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Marshall`s Rules with Aggregate Inputs

Alberto Behar
World Bank

Abstract

We establish the formal link between the separability of inputs in a production function and the aggregate elasticity of demand for those inputs. This validates the implicit assumption used when calculating an aggregate elasticity with aggregated input prices and provides a practical approach to calculating an aggregate elasticity when one has disaggregated prices. We illustrate the approach to add to a thin empirical literature on labor demand elasticities in developing countries by using South African data.

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1. Introduction

Economists are interested in studying labor demand elasticities for a number of reasons. For example, they help understand the employment effects of policies that affect labor costs, like minimum wage legislation or payroll taxes. Cross-elasticities are also important; for example, they can measure whether higher capital costs increase or decrease the demand for labor. Based on early work by Marshall (1920) and Allen (1938), many studies have estimated own- and cross-elasticities of labor demand. See Hamermesh (1993) and Cahuc & Zylberberg (2004) for reviews.

In many cases, data are only available for one homogeneous group. Even if there are a number of labor categories, it is inevitable that these combine different workers within the category, so all studies are aggregating to some degree. To do so, studies implicitly assume some form of separability in the underlying production function. While Berndt & Christensen (1973a) formulate precise statements on separability and the legitimacy of aggregating factors to study the concept of the elasticity of substitution, no equivalent statements are available for the elasticity of factor demand. The purpose of this paper is to produce such statements.

In particular, it establishes the relationship between the separability of inputs and the validity of Marshall's Rules for an aggregate of those inputs. This would legitimize the use of aggregated data to estimate an aggregate elasticity when we know that not all labor is the same. The relationship has practical uses when the researcher has disaggregated data but would nonetheless like to produce an aggregate summary measure. For example, many CGE or similar simulation-based exercises need one summary parameter as an input.

We start by reviewing the basic link between separability and the elasticity of substitution. Our theoretical contribution is the application to factor demand. Thereafter, we use a translog cost function to apply the theory to a data set in which we have four occupation types plus capital and the objective is to produce one elasticity between labor types.

The application has its own merit. The data is from South Africa, which makes this study one of few for developing countries: according to Fajnzylber & Maloney (2001), only two of the nearly 200 studies surveyed by Hamermesh (1993) use establishment data for developing countries. It is also a setting where unemployment is high and it is feared new labor legislation has raised the cost of labor. Our results yield labor demand elasticities of almost unity. While capital and both labor types are substitutes, More- and Less-skilled labor are complements.

2. Theory

2.1 The elasticity of substitution and separability

The elasticity of substitution measures the percentage change in relative demand for two inputs in response to a change in relative factor prices. Our point of departure is homothetic production function $Q = f(x_1, x_2, ..., x_n)$, from which Allen (1938) developed an elasticity measure when there are more than two inputs. Uzawa (1962) uses the dual cost function $C = g(w_1, w_2, ..., w_n, q)$ to express this elasticity of substitution as

$$\sigma_{ij} \equiv -\frac{d \log \frac{x_i}{x_j}}{d \log \frac{w_i}{w_j}} = \frac{C g_{ij}}{g_i g_j}, \tag{1}$$

where g_i, g_j are first derivatives with respect to the prices of factors i, j - w_i, w_j - and g_{ij} is the cross partial derivative. This partial measure assumes relative price of factors i and j

change exogenously but the prices of all other factors and output remain constant. If $\sigma_{ij} > 0$, the factors are substitutes. If $\sigma_{ij} < 0$, they are complements.

