Economics Bulletin

Volume 30, Issue 1

Allocations and Price Trends in Sequential Auctions under Complete Information with Symmetric Bidders

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Abstract

We analyze sequential second-price auctions under complete information involving two or more bidders with similar decreasing marginal valuations. Krishna (1999) designed a 2-bidder numerical example to show the existence of two symmetric equilibria characterized by an asymmetric allocation and weakly declining prices. We generalize Krishna's insights by showing that symmetric (asymmetric) allocations imply constant (weakly declining) price patterns and we derive the necessary conditions supporting symmetric allocations. The conditions become increasingly restrictive as the number of object increases.

Funding from the Canada's research chair program and from FQRSC is gratefully acknowledged.
Citation: Mohamed Jeddy and Bruno Larue and Jean-philippe Gervais, (2010) "Allocations and Price Trends in Sequential Auctions under Complete Information with Symmetric Bidders", *Economics Bulletin*, Vol. 30 no.1 pp. 429-436.
Submitted: Dec 18 2009. **Published:** February 02, 2010.

1. Introduction

Bernheim and Whinston (1986) have argued that the complete information assumption is appropriate for the analysis of frequently-held auctions involving the same bidders. In such settings, the bidders know each other's valuations, but the seller is poorly informed. It is easy to construct a second-price auction under complete information involving two symmetric bidders with declining valuations that will support two symmetric equilibria[1] characterized by a constant price pattern. Consider the outcome tree of the game illustrated in Figure 1. Arrows denote the allocation in each subgame and prices are given next to the paths. At each node, the bidders' gross payoffs are put in parentheses. Each unit could go either to bidder A (left branch) or to bidder B (right branch). The equilibrium outcome is solved by backward induction and bids reflect the opportunity cost of not winning. The outcome tree, unlike the extensive form, features gross payoffs at every node which are obtained through subgame replacement. At nodes associated to the j^{th} object, gross payoffs are defined as the sum of valuations for objects won along the given path minus the sum of prices for objects that would be won among the last n-j+1 objects. For the last object, gross payoffs are the sum of the valuations.

In Figure 1, bidders' valuations for the first and second objects are θ_1 and θ_2. Bidder i has gross payoffs of $\theta_1+\theta_2$, θ_1 and 0 from winning both objects, one object and nothing at the end nodes. Provided bidder A won the first object, he would bid his gross payoff differential θ_2 for the second object. Conditional on bidder A having won the first object, bidder B would have a gross payoff differential of θ_1-0 and would win the second object at price $p_2^B=\theta_2$. Conditional on the first object being won by bidder B, bidder A would win the second object by bidding θ_1 and paying $p_2^A=\theta_2$. Moving up the tree, the payoffs at the two nodes account for allocations and prices derived for the second object: $(\theta_1,\theta_1-\theta_2)$ vs $(\theta_1-\theta_2,\theta_1)$. Both bidders end up bidding θ_2, knowing that if they lose the first object they will get the second at the same price.

As for the Heckscher-Ohlin model in the trade literature, the results of this 2x2 auction are not robust when the number of objects n or the number of bidders increases. In an example of a four-object auction involving two bidders with symmetric valuations, Krishna (1999) uncovered two symmetric equilibria characterized by an asymmetric allocation and declining prices with one bidder winning three objects and the other bidder getting a single object. The multiplicity of equilibria arises because bidders can be interchanged.[2]

The analysis of sequential auctions under complete information with symmetric bidders has been largely ignored in the literature and it is the purpose of this note to shed more light on such auctions. We show that when the number of objects is even, but greater or equal to 4, symmetric allocations and a constant price trend arise under specific

[1] We assume throughout that bidders have identical declining valuations. Equilibria are symmetric when they generate the same sequence of prices. Bidders can then be interchanged.

[2] When the two bidders have asymmetric valuations, Katzman (1999) has shown that the equilibrium is unique, possibly inefficient and that the price pattern may be constant or declining. Gale and Stegeman (2001) have analyzed cases with asymmetric valuations with more than two objects.

conditions about bidders' valuations. Otherwise the allocations are uneven and prices are declining with possibly flat segments. When the number of objects is uneven, allocations are asymmetric and prices are declining.

