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Dividend approach and level consistency for the Derks and Peters value

Yu-Hsien Liao
Department of Mathematics, Chung Jen College of Nursing, Health Science and Management

Abstract

Different from the potential approach of Hart and Mas-Colell (1989), we provide the dividend approach to multi-choice games. Also, we define the level-reduced game by reducing the number of the activity levels and define related consistency on multi-choice games.

Dear Sir, I am sorry to bother you. I would like to submit my paper "Dividend approach and level consistency for the Derks and Peters value" to the Economics Bulletin . I hope that it can be considered for publication in it. Please find attached files of the paper "Dividend approach and level consistency for the Derks and Peters value". Thanks for your patience and assistance! Please acknowledge receipt of this email. With best regards, Yu-Hsien Liao

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1 Introduction

The *Shapley value* (Shapley, 1953) is a well-known solution concept in cooperative game theory. A *multi-choice TU game*, introduced by Hsiao and Raghavan (1992), is a generalization of a TU game in which each player has several activity levels. There are several extensions of the Shapley value in the framework of multi-choice games. Here we focus one of these extensions proposed by Derks and Peters (1993), which we name the *D&P Shapley value*.

Hart and Mas-Colell (1989) were the first to introduce the potential approach to TU games. In consequence, they proved that the Shapley value can result as the vector of marginal contributions of a potential function. The potential approach is also shown to yield a characterization for the Shapley value, particularly in terms of an internal consistency property. In Section 3, we introduce the *dividend approach* to multi-choice games. The dividend approach is a dual view of the potential approach. By the dividend approach, we show that the D&P Shapley value can result as the vector of aggregate of a dividend.

There are two important factors, the players and their activity levels, for multi-choice games. By reducing the number of the players, Hwang and Liao (2008) proposed an extension of the reduced game due to Hart and Mas-Colell (1989) on multi-choice games. By reducing the number of the activity levels, we define the *level-reduced game* and the *level consistency* in Section 4. Finally, we show that the D&P Shapley value satisfies the level consistency based on the dividend approach.

2 Definitions and Notations

Let U be the universe of players. Let $N = \{1, \cdots, n\} \subseteq U$ be a set of players. Suppose each player i has $m_i \in \mathbb{N}$ levels at which he can actively participate. Let $m = (m_1, \cdots, m_n)$ be the vector that describes the number of activity levels for each player, at which he can actively participate. For $i \in U$, we set $M_i = \{0, 1, \cdots, m_i\}$ as the action space of player i, where action 0 means not participating, and $M_i^+ = M_i \setminus \{0\}$. For $N \subseteq U$, $N \neq \emptyset$, let $M^N = \prod_{i \in N} M_i$ be the product set of the action spaces for players N. Denote the zero vector in $\mathbb{R}^N$ by 0_N.

A **multi-choice TU game** is a triple (N, m, v), where N is a non-empty and finite set of players, m is the vector that describes the number of activity levels for each player, and $v : M^N \to \mathbb{R}$ is a characteristic function which assigns to each action vector $x = (x_1, \cdots, x_n) \in M^N$ the worth that the players can obtain when each player i plays at activity level $x_i \in M_i$ with $v(0_N) = 0$. If no confusion can arise a game (N, m, v) will sometimes be denoted by its characteristic function v. Given a multi-choice game (N, m, v) and $x \in M^N$, we write (N, x, v) for the **multi-choice TU subgame** obtained by restricting v to $\{y \in M^N \mid y_i \leq x_i \ \forall i \in N\}$ only. Denote the class of all multi-choice TU games by MC.

Given $(N, m, v) \in MC$, let $L^{N,m} = \{(i, j) \mid i \in N, j \in M_i^+\}$. A **solution** on MC is a map ψ assigning to each $(N, m, v) \in MC$ an element

$$\psi(N, m, v) = \big(\psi_{i,j}(N, m, v)\big)_{(i,j)\in L^{N,m}} \in \mathbb{R}^{L^{N,m}}.$$

Here $\psi_{i,j}(N, m, v)$ is the power index or the value of the player i when he takes action j to play game v. For convenience, given $(N, m, v) \in MC$ and a solution ψ on MC, we define $\psi_{i,0}(N, m, v) = 0$ for all $i \in N$.

