Economics Bulletin

Volume 32, Issue 1

Multiplicity of Equilibria in Multi-Unit Demand Sequential Auctions under Complete Information

Mohamed Jeddy
CREATE, Université Laval

Bruno Larue
CREATE, U. Laval

Abstract

We show that the result on the existence of a unique Nash perfect equilibrium in two-bidder multi-unit sequential second-price auctions under complete information (as in Krishna, 1993; Katzman, 1999; and Gale and Stegeman, 2001) is not robust in higher dimensional auctions. Using an example featuring three bidders competing for four objects, we found two equilibria characterized by different vectors of prices and allocations.

Senior authorship is unassigned. Funding from the SPAA research network is gratefully acknowledged.
Citation: Mohamed Jeddy and Bruno Larue, (2012) "Multiplicity of Equilibria in Multi-Unit Demand Sequential Auctions under Complete Information", *Economics Bulletin*, Vol. 32 No. 1 pp. 456-465.
Contact: Mohamed Jeddy - mohamed.jeddy@fsaa.ulaval.ca, Bruno Larue - Bruno.Larue@eac.ulaval.ca.
Submitted: April 22, 2011. **Published:** January 30, 2012.

1. Introduction

In many real world auctions, the seller is poorly informed while each bidder is well informed, not only about the item(s) being auctioned and hence his own valuation, but also about his competitors' valuations. Bernheim and Whinston (1986) use the example of a few firms relying on a common technology and routinely bidding on construction contracts to justify the complete information assumption. Gale and Stegeman (2001, p.75) argue that the assumption is justified in cases where two well informed sellers bid sequentially for contracts, like for waste disposal, consulting services and military hardware. Electronic livestock auctions are also good examples. The Quebec hog auction was in operation every day except on weekends between 1989 and 2008 as seven meat processors were competing on virtual fixed-size lots of hogs scoring 100 on a quality index.[1]

The analysis of multi-unit demand sequential auction under complete information with more than two asymmetric bidders has been largely ignored in the literature possibly because of a presumption that results for two bidders could be generalized to the *k*-bidder case. Important contributions on multi-unit auctions under complete information by Krishna (1993), Katzman (1999), Gale and Stegeman (2001) and Rodriguez (2009) demonstrate the existence of a unique Nash perfect equilibrium when there are two bidders. Jeddy, Larue and Gervais (2010) analyzed price trends and allocations when *k* bidders have identical decreasing valuations. They found that symmetric allocations and constant price trends are supported by rather stringent conditions on the declining pattern of valuations. Thus, unique asymmetric allocations are the most common equilibrium outcomes in this setting.

As is common in the auction literature (e.g., Engelbrecht-Wiggans, 1999), we rely on a numerical example for a sequential second-price auction involving three bidders and four objects to show that equilibrium uniqueness charactering 2-bidder auctions under complete information, does not hold generally.

2. The model and discussion

Consider a sequence of four second-price auctions where three individual bidders have diminishing marginal valuations such that: $V_1^j > V_2^j > V_3^j > V_4^j \ \forall j = A,\ B,\ C$ where V_i^j is the i^{th} valuation of bidder *j*. They compete for four homogenous objects under complete information. In the example that follows, we will use $V^A = \{20,\ 15,\ 14,\ 12\}$, $V^B = \{18,\ 13,\ 10,\ 5\}$ and $V^C = \{17,\ 11,\ 9,\ 3\}$. The seller is non-strategic and sets a reserve price equal to zero. The strategic behaviour of bidders in second-price multi-unit sequential auctions under complete information was characterized by Krishna (1993, 1999), Katzman (1999), Gale and Stegeman (2001) and Jeddy, Larue and Gervais (2010). Each bidder is assumed to follow the weakly dominant strategy of sincere bidding in the last and 4$^{\text{th}}$ round.

[1] When hogs were delivered to a plant, a quality grid was used to make price adjustments for hogs scoring below or above 100. Therefore, quality issues were internalized. Furthermore, it is not heroic to assume that each meat processor knows the production capacity, cost structure and market opportunities of other meat processors.

For k < 4, it is a weakly dominant strategy for each bidder to place a bid in the k^{th} round that would make him indifferent between winning and losing the k^{th} round, considering the contingent outcomes from the $(k+1)^{th}$ to the 4^{th} rounds. The game is solved by backward induction and in 2-bidder auctions, the equilibrium is unique. As we will see below, higher dimensions bring complexities and multiple equilibria.

