bidders are ex ante symmetric relative to the informational knowledge and believe that their private valuations are drawn from a distribution F over $[\mu - a, \mu + a]$, where a is *common knowledge* whereas the mean μ is *unknown*. Thus, when a bidder, say i, learns his own valuation v_i, he can infer that $\mu \in [v_i - a, v_i + a]$ according to the cumulative F_μ with corresponding density f_μ. Since all the bidders are ex ante symmetric relative to the informational knowledge, bidder i then infers that the evaluation of an opponent, say j, $v_j \in [v_i - 2a, v_i + 2a]$ according to the cumulative distribution F_j with corresponding density f_j. In order to highlight the impact of risk aversion on bidding strategies, let us assume that each bidder i is characterized by a constant relative risk aversion (CRRA) utility function $U(x_i) = x_i^\rho$ (with $0 < \rho \leq 1$), where $1 - \rho$ is the CRRA parameter of each bidder and x_i is bidder i's income.

3 The first-price sealed-bid auction

We first consider an FPA. As noted by VUS, the assumption that a bidder's valuation is drawn from a known distribution of unknown mean introduces a substantial simplification relative to the standard IPV model (where the mean of the distribution is assumed to be known). In the IPV model each bidder's own valuation tells him something about his position relative to his competitors. As a result, the bidder chooses a mark-up (or profit) which depends on his own valuation. In VUS's model, where the mean of the distribution is not known, the fact that a bidder privately knows his own valuation does not reveal him anything about his relative position and so does not affect his winning probability. Therefore, there is no reason why he should let his strategic mark-up depend on his own valuation. His optimal bidding strategy is therefore very simple: bid his own valuation minus a constant mark-up (which is independent of bidders' valuations). Hence, if we consider the case of bidder i, we can assume that his equilibrium bid has the following form

[5]This prior belief is described by Biais and Bossaerts (1998) as the "average opinion rule."

$$B_i\left(v_i\right) = v_i - b_i,$$

where b_i represents his markup (or profit). Then, under an FPA, bidder i's expected utility is given by

$$EU_i = (b_i)^{\rho} P(b_i, b_j), \tag{1}$$

where $P(b_i, b_j)$ reflects his probability of winning when he chooses a strategic mark-up b_i, while the other bidders choose b_j. Differentiating (1) with respect to b_i yields the optimal markup for bidder i

$$\begin{aligned}
\frac{\partial EU_i}{\partial b_i} &= (b_i)^{\rho} P'(b_i, b_j) + \rho {b_i}^{(\rho-1)} P(b_i, b_j) = 0 \\
&\Leftrightarrow \; b_i P'(b_i, b_j) + \rho P(b_i, b_j) = 0 \\
&\Leftrightarrow \; b_i = -\frac{\rho P(b_i, b_j)}{P'(b_i, b_j)}.
\end{aligned} \tag{2}$$

Let us now derive the probability of winning. Bidder i wins against bidder j if $v_i - b_i > v_j - b_j$, *i.e.* if $v_j < v_i - b_i + b_j$, which occurs with probability $F_j(v_i - b_i + b_j)$. Then, ex ante, bidder i wins against $(n-1)$ other bidders with probability P (defined in expectation over the unknown mean μ) such that

$$P = \int_{\mu} (F_j(v_i - b_i + b_j))^{n-1} f_{\mu}(\mu) d\mu.$$

In order to provide an explicit form of the probability of winning, we now consider the special case of a uniform distribution for F (and therefore for F_j andF_{μ}). Then, we have

$$P = \int_{\mu} \left(\frac{a - b_i + b_j + v_i - \mu)}{2a}\right)^{n-1} \frac{1}{2a} d\mu.$$