To state the D&P Shapley value, some more notations will be needed. Given $S \subseteq N$, let $|S|$ be the number of elements in S and let $e^S(N)$ be the binary vector in $\mathbb{R}^N$ whose component $e_i^S(N)$ satisfies

$$e_i^S(N) = \begin{cases} 1 & \text{if } i \in S \ , \\ 0 & \text{otherwise} \ . \end{cases}$$

Note that if no confusion can arise $e_i^S(N)$ will be denoted by e_i^S.

Given $(N, m, v) \in MC$ and $x \in M^N$, we define $\|x\| = \sum_{k\in N} x_k$ and $S(x) = \{k \in N \mid x_k \neq 0\}$.

Let $x, y \in \mathbb{R}^N$, we say $y \leq x$ if $y_i \leq x_i$ for all $i \in N$. The analogue of unanimity games for multi-choice games are **minimal effort games** (N, m, u_N^x), where $x \in M^N$, $x \neq 0_N$, defined by for all $y \in M^N$,

$$u_N^x(y) = \begin{cases} 1 & \text{if } y \geq x \ ; \\ 0 & \text{otherwise.} \end{cases}$$

Hsiao and Raghavan (1992) showed that for $(N, m, v) \in MC$ it holds that $v = \sum\limits_{x\in M^N\setminus\{0_N\}} a^x(v)\ u_N^x$, where $a^x(v) = \sum\limits_{S\subseteq S(x)} (-1)^{|S|}\ v(x - e^S)$.

Definition 1 (Derks and Peters, 1993) *The* **D&P Shapley value** *Θ is the solution on MC which associates with each $(N, m, v) \in MC$ and each player $i \in N$ and each $j \in M_i^+$ the value*

$$\Theta_{i,j}(N, m, v) = \sum_{x\in M^N, x_i\geq j} \frac{a^x(v)}{\|x\|}.$$

Note that the so called **dividend** *$a^x(v)$ is divided equally among the necessary levels.*

3 Potential and Dividend

For $x \in \mathbb{R}^N$, we write x_S to be the restriction of x at S for each $S \subseteq N$. Let $N \subseteq U$, $i \in N$ and $x \in \mathbb{R}^N$, for convenience we introduce the substitution notation x_{-i} to stand for $x_{N\setminus\{i\}}$ and let $y = (x_{-i}, j) \in \mathbb{R}^N$ be defined by $y_{-i} = x_{-i}$ and $y_i = j$.

Given a function $P : MC \longrightarrow \mathbb{R}$ which associates a real number $P(N, m, v)$ to each game (N, m, v). Then for each $(i, j) \in L^{N,m}$ we define

$$D^{i,j}P(N, m, v) = P(N, m, v) - P\big(N, (m_{-i}, j-1), v\big).$$

Definition 2 *A solution ψ on MC admits a* **potential** *if there exists a function $P : MC \to \mathbb{R}$ satisfies for all $(N, m, v) \in MC$ and for all $(i, j) \in L^{N,m}$,*

$$\psi_{i,j}(N, m, v) = D^{i,j}P(N, m, v).$$

Solutions that admit a potential assign a scalar evaluation to each game in such a way that a player's payoff is his marginal contribution to this evaluation. Moreover, a function $P : MC \longrightarrow \mathbb{R}$ is said to be *0-normalized* if $P(N, 0_N, v) = 0$ for each $N \subseteq U$. And we say it is *efficient* if it satisfies the following condition: For all $(N, m, v) \in MC$,

$$\sum_{i \in N} \sum_{j=1}^{m_i} D^{i,j}P(N, m, v) = v(m).$$

Theorem 1 (Hwang and Liao, 2008) *A solution ψ on MC admits a uniquely 0-normalized and efficient potential P if and only if ψ is the D&P Shapley value Θ on MC. For all $(N, m, v) \in MC$ and $(i, j) \in L^{N,m}$,*