When there are $j > 2$ bidders, some bidders may not matter. In a second-price auction of a single object under complete information, the unique equilibrium has bidder A winning the object at price $p = V_1^B$, provided that $V_1^A > V_1^B > V_1^C > ... > V_1^J$. The outcome would be the same whether some bidder(s) j, $j \neq \{A, B\}$, is (are) bidding or not, which contrasts with the case when bidders are incompletely informed about their rivals' valuations. In this one-object auction, bidders C,..., J are in a "non-strategic" position because their valuations have no impact on the equilibrium price. The same insight applies in higher dimensional auctions with J bidders and n objects. Intuitively a bidder with low valuations is more likely to matter once bidders with high valuations have won some objects. Accordingly, if a bidder cannot win or influence the price at one of the J^{n-1} nodes at the bottom of the outcome tree of the game, then the bidder is said to be in a non-strategic position.

Figures 1-3 illustrate three parts of the outcome tree for our 3 bidder-4 object auction. In Figure 1, it is assumed that the 1^{st} object has been won by bidder A. Starting at node A, the 2^{nd} object can be won by bidder A, bidder B or bidder C, hence the nodes AA, AB and AC. At node AAA, it is assumed that the first three objects have been won by A. If bidder A was to win the 4^{th} and last object, his gross payoff would be the sum of his valuations for objects won, 61, minus the sum of prices of objects won in subsequent rounds, which is zero given that this is the last round. Losing the 4^{th} object to bidder B or bidder C entails a payoff of 49. Therefore, bidder A is willing to bid 61-49=12 for the 4^{th} and last object while bidders B and C are willing to bid 18-0=18 and 17-0=17, respectively. Conditional on bidder A winning the first three objects, bidder B would win the last object and pay bidder C's bid of 17, hence the arrow emanating from AAA and pointing toward the gross payoff associated with bidder B winning the last object. At node ACC, the 1^{st} object has been allocated to bidder A and the next two to bidder C. Competing for the last object, bidders, A, B, and C would bid (35-20=15), (18-0=18) and (37-28=9), respectively. Bidder B would win and pay bidder A's bid. Because the bids that matter at nodes AAA and ACC involve bids by all three bidders, we cannot simplify the game by discarding one or more bidders on the ground that they have non-strategic positions.

The arrows emanating from the nodes AAA, AAB, …, CCC in Figures 1-3 show all of the conditional fourth object allocations. Conditional on the allocation of the first two objects and the contingent allocation of the 4^{th} object, we can then analyze the allocation of the 3^{rd} object. To do that, we begin by computing the gross payoffs at nodes AAA, ..., CCC. For example, the gross payoff vector at node AAA is the gross payoff at AAAB adjusted for the price of the 4^{th} object won by bidder B: $\begin{pmatrix} 49 \\ 18-17 \\ 0 \end{pmatrix} = \begin{pmatrix} 49 \\ 1 \\ 0 \end{pmatrix}$. Conditional on A winning the first two objects, the gross payoffs associated with bidder B winning the 3^{rd} object (and C winning the fourth), at node AAB, and C winning the 3^{rd} object (and B winning the fourth), at node AAC, are respectively given by $\begin{pmatrix} 35 \\ 18 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 35 \\ 4 \\ 17 \end{pmatrix}$. Unlike in the last round, the opportunity cost of losing is no longer invariant. Bidder B is willing to pay as much as 14 to prevent C from winning and as much as 17 to prevent A from winning. Similarly, bidder C is willing to spend 17 to prevent A from winning and 14 to prevent B from winning. These variations in the opportunity costs occur because of the contingent allocations of the 4th object. However, bidders B and C know that bidder A will bid only 14 and that they do not need to bid as much as 17 to counter him. It is also clear that they have an incentive to bid $14+\varepsilon$ as close as possible to, but in excess of, 14 given that bidder B would prefer to lose to bidder C and vice versa.[2] Hence, conditional on bidder A winning the first two objects, the price for the 3rd object would be $14+\varepsilon$ and there would be two potential equilibria: one with bidders B and C winning the 3rd and 4th objects and one with bidders C and B winning the 3rd and 4th objects. The adjusted gross payoff vectors at node AA are $\begin{pmatrix} 35 \\ 4-\varepsilon \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 35 \\ 4 \\ 3-\varepsilon \end{pmatrix}$. At node AB, bidders A and C have an incentive to bid in excess of 13 to prevent B from winning, but because bidder C is willing to pay as much as 14 to prevent bidder A from winning, bidder A has no incentive bid in excess of 13. Bidder C wins with a bid of 14 and pays 13. At node AC, the 3rd and 4th objects are won by bidders B and A. Bidders A and B know that bidder C will bid only 11 and hence is not in a strategic position in this subgame. Since bidder B (A) is willing to pay as much as 14 (13) to prevent A (B) from winning, bidder B wins and pay 13. The adjusted gross payoffs at nodes AB and AC are $\begin{pmatrix} 22 \\ 18 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 22 \\ 5 \\ 17 \end{pmatrix}$. Conditional on the 1st object being allocated to bidder A and the contingent allocations for the 3rd and 4th objects, we can then analyze the allocation of the second object, noting from Figure 1 that bidder A is willing to bid 35-22=13 while bidder B is willing to bid $18\text{-}4+\frac{\varepsilon}{2}=14+\frac{\varepsilon}{2}$ to counter A and 18-5=13 to counter C. Bidder C is willing to bid $17\text{-}3+\frac{\varepsilon}{2}=14+\frac{\varepsilon}{2}$ to counter A and 17-4=13 to counter B. This is so because risk neutral bidders rely on expected payoffs to compute their bids. At node AA, either bidder B or C can win the third object by paying $14+\varepsilon$. Thus, the vector