2. The model

The auction is a sequential second-price one involving two completely informed bidders with identical decreasing marginal valuations: $\theta_1 > \theta_2 > ... > \theta_{n-1} > \theta_n$.[3] Part of the 4-object version of the game is illustrated in Figure 2. In this instance, a symmetric allocation with bidders A and B getting two objects each can be achieved through six equilibria provided valuations decrease at a decreasing rate, $\theta_2 - \theta_3 > \theta_3 - \theta_4$: {A,B,A,B}, {B,A,B,A}, {A,A,B,B}, {B,B,A,A}, {A,B,B,A} and {B,A,A,B}. Equilibrium prices are constant and the seller's revenue is $R = 4\theta_3$. If the bidders had symmetric valuations such that $\theta_2' - \theta_3' < \theta_3' - \theta_4'$, prices would weakly decline $p = \{3\theta_3' - \theta_2' - \theta_4', \theta_2' + \theta_4' - \theta_3', \theta_2' + \theta_4' - \theta_3', \theta_4'\}$, one player would get 1 object, the other would get 3 and $R' = \theta_2' + \theta_3' + 2\theta_4' \underset{>}{\overset{<}{-}} R$.[4] Symmetric allocations are also possible in higher-dimensional games. We show that the condition just derived for the $n = 4$ case is a special case of a more general set of conditions.

Proposition 1: *Consider two bidders {A,B} having similar strictly declining marginal valuations and let $k \equiv n/2$ where n is an even number of successive second-price auctions with $n \geq 4$. There are multiple symmetric equilibria with a constant price pattern or weakly declining pattern generating identical payoffs for the two bidders. The bidders get the same number of objects k if and only if the price pattern is constant which requires* $\sum_{m=1}^{k} \theta_m - k\theta_{k+1} > \sum_{m=1}^{k-p} \theta_m - (k-p)\theta_{k+p+1} \; \forall p = 1,...,k-1$.

Proof: Intuitively, bidder A must be indifferent between his allocation and that of bidder B, whether the allocations are symmetric or asymmetric. Under a symmetric allocation derived through backward induction, let us assume that bidder A has won k objects and

[3] The case of endogenous valuations is analyzed by Krishna (1999). In her two-object auction, a snowball effect arises because bidders use the object as inputs and compete on the "output" market. The bidder who won the first object has a higher valuation for the second object because that second object would secure a monopoly position. In our case, we treat valuations as exogenous. This could be rationalized by the existence of alternative marketing mechanisms preventing monopoly outcomes. For example, the daily hog auction in the Canadian province of Quebec involved a small number of bidders. However, they get a large share of their hog supply through a pre-attribution/formula pricing mechanism based on historical market shares.

[4] Consider the following examples with valuations adding up to the same total such that the seller's revenue from selling the 4 objects as a block would be the same: $\theta = \{10,9,6,5\}$, $\theta' = \{10,9,6.7,4.3\}$, $\theta'' = \{10,9,8,3\}$ and $\theta''' = \{10,8,7,5\}$. When the objects are sold sequentially, the first set of valuations produces a symmetric allocation, identical prices $p = 6$ and revenue $R = 24$. For the asymmetric allocations with weakly declining prices, we have $R' = 24.3, R'' = 23, R''' = 25$.

bidder B has won $0 \le j \le k-1$ objects. When $j = k-1$, one object remains to be auctioned. Bidder A bids his valuation for the n^{th} and last object and this is the price that bidder B will pay given that his valuation is higher: $p^B_{k+j+1} = \theta_{k+1} < \theta_k$. Thus, at the $(n\text{-}1)^{th}$ auction, bidders know that if they lose the object they will win the last one and gain $\theta_k - \theta_{k+1}$. Because bidders must be indifferent between winning and losing the $(n\text{-}1)^{th}$ object, prices for the $(n\text{-}1)^{th}$ and n^{th} objects must be the same. This is just like the 2×2 auction in Figure 1.

Consider now $j = k-2$. Bidder B knows that bidder A has used up his first k valuations. Bidder B can win the last two objects by bidding in excess of θ_{k+1} and gain $\theta_{k-1} - \theta_{k+1} + \theta_k - \theta_{k+1}$ for these last two objects, or win one object and gain $\theta_{k-1} - \theta_{k+2}$ or win none and gain nothing. The latter option is dominated because valuations are strictly declining. If bidder B is to win the last two objects, it must be that: $\theta_k - \theta_{k+1} > \theta_{k+1} - \theta_{k+2}$ or valuations must decrease at a decreasing rate at the k^{th} valuation. This is the condition required to have symmetric allocations for the 4-object auction in Figure 2. If it is not met, an asymmetric allocation emerges and prices must decline.

