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Hyperbolic discounting may be time consistent

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Abstract

Using dynamic programming methodology, the paper analyzes the most general conditions for an additive utility functional to represent time consistent preferences. It challenges the conventional wisdom of the domain, which, following Strotz(1956), assume that only exponential discounting is compatible with time consistent behavior. The paper gives some examples of special time consistent hyperbolic discount functions and also discuss the relation between time consistency and stationarity.

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1 Introduction

The purpose of this paper is to clarify the relations between the important notions of time consistency, stationarity and various sorts of parametrical discount functions (exponential, hyperbolic, etc.). The notion of time consistency has been introduced by Strotz (1956). To define properly this notion, it is important to distinguish (1) the *calendar date of decision (planification)* (denoted t) and (2) the *calendar date of the future act of consumption* (denoted s). The difference between those two dates is the *algebraic time distance of the future act of consumption according to the date of decision.* The agent will be time consistent if, in the absence of any new information, the choice of consumption for any future calendar date is independent from the calendar date of decision. The main result of Strotz (1956) is that, considering an additive intertemporal utility functional, and considering a discount factor being a function of the *algebraic time distance* ($\boldsymbol{s}$**-**$\boldsymbol{t}$), the agent will be time consistent, *if and only if*, the log derivative of the discount factor is constant, *i.e.* the discount factor is exponential, and the discount rate is constant. Strotz's methodology was to compare the solutions of continuous time programs of intertemporal utility maximization under constraint solved at different decision dates. Strotz's notion of time consistency has proven to be seminal. From the empirical point of view, following the work of psychologists (Ainslie (1975)) many evidence where collected against the exponential discounting model (Thaler (1981), see Frederick et al. (2002) for a survey). Even if there are still some debate (Harrison and Lau (2005) for example) the idea of hyperbolic discounting, which is generally considered as implying time inconsistent behavior, is now very standard in the experimental literature. We will see that this implication is not true. From the theoretical point of view, following (Laibson (1997), Harris and Laibson (2001), the notion of hyperbolic discounting has been used to explain some irrational behavior as procrastination and under-savings (see Salanié and Treich (2006) for a discussion).

The notion of stationarity has been introduced by Koopmans (1960, 1972). If, for a given decision date, the preference order between any two consumption streams is not modified when those streams are anticipated or postponed by a same amount of time, then the preference relation fulfills stationarity. It can be represented by an additive intertemporal utility functional, with the same per "period utility function" for each calendar date weighted by an exponential discount factor (*i.e.* a constant rate of discount). Koopmans's methodology is fully axiomatic, and it provides strong foundations for the exponential discounted utility model proposed by Samuelson (1937). Bleichrodt et al. (2008) generalize and clarify Koopmans axiomatic methodology.

The two notions of time consistency and stationarity have been progressively melted in a sort of *conventional wisdom* in the field of intertemporal choice. The claim of this paper is that this conventional wisdom sometimes abusively sums up some results which do not have the same domain of validity and have been established using different methodologies. This paper is organized as follows: section 2 discussed the notion of time consistency within the dynamic programming methodology (Strotz (1956)'s original methodology); section 3 discuss the time consistency of some parametrical or semi parametrical discount functions; and at last section 4 concludes.

2 Time consistent dynamic programming for a naive decision maker.

Let's denote the "calendar date of decision" as t. A consumer is endowed with an initial capital, $K(t)$, and a planning horizon T. At every moment, this capital brings interest at a constant rate, r and can be used to finance consumption. We will consider $\mathscr{C}_t$ the set of all functions $c : [t, T] \longrightarrow \mathbb{R}^+$ continuous and derivable. We assume that the consumer's intertemporal preferences at date t are represented by an additive utility functional,

$$V(c, t) = \int_t^T v(c(s), s, t) ds \tag{1}$$

with v, the "per period felicity function", continuous and derivable in $c(s)$ and s, and strictly concave in $c(s)$. Taking s as a variable of the "per period felicity function" allows us to include in the analysis the possibility of tastes evolving with age (non stationarity). Taking t as a variable of the intertemporal utility functional allows us to include in the analysis the possibility of changing taste at different decision date. This specification is thus the most general possible within the additive framework.