[2] Bidder B (C) would bid 14 if he was sure that bidder C (B) would bid more than 14. Since bidders bid simultaneously and noncooperatively, bidders B and C must insure that bidder A does not win and cannot take the chance to bid only 14. Because they are concerned with the winner's curse, they bid $14+\varepsilon$, where ε is the smallest possible increment and each has a probability of winning of 0.5

of expected payoffs at this node is $\begin{pmatrix} \frac{35+35}{2} \\ \frac{4-\varepsilon+4}{2} \\ \frac{3+3-\varepsilon}{2} \end{pmatrix} = \begin{pmatrix} 35 \\ 4-\frac{\varepsilon}{2} \\ 3-\frac{\varepsilon}{2} \end{pmatrix}$. Knowing that a bid of 13+ε as close as possible to, but in excess of, 13 is best, both bidders B and C can win the 2nd object, conditional on bidder A winning the 1st object. There are two possible equilibria: one with bidders B, C and A winning the 2nd , 3rd and 4th objects and one with bidders C, B and A winning the 2nd, 3rd and 4th objects. The corresponding adjusted gross payoff vectors at node A are $\begin{pmatrix} 22 \\ 5-\varepsilon \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 22 \\ 5 \\ 4-\varepsilon \end{pmatrix}$, respectively.

The same sort of reasoning is used to move up the part of the outcome tree depicted by Figure 2. Conditional on the first two objects being allocated to bidders B and C (i.e., node BC), there are two possible allocation paths for the 3rd and 4th objects. Bidder A is willing to pay as much as 15 to prevent bidder C from winning the 3rd object, but bidder C's bid of 11 can be beat by a bid of 13 which is what bidder A is willing to pay to prevent bidder B from getting the 3rd object. Since 13 is also the bid of bidder B, bidders A and B are equally likely to win the third object, conditional on bidders B and C winning the first two. At node BA, bidder B's bid is 13. Bidder C is willing to pay as much as 14 (15) to prevent bidder A (C) from winning the object. Knowing that C will outbid B, bidder A is indifferent between paying 13 and letting C win. Hence C wins and pays 13. Conditional on B winning the first object (i.e, node B), bidders A and B bid 13. Bidder C's payoff is highest if bidder B wins and hence he must bid strictly less than 13. Thus, conditional on bidder B getting the first object, the second object would be won by bidders A and B with equal probabilities.

In Figure 3, conditional on the first object being allocated to bidder C, the remaining three objects would be allocated either to bidders A, B and A or to bidders B, A and A at a constant price of 13; or to bidders B, B and A at prices 13, 13 and 11, respectively.