In this 4 object-auction, if bidder B is to win only one object, his maximum payoff is achieved by having bidder A get the first three objects. Hence, $\pi^B = \theta_1 - \theta_4$ which must equal $\sum_{m=1}^{3}(\theta_m - p_m)$. Clearly the average price on the first three objects must be above θ_4. Furthermore, if one of the first three objects was to be sold below θ_4, bidder B would prefer getting this object instead of the fourth object. But bidder A would prefer bidder B's payoff and so a price below θ_4 cannot be observed. Therefore, prices must be weakly declining. Consider now the case j=0 (i.e., bidder A has won the first k objects and k others remain to be auctioned). A symmetric allocation requires that bidder B wins the last k objects and that both bidders get the same payoff. This requires that $\sum_{m=1}^{k}\theta_m - k\theta_{k+1} > \sum_{m=1}^{k-p}\theta_m - (k-p)\theta_{k+p+1}\ \forall p = 1,...,k-1$. **QED**

The number of conditions increases with k (or n) because the symmetric allocation is pitted against a larger number of potential asymmetric allocations. Furthermore, the conditions supporting a symmetric allocation become increasingly stringent when the number of objects increases. For $k=3$ $(n=6)$, it must be that $\sum_{m=1}^{3}\theta_m - 3\theta_4 > \max\left(\sum_{m=1}^{2}\theta_m - 2\theta_5, \theta_1 - \theta_6\right)$. These inequality restrictions can be rearranged as: $Min(\theta_3 + 2\theta_5, \theta_2 + \theta_3 + \theta_6) \ge 3\theta_4$. Clearly the differences between the first three valuations and the fourth one must be large compared to the differences between the 4^{th} and the 5^{th} and 6^{th}. For $k=5$ $(n=10)$, one of the necessary conditions is $\theta_5 - \theta_6 \ge 4(\theta_6 - \theta_7)$. Clearly $(\theta_k - \theta_{k+1}) - (\theta_{k+1} - \theta_{k+2})$ must increase significantly as the number of objects increases if a symmetric allocation is to be observed.

Proposition 2: *When n, the number of successive second-price auctions with two bidders {A,B} having similar declining marginal valuations, is uneven, the allocation is asymmetric and the price pattern is always declining with possibly flat segments.*

Proof. As for an asymmetric allocation when the number of objects is even in proposition 1, prices must be weakly declining because of payoff symmetry. Consider an auction with n = 3 and bidder A winning 2 objects and bidder B winning only one. Bidder B "waits" to win the last object for a payoff of $\pi^B = \theta_1 - \theta_3$. Bidder A must be indifferent between winning the first two objects or taking bidder B's place as the winner of a single object and vice versa. Furthermore, when the second object is put for sale, bidder B must be indifferent between his payoffs from waiting for the third object or getting the second object. Bidder A knows that and the price for the second and third objects is the same: θ_3 which explains the flat segment. Therefore, payoff symmetry requires that the price sequence be: $p = \{\theta_2, \theta_3, \theta_3\}$. Because players A and B can be interchanged, there are two symmetric equilibria with the same weakly declining price pattern. **QED**

Figure 3 illustrates the results of proposition 1 and 2 via a few examples. The first example illustrates the case for 4 objects with declining valuations equal to $\{10,7,5,4\}$ for each bidder. The condition in proposition 1 is met and the equilibrium is characterized by a constant price. The 5-object example with bidders' valuations equal to $\{20,15,12,10,2\}$ generates weakly declining prices: $p = \{16,8,8,8,3\}$. A similar outcome also emerges with our 6-object example with bidders' valuations equal to $\{20,15,12,10,7,6\}$. Even though bidders have symmetric valuations, they can safely exploit rapid declines in valuations through asymmetric allocations. For the same reason, a symmetric (inefficient) allocation can arise when bidders have asymmetric valuations as shown by Katzman (1999).