To compute this probability, assume[6] e.g. that $b_i \geq b_j$. Then $P \geq 0$ if $\mu \leq v_i + a - b_i + b_j$ and $P \leq 1$ if $\mu \geq v_i - a - b_i + b_j$. Since $b_i \geq b_j$ and given that μ lies over the interval $[v_i - a, v_i + a]$, one thus has to integrate only over the interval $[v_i - a, v_i + a - b_i + b_j]$. Then P becomes

$$\begin{aligned}
P &= \int_{v_i - a}^{v_i + a - b_i + b_j} \frac{1}{2a} \left(\frac{a - b_i + b_j + v_i - \mu)}{2a}\right)^{n-1} d\mu \\
&= \frac{1}{n} \left(\frac{2a - (b_i - b_j)}{2a}\right)^{n}.
\end{aligned} \tag{3}$$

Given (3) and the first order condition (2), the unique symmetric equilibrium for the strategic markups satisfies[7]

$$b_i = b_j = \frac{2a\rho}{n}.$$

Note that the optimal strategic markup of a bidder is increasing with respect to the uncertainty parameter a, decreasing with his risk aversion level and decreasing with the

[6] The same reasoning can be applied with $b_i < b_j$.

[7] We check that the second order conditions are satisfied.

number of bidders. These results are consistent with intuition and conventional results in auction theory.[8]

Given the bidders' optimal strategies, we can now compute the seller's expected revenue from an FPA

$$\begin{aligned} ER_{FPA} &= \int_{\mu-a}^{\mu+a} vnf(v)(F(v))^{n-1}\,dv - \frac{2a\rho}{n} \\ &= \mu + a - \frac{2a}{n+1} - \frac{2a\rho}{n}, \end{aligned} \tag{4}$$

where $nf(v)(F(v))^{n-1}$ is the frequency distribution of the highest valuation among n bidders.

4 The second-price sealed-bid auction

Under an SPA, the winner pays a price equal to the second highest bid. Given this payment rule, it is well known that it is a dominant strategy for a bidder to place a bid equal to his true valuation. Therefore, the seller gets a price equal to the expected second-highest valuation. The frequency distribution of the second-highest valuation is

$$n(n-1)f(v)(1-F(v))(F(v))^{n-2}.$$

Then, the expected revenue from an SPA can be written as

$$\begin{aligned} ER_{SPA} &= \int_{\mu-a}^{\mu+a} vn(n-1)f(v)(1-F(v))(F(v))^{n-2}\,dv \\ &= \mu + a - \frac{4a}{n+1}. \end{aligned} \tag{5}$$

5 The revenue comparison

We now compare the seller's expected revenue from both auctions. We first consider the case of risk neutral bidders (*i.e.* $\rho = 1$). We can compute

$$ER_{FPA} - ER_{SPA} = \frac{-2a}{n(n+1)} < 0. \tag{6}$$

Notice that even if the seller's expected revenue from an FPA or an SPA depends on the true value of the mean, the difference $ER_{FPA} - ER_{SPA}$ does not depend on this parameter. So, when the bidders observe the procedure chosen by the seller, they cannot infer anything concerning μ. Therefore, the strategies depicted in the previous sections remain the optimal ones. From (6), we have the following lemma:

Lemma 1 *In a setting with risk neutral bidders whose valuations are drawn from a uniform distribution, the SPA yields a higher expected revenue than the FPA.*

This lemma can be easily interpreted. In an FPA, when $\rho = 1$, each bidder chooses a markup equal to $2a/n$. Then, from (4), the expected revenue from an FPA is equal to the expected highest valuation minus $2a/n$. In an SPA, the seller's expected revenue

[8] See among others Klemperer (2001) or Krishna (2002).

is equal the expected second highest valuation among n bidders whose valuations are uniformly distributed in the interval $[\mu - a, \mu + a]$ of length $2a$. This expected second highest valuation is equal to the expected highest valuation minus $2a/(n+1)$. Since $2a/n > 2a/(n+1)$, the seller's expected revenue is higher in an SPA than in an FPA.