To determine the allocation of the first object and hence the equilibrium paths, we need to compare adjusted gross payoff vectors at nodes A, B and C: $\begin{pmatrix} 22 \\ 5-\varepsilon \\ 4 \end{pmatrix}$ or $\begin{pmatrix} 22 \\ 5 \\ 4-\varepsilon \end{pmatrix}$ if bidder A wins, $\begin{pmatrix} 9 \\ 18 \\ 4 \end{pmatrix}$ or $\begin{pmatrix} 9 \\ 18 \\ 6 \end{pmatrix}$ if bidder B wins and $\begin{pmatrix} 9 \\ 5 \\ 17 \end{pmatrix}$ if bidder C wins. Because bidders are risk neutral and rely on expected payoffs to compute their bids, we can infer that bidder A has an incentive to bid 13 to counter bidders B and C, but bidder B is willing to pay 13 to prevent C from winning and 13+$\frac{\varepsilon}{2}$ to counter bidder A. Bidder C has an expected payoff of 5 if B wins and hence is willing to pay only 12 to prevent B from winning. Like B, bidder C is willing to pay $13+\frac{\varepsilon}{2}$ to prevent A from winning. Accordingly, bidder C has no incentive to bid as high as bidder B. Therefore, bidder B wins the first object by bidding $13+\frac{\varepsilon}{2}$ and pays 13, the bids of the other two bidders. The entire game has two equilibrium allocations given by: E_1 = (B, A, C, A) with vector of prices (13, 13, 13, 13) and E_2 = (B, B, C, A) with vector of prices (13, 13, 11, 11) generating payoff vectors $\begin{pmatrix} 9 \\ 5 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 9 \\ 5 \\ 6 \end{pmatrix}$.

Proposition. *In multi-unit demand second-price sequential auction under complete information, there can be more than one pure strategy Nash perfect equilibrium.*

The uniqueness property in Katzman (1999) 2-bidder and 2-object sequential auction and in Gale and Stegeman (2001) 2-bidder and *n* object auctions does not generalize,[3] but as in Katzman (1999), an inefficient equilibrium characterized by a declining price pattern can emerge.

As in the 2x2 auctions, bidders in *J*-bidder auctions may find it optimal to concede objects to exploit the declining valuations of rivals. In a 2x2 auction, this sort of strategy is tantamount to choosing an allocation path. When there are several bidders, a bidder conceding an object may only increase the probability of his preferred path even if bidders have asymmetric valuations as in our example. As a result, multiple equilibria are likely.

3. Conclusion

We analyze multi-unit demand sequential second-price auction under complete information with asymmetric bidders. We rely on a three bidder - four object example to show that the result about equilibrium uniqueness in 2 bidder - *n* object case (e.g., Gale and Stegeman, 2001) is not robust. The implication is that different allocations may be observed in frequently repeated auctions involving the same bidders even if their valuations do not change. Casual empirical evidence from the Quebec daily hog auctions between February 1st of 2006 and August 31th of 2006 supports this hypothesis. The coefficient of variation for U.S. hog price over this period is 0.09. Given that the Canadian and US markets are highly integrated, the US price is a proxy for the variability of the market. The relative stability of the market over this short period suggests that processors' valuations probably did not change much. Yet, the coefficient of variation of the Herfindahl index, which captures changes in the allocations on the auction, is 0.23. This evidence does not constitute a formal test, but it is consistent with multiplicity of equilibrium allocations in the daily sequential auctions.

[3] Cai et al. (2007) show that a pure strategic symmetric equilibrium does not exist in sequential auctions in which all bids are revealed after each auction and bidders have single-unit demand.

4. References

Bernheim, B. D. and M. D. Whinston (1986) “Menu Auctions, Resource Allocation, and Economic Influence” *Quarterly Journal of Economics* **101**, 1-32.

Cai, G., P. R. Wurman and X. Chao (2007) “The Non-existence of Equilibrium in Sequential Auctions When Bids are Revealed” *Journal of Electronic Commerce* **8**, 141-157.

Engelbrecht-Wiggans, R. (1999) “An example of Multi-Unit Auctions with Atypically Many Equilibria” *Economics Letters* **65**, 67-69.

Gale I. L. and M. Stegeman (2001) “Sequential Auctions of Endogenously Valued Objects” *Games and Economic Behavior* **36**, 74-103.