Our analysis can be generalized for cases involving more than 2 bidders. In the 3-bidder case with n a multiple of 3, the symmetric allocation entails having bidders A,B,C winning $k \equiv n/3$ objects at a constant price $p = \theta_{k+1}$. When the game is at a point where n-3 objects have been sold such that bidders A,B,C have $\{k, k, k-3\}$ objects, then bidder C must decide whether it is best to get the last three objects or to get only one and letting the other bidders get one as well: $\sum_{i=k-2}^{k} \theta_i - 3\theta_{k+1} \geq \theta_{k-2} - \theta_{k+2}$. This is a necessary, but not sufficient condition. However, if $n = 9$, we are comparing allocations $\{3,3,3\}$ and $\{4,4,1\}$ and our necessary condition for a symmetric allocation is $\theta_2 + \theta_3 + \theta_5 > 3\theta_4$. Other asymmetric allocations, $\{5,2,2\}$ and $\{7,1,1\}$ impose additional conditions, namely: $\sum_{i=1}^{3} \theta_i - 3\theta_4 > \max\left(\sum_{i=1}^{2} \theta_i - 2\theta_6, \theta_1 - \theta_8\right)$ or $Min\left(\theta_3 + 2\theta_6, \theta_2 + \theta_3 + \theta_8\right) > 3\theta_4$. The drop in valuation between the k^{th} and k+1$^{\text{th}}$ objects must be large, as shown for the 2-bidder cases.

3. Conclusion

We analyze sequential second-price auctions under complete information when bidders have identical decreasing marginal valuations over *n* objects ($\theta_1 > ... > \theta_n$). We show that a symmetric (asymmetric) allocation with each is bidder getting *k* objects is characterized by constant (weakly declining) prices. Generally, symmetric allocations require that valuations be such that $\theta_k - \theta_{k+1}$ be larger than $\theta_{k+1} - \theta_{k+2}$. The decreases in valuations from the k+1th object must be increasingly small relative to the decrease in valuation between the k^{th} and k+1th objects as the number of objects auctioned increases, thus making asymmetric allocations more likely when the number of objects is large.

4. References

Bernheim, B.D. and M.D. Whinston (1986) "Menu Auction, Resource Allocation and Economic Influence" *Quarterly Journal of Economics* **101**, 1-32.

Gale, I. L. and M. Stegeman (2001) "Sequential Auctions of Endogenously Valued Objects" *Games and Economic Behaviour* **36**, 74-103.

Katzman B. (1999) "A Two Stage Sequential Auction with Multi-Unit Demands" *Journal of Economic Theory* **86**, 77-99.

Krishna, K. (1999) "Auctions with Endogenous Valuations: The Snowball Effect Revisited" *Economic Theory* **13**, 377-391.

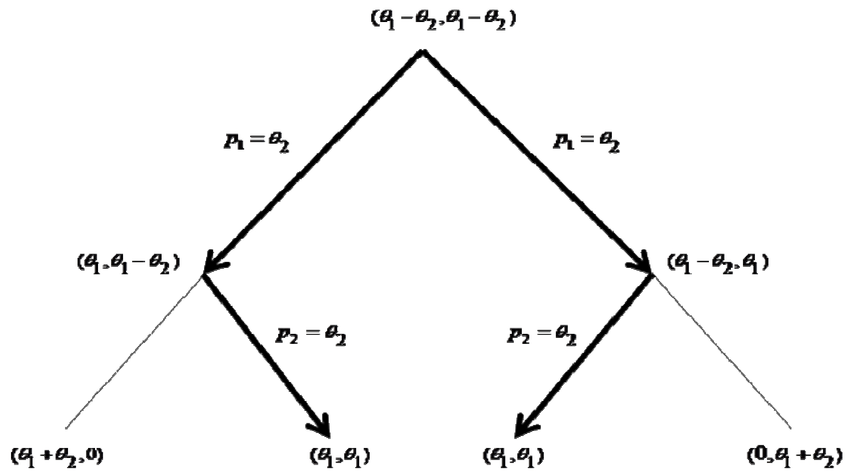

Figure 1. The complete information two-bidder two-object second-price auction with symmetric valuations.