The choice of the consumer at date t is the function $c_t \in \mathscr{C}_t$ solution of the program:

$$\mathcal{P}_t \quad \begin{cases} Max \int_t^T v(c(s), s, t) ds \\ s.t. \forall s \in [t, T], \quad \dot{K}(s) = rK(s) - c(s) \\ K(t) \text{ given and } K(T) \geq 0 \end{cases}$$

Thus $c_t(s)$ is the optimal consumption at age s planed at date t (the value of the control variable at age s) and $K_t(s) = \int_t^s (rK(\tau) - c_t(\tau)) d\tau$ the remaining capital at age s for an agent planing his consumption at date t (the value of the state variable at age s).

Let's now assume that at calendar date $t' \in (t, T)$, the consumer is allowed to plan again his consumption for his remaining time. Following Strotz (1956), we will call the consumer "naive" because when he plans his consumption at date t, he does not anticipate in any way that he will be allowed to plan again at any subsequent date.

Keeping all assumptions made before, the choice of the consumer at date t' is the function $c_{t'} \in \mathscr{C}_{t'}$ solution of the program:

$$\mathcal{P}_{t'} \quad \begin{cases} Max \int_{t'}^T v(c(s), s, t') ds \\ s.t. \forall s \in [t', T], \quad \dot{K}(s) = rK(s) - c(s) \\ K(t') = K_t(t') \text{ and } K(T) \geq 0 \end{cases}$$

We now have all the material needed to define properly time consistency within the dynamic programming approach. Our definition is essentially the same as the one given by Strotz (1956) in his seminal paper.

Definition 1. *The agent is **time consistent** if and only if:*

$$\forall t' \in [t, T], \forall s \in [t', T], c_t(s) = c_{t'}(s) \equiv c(s)$$

Within the set of utility functions defined by Equation 1, what are the specificity of the one compatible with time consistent behavior ?

Proposition 1. *An agent is time consistent if and only if v is of the form:*

$$v(c(s), s, t) = \alpha(t) w(c(s), s) + \beta(t) \tag{2}$$

Proof: It can easily be proven that the terminal constraint is binding ($K(T) = 0$) for both programs and because of the strict concavity of v in $c(s)$, c_t and $c_{t'}$ are both unique. The dynamic constraint on the state variable are the same for the two programs over the interval $[t', T]$, with the same initial and terminal value. Thus we have:

$$K(t') = \int_{t'}^{T} \exp[-r(s-t')]c_t(s)ds = \int_{t'}^{T} \exp[-r(s-t')]c_{t'}(s)ds \tag{3}$$

Moreover, using Pontryagin maximum principle it is easy to show that $\forall s \in [t', T]$, first order conditions of programs $\mathcal{P}_t$ and $\mathcal{P}_{t'}$ imply:

$$\dot{c}_t(s) = \frac{r + \frac{v''_{12}(c_t(s),s,t)}{v'_1(c_t(s),s,t)}}{-\frac{v''_{11}(c_t(s),s,t)}{v'_1(c_t(s),s,t)}} \text{ and } \dot{c}_{t'}(s) = \frac{r + \frac{v''_{12}(c_{t'}(s),s,t')}{v'_1(c_{t'}(s),s,t')}}{-\frac{v''_{11}(c_{t'}(s),s,t')}{v'_1(c_{t'}(s),s,t')}} \tag{4}$$

If $\forall s,\ c_t(s) = c_{t'}(s)$ then we must have $\forall s,\ \dot{c}_t(s) = \dot{c}_{t'}(s)$. Proposition 1 is straightforward.

Proposition 1 is the most general statement concerning time consistency we can make within the additive utility framework. As we can see, it is much more general than standard claims concerning time consistency. In the next section we will discuss this result according to special parametrical form of the discounted utility functional. But we first have to set some definitions.

Definition 2. *We can define the* ***generalized subjective rate of discount*** *as:*

$$\Theta(c(s), s, t) \equiv -\frac{v''_{12}(c(s), s, t)}{v'_1(c(s), s, t)}$$

and the ***rate of absolute resistance to intertemporal substitution*** *as:*

$$\rho(c(s), s, t) \equiv -\frac{v''_{11}(c(s), s, t)}{v'_1(c(s), s, t)}$$

According to those definitions we can rewrite the derivative of the optimal consumption plan decideed at date t (Equation 4) as:

$$\dot{c}_t(s) = \frac{r - \Theta(c_t(s), s, t)}{\rho(c_t(s), s, t)} \tag{5}$$

The slope of the planed consumption function is equal to the ratio of the difference between the market rate of discount (*i.e.* the rate of interest) and the generalized subjective rate of discount, on one hand, with rate of absolute resistance to intertemporal substitution, on the other hand. This result is a generalisation of the proposition of Yaari (1964).