Jeddy M., B. Larue and J-P. Gervais (2010) “Allocations and Price Trends in Sequential Auctions under Complete Information with Symmetric Bidders” *Economics Bulletin* **30**, 429-436.

Katzman B. (1999) “A Two Stage Sequential Auction with Multi-Unit Demands” *Journal of Economic Theory* **86**, 77-99.

Krishna, K. (1993) “Auctions with Endogenous Valuations: The Persistence of Monopoly Revisited” *American Economic Review* **83**, 147-60.

Krishna, K. (1999) “Auctions with Endogenous Valuations: The Snowball Effect Revisited” *Economic Theory* **13**, 377-391.

Rodriguez, G. E. (2009) “Sequential Auctions with Multi-Unit Demands” *The B.E. Journal of Theoretical Economics* **Volume 9**: **Issue 1** (Contributions), Article 45.

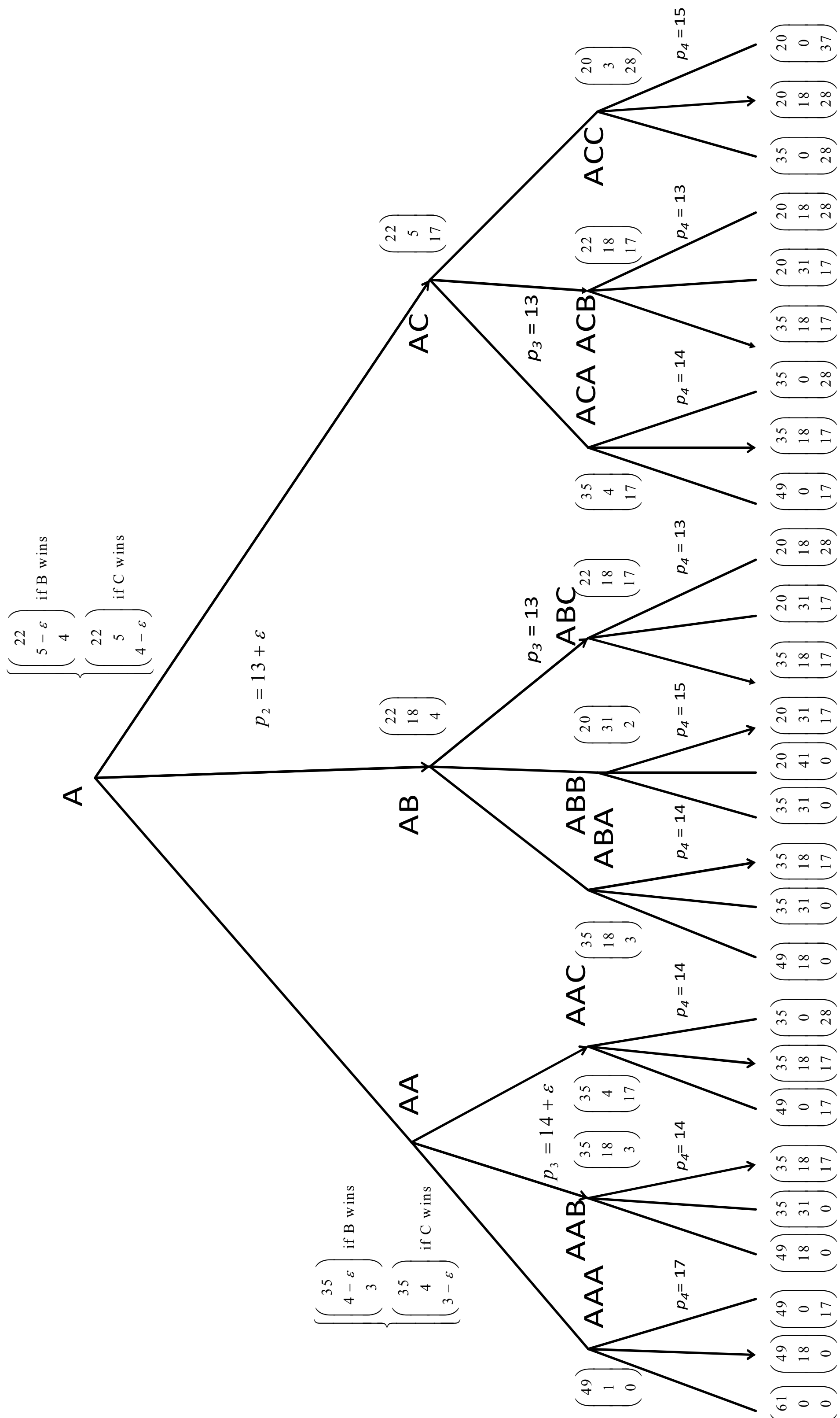

Figure 1. Part of the outcome tree, given the first object is allocated to bidder A.