$$\Theta_{i,j}(N, m, v) = D^{i,j}P(N, m, v).$$

Given a function $d : MC \longrightarrow \mathbb{R}$ which associates a real number $d(N, m, v)$ to each $(N, m, v) \in MC$. Then for each $(i, j) \in L^{N,m}$ we define

$$\int_{i,j} d(N, m, v) = \sum_{x \in M^N, x_i \geq j} d(N, x, v).$$

Definition 3 *A solution ψ on MC admits a* **dividend** *if there exists a function $d : MC \to \mathbb{R}$ satisfies for all $(N, m, v) \in MC$, $N \neq \emptyset$ and for all $(i, j) \in L^{N,m}$,*

$$\psi_{i,j}(N, m, v) = \int_{i,j} d(N, m, v).$$

Solutions that admit a dividend assign a scalar evaluation to each game in such a way that a player's payoff is his *marginal accumulation* to this evaluation. Moreover, a function $d : MC \longrightarrow \mathbb{R}$ is said to be *0-normalized* if $d(N, 0_N, v) = 0$ for each $N \subseteq U$. And we say it is *efficient* if it satisfies the following condition: For all $(N, m, v) \in MC$,

$$\sum_{i \in N} \sum_{j=1}^{m_i} \int_{i,j} d(N, m, v) = v(m).$$

Theorem 2 *Let ψ be a solution on MC. ψ admits a potential if and only if ψ admits a dividend.*

Proof. Assume that ψ be a solution on MC and ψ admits a dividend d. Define $P : MC \longrightarrow \mathbb{R}$ to be that $P(N, m, v) = \sum_{x \in M^N} d(N, x, v)$ for all $(N, m, v) \in MC$. Since ψ admits the dividend d, for all $(N, m, v) \in MC$ and $(i, j) \in L^{N,m}$,

$$\begin{aligned} \psi_{i,j}(N, m, v) &= \sum_{x \in M^N, x_i \geq j} d(N, x, v) \\ &= \sum_{x \in M^N} d(N, x, v) - \sum_{x \in M^N, x_i \leq j-1} d(N, x, v) \\ &= P(N, m, v) - P\big(N, (m_{-i}, j-1)\big). \end{aligned}$$

Hence, ψ admits the potential P.

Assume that ψ be a solution on MC and ψ admits a potential P. Define $d : MC \longrightarrow \mathbb{R}$ to be that for all $(N, m, v) \in MC$, $d(N, m, v) = \sum_{T \subseteq S(m)} (-1)^{|T|} P(N, m - e^T, v)$. It is easy to check that $P(N, m, v) = \sum_{x \in M^N} d(N, x, v)$. Since ψ admits the potential P, for all $(N, m, v) \in MC$ and $(i, j) \in L^{N,m}$,

$$\begin{aligned} \psi_{i,j}(N, m, v) &= P(N, m, v) - P\big(N, (m_{-i}, j-1)\big) \\ &= \sum_{x \in M^N} d(N, x, v) - \sum_{x \in M^N, x_i \leq j-1} d(N, x, v) \\ &= \sum_{x \in M^N, x_i \geq j} d(N, x, v). \end{aligned}$$

Hence, ψ admits the dividend d. The proof is completed. ■

Theorem 3 *A solution ψ on MC admits a uniquely 0-normalized and efficient dividend d if and only if ψ is the D&P Shapley value Θ on MC. For each multi-choice game $(N, m, v) \in MC$ and $(i, j) \in L^{N,m}$*

$$\Theta_{i,j}(N, m, v) = \sum_{x \in M^N, x_i \geq j} d(N, x, v).$$

Proof. It is easy to derive this result by Theorems 1 and 2. ■

Remark 1 *Given $(N, m, v) \in MC$, by Definition 1 and Theorem 3,*

$$d(N, x, v) = \frac{a^x(v)}{\|x\|} \quad \text{for all } x \in M^N.$$

4 Level-reduced Game and Level Consistency

In this section we define the level-reduced game and the level consistency which are putting the accent on " *level* ". Based on the dividend approach, we also show that the D&P value satisfies the level consistency.