Corollary 1-a. *The agent is time consistent if and only if the generalized subjective rate of discount and the rate of absolute resistance to intertemporal substitution are both invariant according to t,* i. e.

$$\Theta(c(t), s, t) = -\frac{w''_{12}(c(s), s)}{w'_1(c(s), s)} \text{ and } \rho(c(t), s, t) = -\frac{w''_{11}(c(s), s)}{w'_1(c(s), s)}$$

At this point it is important to notice that the subjective rate of discount at a given age has to be independent from the decision date, but not from the age. Constancy of the subjective rate of discount is not a requisite for time consistency. We will discuss further this point in the next section.

3 Time consistency and parametrical or semi-parametrical discount functions

Since the seminal paper of Samuelson (1937), there is a tradition for separating the discount factor and the "per period" utility. Within our general formulation it just means that v is multiplicatively separable in $c(s)$ *i. e.* $v(c(s), s, t) = f(s, t)u(c(s))$. Moreover we can assume that the discount factor is a function of the algebraic time distance. In this special case proposition 1 has strong implications.

Corollary 1-b (Exponential discounting model - Samuelson (1937))**.** *If the intertemporal utility functional takes the form* $V(c, t) = \int_t^T f(s-t)u(c(s))ds$*, with* f *a positive and bounded function, then the agent is time consistent if and only if* $f(s-t) = \exp[-\theta(s-t)]$*. In this special case* $\Theta(c(t), s, t) = \theta$

Proposition 1 shows that time consistency implies that the "per period felicity function" is multiplicatively separable in t. Obviously, if the discount factor is a function of the algebraic time distance, then time consistency implies that the discount factor is necessarily exponential, because only exponential functions transform sums in products.

However if we adopt the more general formulation of the discounted utility model, we get:

Corollary 1-c (Generalized discount factor - Burness (1976))**.** *If the intertemporal utility functional takes the form* $V(c, t) = \int_t^T f(s, t)u(c(s))ds$*, with* f *a positive and bounded function, then the agent is time consistent if and only if* $f(s, t)$ *is multiplicatively separable in* s*, i. e.* $f(s, t) = \alpha(s)\beta(t)$

Corollary 1-b characterizes the "conventional wisdom" in the domain. Corollary 1-c shows that this "conventional wisdom" lacks of generality. Exponential discounting is not a necessary condition for time consistency. Many examples can be given of time consistent non-exponential discount function.

Example 1 (An hyperbolic time consistent utility functional)**.**

$$V(c, t) = \int_t^T \frac{t}{s} u(c(s))ds(\textit{with } t > 0)$$

This intertemporal utility functional is "hyperbolic" because the discount factor is an hyperbolic function. It implies time consistency because it satisfies the conditions of Proposition 1(and Corollary 1-c). The subjective rate of time preference $\Theta(s) = 1/s$ is also hyperbolic and is independent from t (Corollary 1-a). Moreover, the discount factor equals to one in t and decline with s (*i.e.* the agent shows decreasing impatience). In fact, this special discount function was the one proposed in Ainslie (1975)'s seminal paper on hyperbolic discounting, suggesting that general shape of this function what not too far from the empirical findings of the author. Actually experimentalist use most of the time other forms of hyperbolic or quasi hyperbolic discounting. But because they believe that a decreasing rate of discount is a sufficient condition for time inconsistency, they don't see the necessity to test functional form respecting the property of corollary 1-c against more standard ones. The choice of this quite provocative example for giving the title of this paper is precisely a call for more precise empirical investigation on this topic.

Example 2. *A cyclothymic time consistent utility functional:*

$$V(c,t) = \int_t^T \frac{\cos[t] + \delta}{\cos[s] + \delta} u(c(s))ds (with\ \delta > 1)$$

In this second example we have seasonal effects. For example a consumer may value less consumption in winter than in summer. Those intertemporal preferences are thus characterized by cyclical discount rate and factor, and also cyclical consumption. Nevertheless we fulfill the requirement of Proposition 1 (and Corollary 1-c) and thus an individual endowed with such preferences will be time consistent.

The first demonstration of Corollary 1-b is generally attributed to Strotz (1956). But Strotz's result was in fact slightly more general, because he was not assuming that the "per period" utility function was invariant with age and thus not assuming stationarity or a weaker hypothesis of that kind.