Following Berndt & Christensen (1973a), partition the n inputs in $f(\cdot)$ into R mutually exclusive and exhaustive subsets $[X^1, ..., X^R]$, which we call partition P. The production function is weakly separable with respect to partition P if the marginal rate of technical substitution between any pair of inputs x_i, x_j from any subset X^S is independent of the quantity of any input outside X^S. That is, $\frac{d}{dx_k}\left(\frac{f_i}{f_j}\right) = 0 \ \forall i, j \epsilon X^S, k \notin X^S$, where f_i, f_j are the marginal products of inputs x_i, x_j. Differentiation gives:

$$f_j f_{ik} - f_i f_{jk} = 0 \ \forall i, j \epsilon X^S, k \notin X^S \tag{2}$$

Weak separability is necessary and sufficient for $f(\cdot)$ to be legitimately written as $q = F[X_1, ..., X_R]$, where X_S is a function of the elements of X^S only. Lau (1969) shows weak separability in the production function with respect to partition P implies weak separability in the cost function (and vice versa), so $g(\cdot)$ can after partition P consist of R subsets. Then, the cost function can be written as $C = G[W_1, ..., W_R, q]$, where W_S is a function of the prices of the inputs in X^S, which comprise set W^S. The analogue to (2) is:

$$g_j g_{ik} - g_i g_{jk} = 0 \ \forall i, j \epsilon X^S, k \notin X^S \tag{3}$$

Berndt & Christensen (1973a:407) build on these conditions to establish that separability of factors x_i and x_j from all others in the production function is equivalent to:

$$\sigma_{ik} = \sigma_{jk} \ \forall k : k \neq i, k \neq j \tag{4}$$

In other words, the elasticity of substitution between some aggregate of x_i and x_j, which we call X_I, and a third input x_k is $\sigma_{Ik} = \sigma_{ik} = \sigma_{jk}$. They also show this is equivalent to the legitimate construction of an aggregate index of factors x_i and x_j or their prices.

2.2 Aggregation and the elasticity of factor demand

Based on Marshall's Rules (1920:383), we can write the compensated cross-elasticity of demand in terms of factor shares and the elasticity of substitution. When the prices of all other factors and output remain constant (Hamermesh, 1986),

$$\bar{\lambda}_{ij} = \frac{d \log x_i}{d \log w_j} = s_j \sigma_{ij}, \tag{5}$$

where s_j is the cost share of factor j. While Berndt & Christensen (1973ab, 1974) make statements about separability that allow us to produce aggregated elasticities of substitution (equation (4)), the same has not been said about the cross and own-elasticities of factor demand.

Assume each of the disaggregated input quantities change by the same proportion and that each of the disaggregated input prices also change by the same proportion. Informally, we can say that the elasticity of an aggregate of one set of input quantities with respect to an aggregate of another set of input prices is the *sum* of the elasticities of *one* of the input quantities with respect to each of the input prices. Equivalently, the elasticity of the aggregate of one set of input quantities with respect to an aggregate of another set of input prices is the elasticity of substitution between the aggregates multiplied by the cost share of the aggregate input whose price has changed.

Formally, let W_I be a legitimate aggregate of one or more input prices ie all $w_i \in W^I$ are weakly separable from $w_i \notin W^I$. Write $d\log w_j = \hat{w}\ \forall j \in W^J$ and $d\log x_i = \hat{x}\ \forall i \in X^I$. Define the constant output cross-elasticities as follows:

- $\bar{\lambda}_{ij} \equiv \frac{d\log x_i}{d\log w_j}$
- $\bar{\lambda}_{iJ} \equiv \frac{d\log x_i}{d\log W_J}$
- $\bar{\lambda}_{IJ} \equiv \frac{d\log X_I}{d\log W_J}$ (The aggregate elasticity of factor demand.)

$S_J \equiv \sum_j s_j\ \forall j \in W^J$ such that the factor share of an aggregate is the sum of the disaggregated shares. σ_{ij} is the disaggregated elasticity of substitution, σ_{IJ} is the aggregate elasticity of substitution and σ_{iJ} is the elasticity of substitution between a disaggregated input and an aggregated input.