The result of Lemma 1 is consistent with the more general result of Milgrom and Weber (1982) which shows that when bidders are risk neutral and have affiliated values, the SPA yields a higher expected revenue than the FPA. However, as noted by Milgrom and Weber (1982), there is no clear qualitative comparison between FPA and SPA in the presence of risk aversion. In order to compare these two auction mechanisms in the case of risk averse bidders (*i.e.* $0 < \rho < 1$) having private affiliated values, we can compute

$$ER_{FPA} - ER_{SPA} = 2a\left(\frac{1}{n+1} - \frac{\rho}{n}\right).$$

Then, we have the following proposition:

Proposition 1 *Suppose that bidders have a CRRA utility function and that their valuations are drawn from a uniform distribution. When $\rho <$ (resp. $>$)$\frac{n}{n+1}$, the FPA yields a higher (resp. lower) expected revenue than the SPA.*

This result is consistent with the experiments of Kagel et al. (1987) which show that the revenue-raising possibilities inherent in second price auction, relative to a first price auction, with positively affiliated values, are severely compromised by the potential for risk averse bidding. Besides, we can give an interpretation of proposition 1. In an SPA, the seller's expected revenue does not depend on the bidders' level of risk aversion. However, in an FPA, when bidders are risk averse, the markup is now equal to $2a\rho/n$; raising one's bid slightly in an FPA is analogous to buying partial insurance. Following the same interpretation as for lemma 1, the FPA yields a lower expected revenue for the seller if $\frac{2a\rho}{n} < \frac{2a}{n+1}$. Notice that the cut-off value of ρ (i.e. $\frac{n}{n+1}$) is increasing with n. Actually, when n is increasing, the bidders' markup are decreasing. This is the result of a competition effect. So, the difference between the seller's expected revenues from both auctions is also decreasing. Then, with a large number of bidders, the seller is better off with an FPA even if the bidders exhibit a low level of risk aversion.

6 Conclusion

This paper has considered a revenue comparison between an FPA and an SPA when bidders are risk averse and have private affiliated values. Under a specific informational framework which combines the independent private value model and the common value model, we have provided conditions under which one auction mechanism out-performs the other.

References

[1] Bajari P. and Hortacsu A. (2005) "Are Structural Estimates of Auction Models Reasonable? Evidence from Experimental Data", *Journal of Political Economy*, Vol. 113 (4), pp. 703-741.

[2] Biais B. and Bossaerts P. (1998) "Asset Prices and Trading Volume in a Beauty Contest", *Review of Economic Studies*, Vol. 65, pp. 307-340.

[3] Campo S., Perrigne I., and Vuong Q. (2003) "Asymetry in first-price Auctions with Affiliated Private Values", *Journal of Applied Econometrics,* Vol. 18, pp. 179-207.

[4] Kagel J., Harstad R. and Levin D. (1987) "Information Impact and Allocation rules in Auctions with Affiliated Private Values: A Laboratory Study", *Econometrica,* Vol. 55, pp. 1275-1304.

[5] Klemperer, P.D. (1999) "Auction theory: A guide to the literature", *Journal of Economic Surveys*, Vol. 13, pp. 227–286.

[6] Krishna, V. (2002) *Auction theory*, Academic Press.

[7] Li T., Perrigne I., and Vuong Q. (2002) "Structural estimation of the affiliated private values model", *Rand Journal of Economics,* Vol. 33, pp. 171-193.

[8] Maskin E., and Riley J. (1984) "Optimal Auctions with Risk Averse Buyers", *Econometrica,* Vol. 52 (6), pp. 1473-1518.

[9] Matthews S.A. (1983) "Selling to Risk Averse Buyers with unobservable Tastes", *Journal of Economic Theory,* Vol. 30 (3), pp. 370-400.

[10] Milgrom E., and Weber J. (1982) "A Theory of Auctions and Competitive Bidding", *Econometrica,* Vol. 50 (5), pp. 1089-1122.

[11] Perrigne I., and Vuong Q. (2007) "Identification and Estimation of Risk Aversion in Auctions", *American Economic Review, Papers and Proceedings,* Vol. 97 (2), pp. 444-448.

[12] Vickrey W. (1961) "Couterspeculation, Auctions and Competitive Sealed Tenders", *Journal of Finance,* Vol. 16, pp. 8-37.

[13] Von Ungern-Sternberg T. (1991) "Swiss Auctions", *Economica,* Vol. 58, pp. 341-357.