By reducing the number of the players, Hwang and Liao (2008) proposed an extension of the reduced game due to Hart and Mas-Colell (1989) on multi-choice games as follows. For $S \subseteq N$, we denote $S^c = N \setminus S$. Given a solution ψ, a game $(N, m, v) \in MC$, and $S \subseteq N$, the **reduced game** $\big(N, (m_S, 0_{S^c}), v^{\psi}_{S,m}\big)$ with respect to ψ, S and m is defined by for all $x \in M^S$,

$$v^{\psi}_{S,m}(x, 0_{S^c}) = v(x, m_{S^c}) - \sum_{i \in S^c} \sum_{j=1}^{m_i} \psi_{i,j}\big(N, (x, m_{S^c}), v\big).$$

Definition 4 (Hwang and Liao, 2008) *A solution ψ on MC satisfies* **consistency (CON)** *if for all $(N, m, v) \in MC$, for all $S \subseteq N$ and for all $(i, j) \in L^{S, m_S}$, $\psi_{i,j}\big(N, (m_S, 0_{S^c}), v^{\psi}_{S,m}\big) = \psi_{i,j}(N, m, v)$.*

Theorem 4 (Hwang and Liao, 2008) *The solution Θ satisfies CON.*

Inspired by Hsiao, Yeh and Mo (1994), we define an alternative level-reduced game and related consistency as follows. Given $(N, m, v) \in MC$ and a solution ψ. For each $z \in M^N \setminus \{0_N\}$, we define an action vector $z^* = (z^*_i)_{i \in N}$ where

$$\begin{cases} z^*_i = m_i & \text{if } \ z_i < m_i \\ z^*_i = 0 & \text{if } \ z_i = m_i. \end{cases}$$

Furthermore, we define a new game v^{ψ}_z with $z \in M^N \setminus \{0_N\}$ such that for all $y \leq z$,

$$v^{\psi}_z(y) = v(y \vee z^*) - \sum_{k \in S(z^*)} \sum_{t=1}^{m_k} \psi_{k,t}(N, (y \vee z^*), v).$$

where $(y \vee z^*)_i = \max\{y_i, z^*_i\}$ for all $i \in N$. We call (N, z, v^{ψ}_z) a **level-reduced game** of v with respect to z and the solution ψ.

Definition 5 *A solution ψ on MC satisfies* **level consistency (LCON)** *if for all $i \in N \setminus S(z^*)$ and for all $j \leq z_i$, $\psi_{i,j}(N,m,v) = \psi_{i,j}(N,z,v_z^{\psi})$.*

Remark 2 *Given $(N,m,v) \in MC$, $S \subseteq N$ and a solution ψ. Let $z = (m_S, 0_{S^c})$, by definitions of v_z^{ψ} and $v_{S,m}^{\psi}$, $v_z^{\psi}(y) = v_{S,m}^{\psi}(y)$ for all $y \leq z$. Hence, if a solution ψ satisfies LCON, then ψ satisfies CON.*

It is known that each $(N,m,v) \in MC$ can be expressed as a linear combination of minimal effort games and this decomposition exists uniquely. The following lemma relates the relation of coefficients of expressions between a $(N,m,v) \in MC$ and its level-reduced game (N,z,v_z^{Θ}) with respect to Θ and z.

Lemma 1 *Let $(N,m,v) \in MC$ and (N,z,v_z^{Θ}) be the level-reduced game of (N,m,v) with respect to Θ and $z \in M^N \setminus \{0_N\}$. Obviously, z can be written by $z = (m_S, z_{S^c})$ for some $S \subseteq N$. If $v = \sum\limits_{y \in M^N \setminus \{0_N\}} a^y(v) \cdot u_N^y$, then v_z^{Θ} can be expressed to be $v_z^{\Theta} = \sum\limits_{0_N \neq y \leq z} a^y(v_z^{\Theta}) \cdot u_N^y$, where for all $y \leq z$,*

$$a^y(v_z^{\Theta}) = \begin{cases} \sum\limits_{t \leq m_{S^c}} \frac{\|(y_S,0_{S^c})\|}{\|(y_S,0_{S^c})\| + \|(t,0_S)\|} a^{(y_S,t)}(v) & \text{if } y = (y_S, 0_{S^c}) \ , \\ 0 & \text{if } y = (y_S, y_{S^c}) \text{ with } |S(y_{S^c}, 0_S)| \neq 0. \end{cases}$$