Lemma 1 *Weak separability with respect to partition P implies $\bar{\lambda}_{iJ} = \sum_{j\in W^J} \bar{\lambda}_{ij}$.*

Proof. By equation (5), $d\log x_i = \sum_j s_j\sigma_{ij} d\log w_j$ when output is constant. In particular, if only the prices in the aggregate W_J change, $d\log x_i = \sum_{j\in W^J} s_j\sigma_{ij} d\log w_j$. However, $d\log w_j = \hat{w}\ \forall j \in W^J \rightarrow \sum_{j\in W^J} d\log w_j = \frac{\sum_{j\in W^J} dw_j}{\sum_{j\in W^J} w_j} = \hat{w} = d\log W_J$. Therefore $\bar{\lambda}_{iJ} = \sum_{j\in W^J} s_j\sigma_{ij} = \sum_{j\in W^J} \bar{\lambda}_{ij}$. ■

Lemma 2 *Weak separability with respect to the partition P implies $\bar{\lambda}_{iJ} = S_J\sigma_{iJ}$.*

Proof. As shown in Berndt & Christensen (1973a), $\sigma_{ij} = \sigma_{iJ}\ \forall j \in W^J$. By Lemma 1, $\bar{\lambda}_{iJ} = \sum_{j\in W^J} s_j\sigma_{iJ}$. Therefore $\bar{\lambda}_{iJ} = S_J\sigma_{iJ}$ ■

Lemma 3 *Weak separability with respect to the partition P implies $\bar{\lambda}_{iJ} = S_J\sigma_{IJ}$.*

Proof. Using $\sigma_{ij} = \sigma_{iJ} \forall\ j \in W^J$, this follows trivially from Lemma 2. ■

Lemma 4 *Weak separability with respect to the partition P implies $\bar{\lambda}_{IJ} = S_J\sigma_{IJ}$.*

Proof. If only the prices in W_J change, by equation (5), $\sum_{i\in X^I} d\log x_i = \sum_{i\in X^I}\sum_{j\in W^J} s_j\sigma_{ij} d\log w_j$. By Lemma 2, $\sum_{i\in X^I} d\log x_i = \sum_{i\in X^I} S_J\sigma_{iJ} d\log W_J$. But $d\log x_i = \hat{x}\ \forall i \in X^I \rightarrow \sum_{i\in X^I} d\log x_i = \frac{\sum_{i\in X^I} dx_i}{\sum_{i\in X^I} x_i} = \hat{x} = d\log X_I$. Therefore $\bar{\lambda}_{IJ} = S_J\sigma_{iJ}$. It follows from Berndt & Christensen (1973a) that $\sigma_{iJ} = \sigma_{IJ}\ \forall i \in X^I$ and therefore $\bar{\lambda}_{IJ} = S_J\sigma_{IJ}$. ■

The results are summarised as follows:

Proposition 1 *Weak separability with respect to the partition P implies $\bar{\lambda}_{IJ} = \bar{\lambda}_{iJ} = S_J\sigma_{IJ} = S_J\sigma_{iJ} = \sum_{j\in W^J} \bar{\lambda}_{ij}$.*

Proof. This follows from the lemmata. ■

The proposition presents Marshall's Rules for aggregate inputs and shows the aggregate elasticity can be calculated by summing the disaggregated elasticities. We can also make a statement about aggregated own-price elasticities:

Corollary 1 *Weak separability with respect to the partition P implies $\bar{\lambda}_{II} = -\sum_{J:J\neq I} \bar{\lambda}_{IJ}$.*

Proof. Using linear price homogeneity, Sato & Koizumi (1973) show $\sum_j \bar{\lambda}_{ij} = 0$. Dividing variables w_j into those that are together with w_i in aggregate W_I and those that are not, we have $\sum_{j\in W^I} \bar{\lambda}_{ij} = -\sum_{j\notin W^I} \bar{\lambda}_{ij}$. By Proposition 1, $\sum_{j\in W^I} \bar{\lambda}_{ij} = -\lambda_{II}$ and $\sum_{j\notin W^I} \bar{\lambda}_{ij} = -\lambda_{IJ}$. Therefore$\bar{\lambda}_{II} = -\sum_{J:J\neq I} \bar{\lambda}_{IJ}$. ■

3. Empirics

3.1 Data and background

South African unemployment has been "*literally off the charts*" compared to other developing countries (Nattrass, 2004:90). Those unemployed under the narrow ILO definition comprise about 25% of the labor force and as many as 40% are unemployed according to the expanded definition (Statistics South Africa, 2005, 2009). Commentators fear South Africa's wage bargaining institutions and the introduction of new labor legislation in 1995 may be raising the costs of labor and contributing to unemployment (Fedderke et. al., 2001). South Africa is therefore an appropriate setting in which to gauge the potential impact of labor costs on employment.