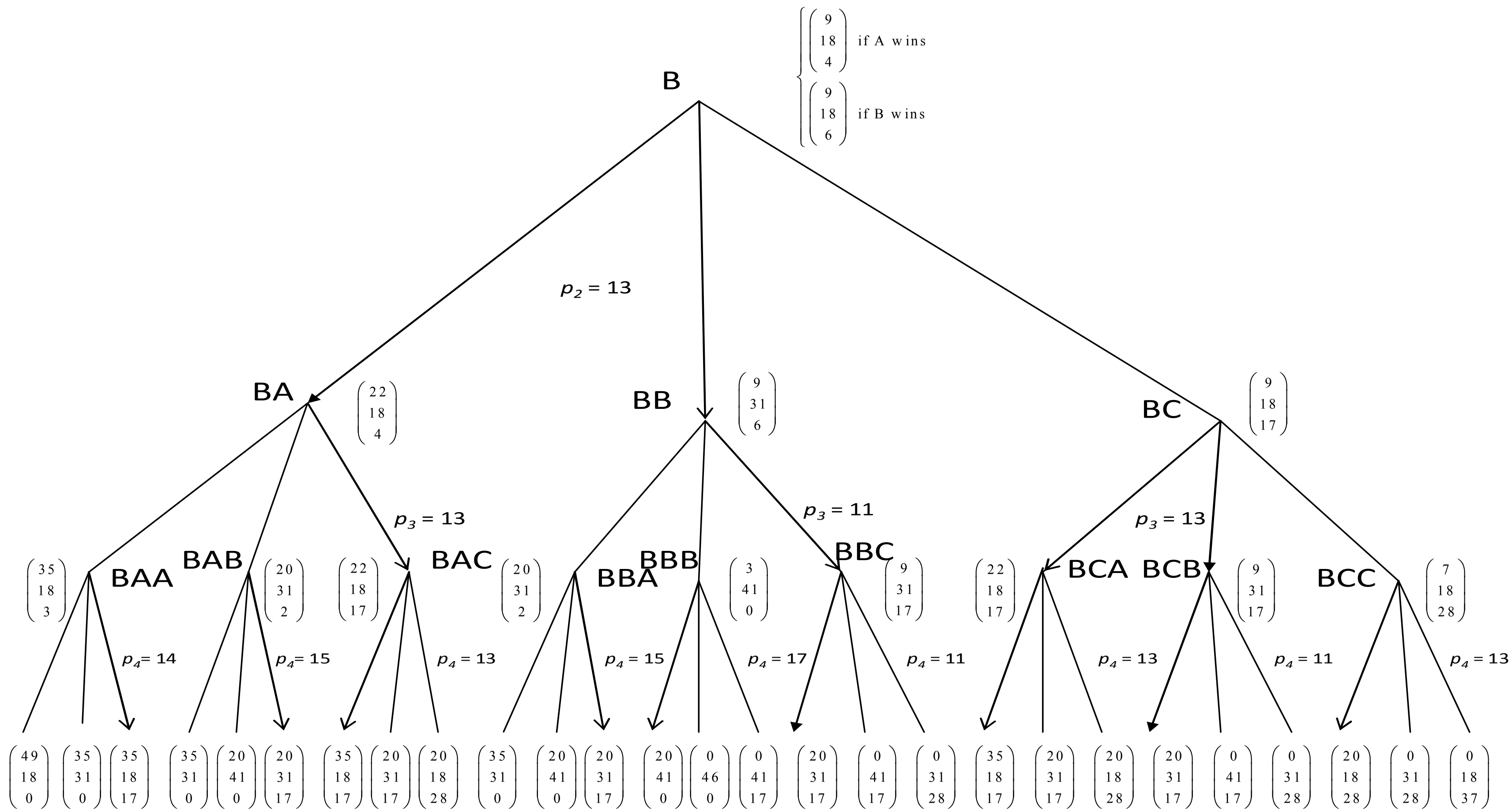

Figure 2. Part of the outcome tree, given the first object is allocated to bidder B.