Corollary 1-d (Strotz (1956))**.** *If the intertemporal utility functional takes the form* $V(c,t) = \int_t^T f(s-t)u(c(s),s)ds$, *with* f *a positive and bounded function, then the agent is time consistent if and only if* $f(s-t) = \exp[-\theta(s-t)]$

A lot has be written on that result. But it is important to notice that, writing $u(c(s),s) = w(c(s),s)\exp[\theta s]$, Strotz's time consistent utility functional can be rewritten

$$V(c,t) = \exp[\theta t] \int_t^T w(c(s),s)ds$$

proving that **the exponential formulation was a mathematical artefact**. There is a kind of paradox here. Strotz's result was in fact very general. Rigorously, he demonstrates that additivity is a sufficient condition for time consistency. But his formulation has proven to be misleading in the sense that most of subsequent works on the question have focused on the exponential formulation and forgotten the possible variability of the "per period" utility function through time. More generally, the point here is that, as soon as one works with an intertemporal utility functional with "per period" utility varying with age, the discount function can no more be defined in a unique fashion and thus becomes a useless concept. However in this case our notion of generalized subjective rate of discount remains valid and allows to discuss time preference, impatience, decreasing impatience and many of the concepts that have been at stack in the recent years.

4 Conclusion

The idea that agents "behave rationally" is the corner stone of normative decision theory. In the domain of intertemporal choice, time consistency is certainly the ultimate criteria for characterizing rational behavior. For a long time, alternatives to the exponential discounted utility model have been discarded because of the (false) belief that they where incompatible with rational behavior. More recently, on the basis of experimental evidence rejecting the exponential discounting model, many behavioral economist have claimed that it was the proof that deviations from rationality were standard in the domain of intertemporal choice. The objective of this note is certainly not to settle this debate. Our only claim is that, if we want to go further on this question, we shall build on a more solid ground. The exponential discounting is not the only one compatible with time consistency. A non constant subjective rate of discounting does not necessarily imply non rational behavior. Even some kind of hyperbolic discounting may be compatible with time consistency. Stationarity implies time consistency, but the reverse is not true. It means in particular that stationarity (which implies that "per period utility function" does not change with age) is a behavioral hypothesis that can be true or false, but it is not a criteria of rationality. The huge domain of non exponential time consistent intertemporal model of choice is still to be explored. Maybe it is time to begin.

References

Ainslie, G. (1975). Specious reward: A behavioural theory of impulsiveness and impulse control. *Psychological Bulletin 82*(4), 463–496.

Bleichrodt, H., K. I. M. Rohde, and P. P. Wakker (2008). Koopmans' constant discounting for intertemporal choice: A simplification and a generalisation. *Journal of Mathematical Psychology 52*, 341–347.

Burness, H. S. (1976). A note on consistent naive intertemporal decision making and an application to the case of uncertain lifetime. *The Review of Economic Studies 43*(3), 547–549.

Frederick, S., G. Loewenstein, and T. O'Donoghue (2002). Time discounting and time preference: A critical review. *Journal of Economic Literature 40*(2), 351–401.

Harris, C. and D. Laibson (2001). Dynamic choices of hyperbolic consumers. *Econometrica 69*(4), 935–957.

Harrison, G. W. and M. I. Lau (2005). Is the evidence for hyperbolic discounting just an experimental artifact. *Behavioural and Brain Sciences 28*(5), 657.

Koopmans, T. C. (1960). Stationary ordinal utility and impatience. *Econometrica 28*(2), 287–309.

Koopmans, T. C. (1972). Representation of preference orderings over time. In McGUIRE and RADNER (Eds.), *Decision and Organization, a Volume in Honor of Jacob Marschak.*, pp. 79–100.

Laibson, D. (1997). Golden eggs and hyperbolic discounting. *Quarterly Journal of Economics 112*(2), 443–477.

Salanié, F. and N. Treich (2006). Over-savings and hyperbolic discounting. *European Economic Review 50*, 1557–1570.

Samuelson, P. A. (1937). A note on measurement of utility. *Review of Economic Studies 4*, 155–161.

Strotz, R. H. (1956). Myopia and inconsistency in dynamic utility maximisation. *Review of Economic Studies 23*(2), 165–180.

Thaler, R. (1981). Some empirical evidence on dynamic inconsistency. *Economics Letters 8*, 201–207.

Yaari, M. E. (1964). On the consumer's lifetime allocation process. *International Economic Review 5*(3), 304–317.