The two sources of data are manufacturing data for about 300 firms from the National Enterprise Survey conducted in 1998, which has been merged with data from the 1999 October Household Survey. We use four occupation types from the firm-level data, namely the Managerial/Professional and Skilled/Artisanal occupations (More skilled) and the Semi-skilled and Unskilled occupations (Less skilled). For further motivation and description of the procedure used to combine the data, see Behar (2010).

3.2 Translog functions

With origins due to Christensen, Jorgenson & Lau (1973), we follow Teal (2000) and represent $g(\cdot)$ by means of a translog cost function,

$$\begin{aligned} \log C &= \log a_0 + \sum_i a_i \log w_i + \frac{1}{2}\sum_i \sum_j B_{ij} \log w_i \log w_j + a_q \log q \\ &\quad + B_q \log^2 q + \sum_i B_{iq} \log q \log w_i, \end{aligned} \tag{6}$$

which will be estimated together with the associated factor share equations to improve efficiency using a seemingly unrelated regression method:

$$s_i = a_i + \sum_j B_{ij} \ln w_j + B_{iq} \ln q \tag{7}$$

See Berndt (1991) for details. We impose restrictions on the coefficients consistent with cost minimizing behaviour (Berndt & Khaled, 1979). Slutsky symmetry requires $B_{ij} = B_{ji}$ while linear price homogeneity requires $\sum_i B_{ij} = 0, \sum_j B_{ij} = 0, \sum_i a_i = 1$ and $\sum_i B_{iq} = 0$. The elasticities of factor demand are (Binswanger, 1974):

$$\bar{\lambda}_{ij} = \frac{B_{ij}}{s_i} + s_j \tag{8}$$

$$\bar{\lambda}_{ii} = \frac{B_{ii}}{s_i} + s_i - 1 \tag{9}$$

By applying (2) or (3) to a translog function, one can test for separability of factors x_i and x_j from all others by means of the following restrictions (Berndt & Christensen, 1974):

$$s_i B_{jk} - s_j B_{ik} = 0 \forall k : k \neq i, k \neq j \tag{10}$$

Separability implies the existence of a valid price index, but it doesn't solve the problem of how best to perform the aggregation. One might conjecture that an average of the prices, weighted in some way by their relative shares, would be appropriate. This corresponds to the conditions for separability in equation (10). Thus, we operationalise separability by running a *disaggregated* regression while imposing these restrictions. For simplicity, we use the sample average of s for the restrictions and for the elasticity calculations.

3.3 Results

Regressions

This section presents the results from estimating (6) with appropriate restrictions of the form (10) imposed. As a preliminary step, we ran an unrestricted model in which we did not impose separability / aggregate the inputs and, using Wald tests of equation (10), failed to reject the restrictions. The restrictions imply our disaggregated cost function $C = g(w_1, w_{2,} w_3, w_4, w_5, q)$ can be written as $C = G(W_M, W_L, W_5, q)$, where W_5 is capital, W_M is more skilled labor and W_L is less skilled labor. The regression results (with restrictions imposed) are presented in Table 1. The overall fit of the regression is good. Our specification also rejects homotheticity, so our results are only valid on the assumption of a locally homothetic technology. We find the B_{ij} jointly significant at 10%, which rejects the null hypothesis of a Cobb-Douglas technology.