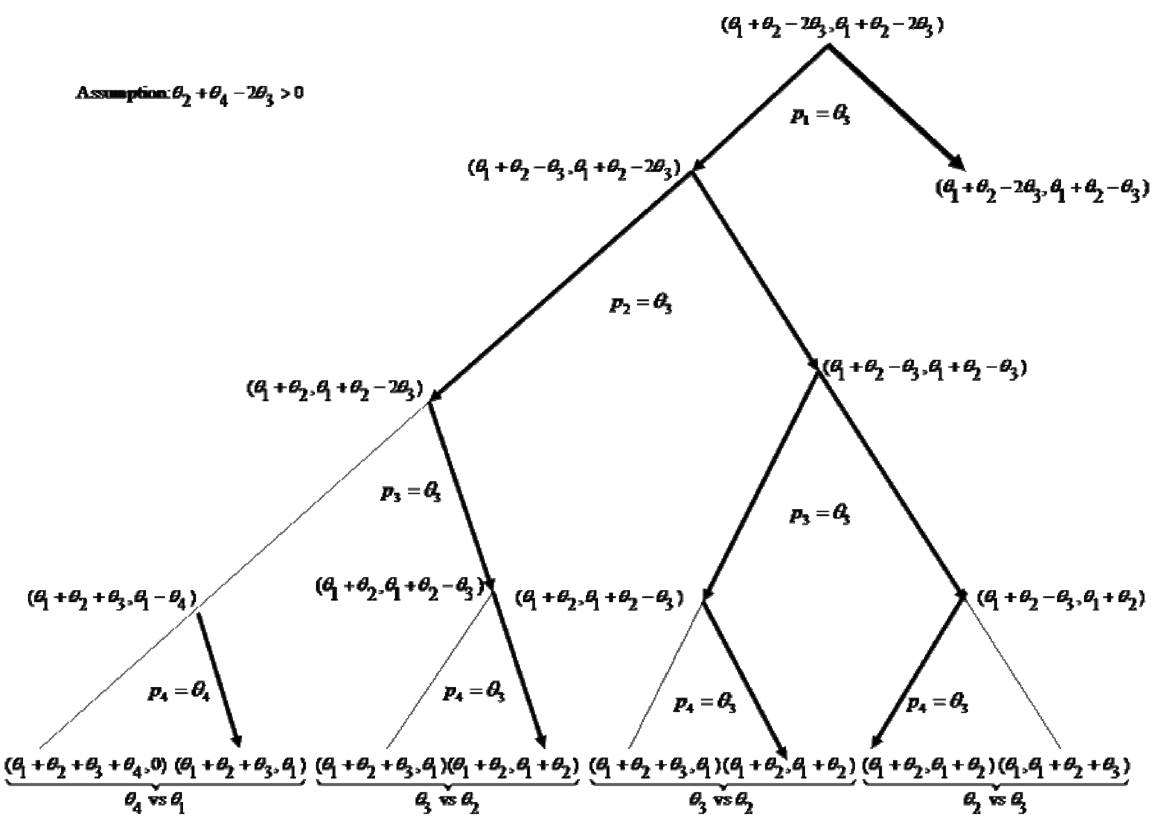

Figure 2. A 2-bidder 4-object auction with symmetric allocations.

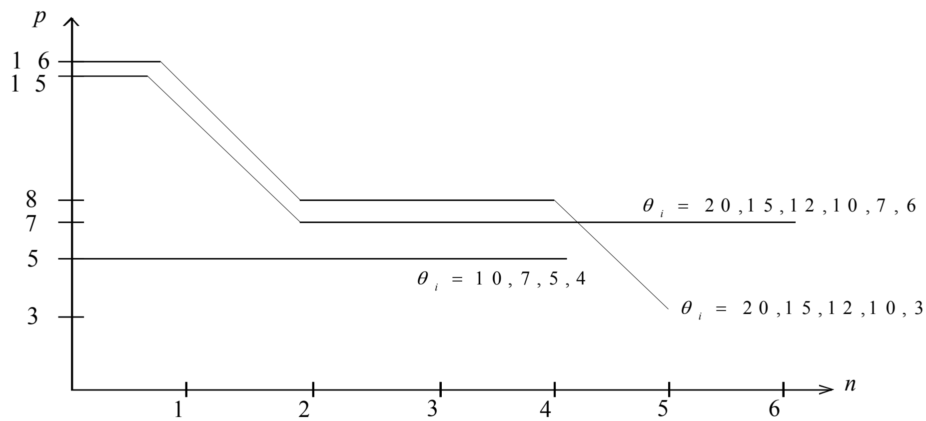

Figure 3. Examples of price patterns when bidders are symmetric.