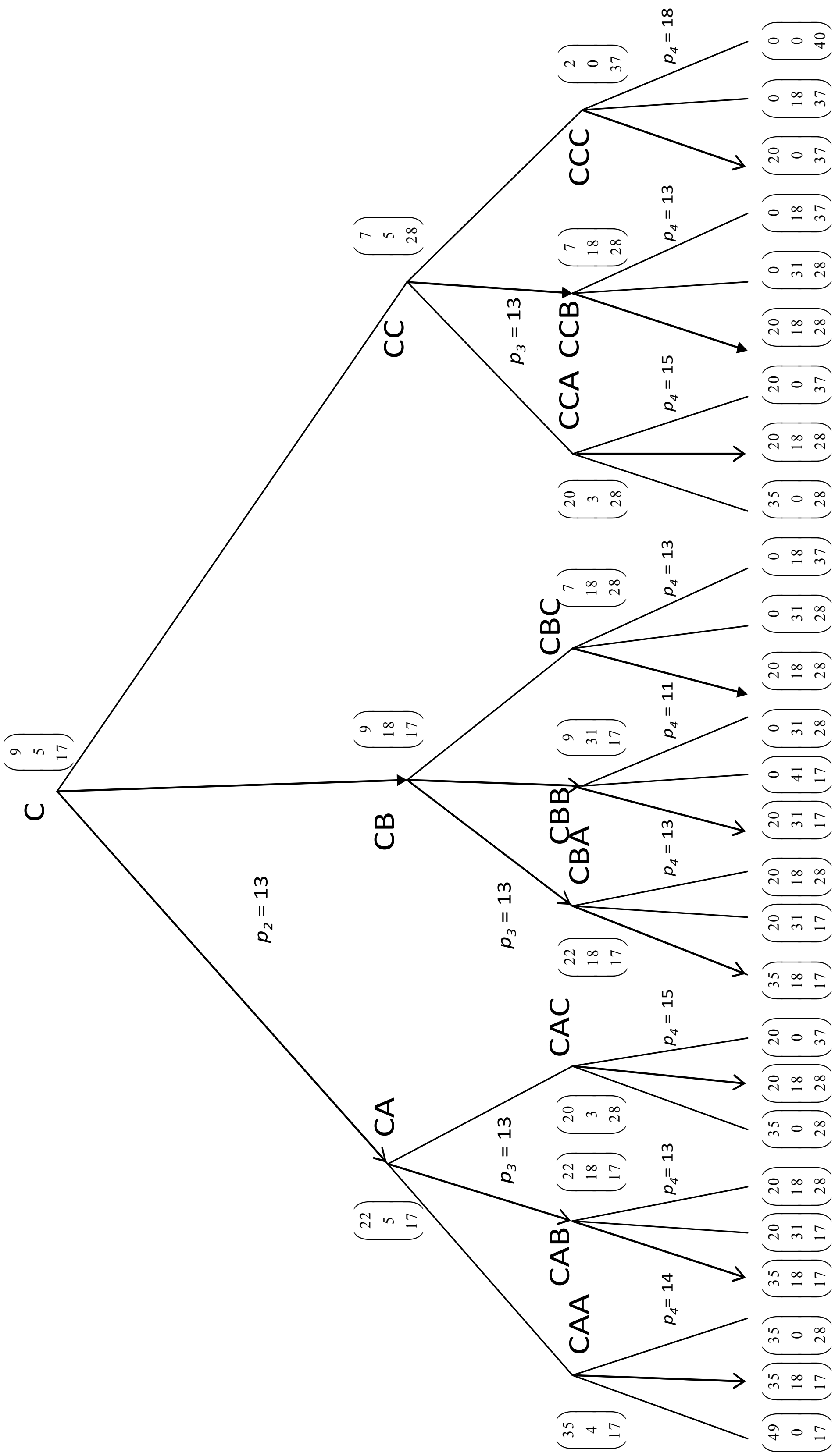

Figure 3. The outcome tree, given the first object is allocated to bidder C.