Elasticities

The elasticities were confirmed to be exactly equal for separable inputs, for example $\sigma_{15} = \sigma_{25} = \sigma_{M5.}$ We present the three aggregated elasticities in Table 2. Capital and both skill types are found to be roughly equally substitutable such that a rise in the cost of labor relative to capital would lead to a relative fall in its employment quantity. More-skilled and Less-skilled labor are complements with an elasticity of substitution of -1.71. Table 3 reports the compensated elasticities of factor demand. Concurring with the review in Hamermesh (1993), more skilled labor demand is less elastic than less skilled labor demand. These aggregate own-elasticities suggest wage push would have contributed to decreased employment levels. Furthermore, the cross elasticities of factor price of -0.45 and 0.58 imply a rise in wages for less skilled workers reduces demand for their skilled counterparts and increases demand for capital.

4. Conclusion

Our substantive contribution has been to estimate elasticities of demand for Capital, More-skilled and Less-skilled labor for South Africa; estimates which are scarce for developing countries. Labor demand elasticities are almost unity and the two labor types are complements. To do this using disaggregated data, we imposed the relevant restrictions implied by separability when estimating a translog function. This is only legitimate because we confirmed the equivalence of the separability of two inputs with respect to other inputs and the calculation of an aggregate elasticity for those inputs. Our result also justifies the notion of a homogeneous elasticity when labor is not homogeneous.

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Table 1: Cost Function Parameter Estimates and System Diagnostics

Dependent variable: Cost					
Variable	Coefficient	p-value	Variable	Coefficient	p-value
Capital	0.09	0.90	value added	0.28	0.00
Man/Prof	0.26	0.42	0.5*(value added)^2	.013	0.00
Skil/Art	0.15	0.42	(value added)*Cap	0.01	0.47
Semi	0.30	0.46	(value added)*Man/Prof	-0.02	0.00
Un	0.20	0.41	(value added)*Skil/Art	-0.004	0.45
value added	0.28	0.00	(value added)*Semi	0.004	0.62
0.5*Capital^2	-0.27	0.10	(value added)*Un	0.005	0.48
Capital*Man/Prof	0.06	0.19	Observations		307
Capital*Skil/Art	0.05	0.19	RMSE		0.54
Capital*Semi	0.10	0.17	"R^2"		0.85
Capital*Un	0.07	0.17	χ^2 for regression significance		2018
0.5*Man/Prof^2	0.03	0.24	p value		0.00
Man/Prof*Skil/Art	-0.03	0.09	Joint significance of $B_{ij} = 0$		0.09
Man/Prof*Semi	-0.03	0.06	Homotheticity p value		0.00
Man/Prof*Un	-0.02	0.06	Constant and controls for exports as a share of output, raw materials as a share of cost, ease of recruitment, training expenditure, a market conditions index, a large-firm dummy, computer investment as a share of output, a dummy for owner-managed firms, a productivity dissatisfaction measure, firm age, province and location dummies and an indicator for technology intensity are not presented.		
0.5*Skil/Art^2	0.03	0.12			
Skil/Art*Semi	-0.03	0.06			
Skil/Art*Un	-0.02	0.06			
0.5*Semi^2	0.02	0.74			
Semi*Un	-0.05	0.26			
0.5*Un^2	0.03	0.51			
System Diagnostics					
Share Equation	Obs	RMSE	"R^2"	χ^2	p
Managerial/Professional	307	0.06	0.43	236.06	0.00
Skilled/Artisanal	307	0.08	0.17	71.06	0.00
Semiskilled	307	0.13	0.16	60.85	0.00
Unskilled	307	0.11	0.11	41	0.01

Table 2: Elasticities of Substitution

σ_{IJ}	
Factor pairing	Elasticity
Capital / More	2.40
Capital / Less	2.19
More / Less	-1.71

Table 3: Elasticities of Factor Demand

$\bar{\lambda}_{IJ}$		J		
		Capital	More	Less
	Capital	-0.94	0.35	0.58
I	More	1.26	-0.80	-0.45
	Less	1.14	-0.25	-0.90