Proof. Let $(N,m,v) \in MC$ and $z \in M^N \setminus \{0_N\}$. Obviously, $z = (m_S, z_{S^c})$ for some $S \subseteq N$ where $z_i \neq m_i$ for all $i \in S^c$. For any $y \leq z$,

$$v_z^{\Theta}(y) \ = \ v(y \vee z^*) - \sum_{k \in S(z^*)} \sum_{t=1}^{m_k} \Theta_{k,t}\big(N, (y \vee z^*), v\big). \tag{1}$$

Clearly, $v_z^{\Theta}(y) = 0$ if $y = 0_N$. Since $z^* = (0_S, m_{S^c})$,

$$\begin{aligned}
(1) \ &= \ v(y_S, m_{S^c}) - \textstyle\sum_{k \in S^c} \sum\limits_{t=1}^{m_k} \Theta_{k,m_k}\big(N, (y_S, m_{S^c}), v\big) \\
&= \ \textstyle\sum_{k \in S(y_S, 0_{S^c})} \sum\limits_{t=1}^{y_k} \Theta_{k,t}(N, (y \vee z^*), v) \\
&= \ \textstyle\sum_{k \in S(y_S, 0_{S^c})} \sum\limits_{t=1}^{y_k} \sum\limits_{\substack{x \leq (y \vee z^*) \\ x_k \geq t}} \frac{a^x(v)}{\|x\|} \\
&= \ \textstyle\sum_{k \in S(y_S, 0_{S^c})} \Big[\sum\limits_{\substack{x \leq (y, m_{S^c}) \\ x_k \geq 1}} \frac{a^x(v)}{\|x\|} + \cdots + \sum\limits_{\substack{x \leq (y, m_{S^c}) \\ x_k = y_k}} \frac{a^x(v)}{\|x\|} \Big] \\
&= \ \textstyle\sum_{k \in S(y_S, 0_{S^c})} \Big[\sum_{\substack{p \leq y_S \\ p_k \geq 1}} \sum\limits_{t \leq m_S} \frac{a^{(p,t)}(v)}{\|(p,0_{S^c})\| + \|(t,0_S)\|} + \cdots + \sum_{\substack{p \leq y_S \\ p_k = y_k}} \sum\limits_{t \leq m_S} \frac{a^{(p,t)}(v)}{\|(p,0_{S^c})\| + \|(t,0_S)\|} \Big] \\
&= \ \sum_{x \leq y} \sum_{t \leq m_S} \frac{\|(x_S,0_{S^c})\|}{\|(x_S,0_{S^c})\| + \|(t,0_S)\|} \cdot a^{(x_S,t)}(v).
\end{aligned} \tag{2}$$

By definition of v_z^Θ, for any $y \leq z$,

$$v_z^\Theta(y) = v_z^\Theta(y_S, 0_{S^c}). \tag{3}$$

Set

$$a^y(v_z^\Theta) = \begin{cases} \sum\limits_{t \leq m_{S^c}} \frac{\|(y_S, 0_{S^c})\|}{\|(y_S, 0_{S^c})\| + \|(t, 0_S)\|} a^{(y_S, t)}(v) & \text{if } y = (y_S, 0_{S^c}) \ , \\ 0 & \text{if } y = (y_S, y_{S^c}) \text{ with } |S(y_{S^c}, 0_S)| \neq 0. \end{cases}$$

By (2) and (3), for all $y \leq z$,

$$v_z^\Theta(y) = \sum_{x \leq y} \sum_{t \leq m_S} \frac{\|(x_S, 0_{S^c})\|}{\|(x_S, 0_{S^c})\| + \|(t, 0_S)\|} \cdot a^{(x_S, t)}(v) = \sum_{x \leq y} a^x(v_z^\Theta).$$

Hence v_z^Θ can be expressed to be $v_z^\Theta = \sum\limits_{0_N \neq y \leq z} a^y(v_z^\Theta) \cdot u_N^y$. ■

Theorem 5 *The solution Θ satisfies LCON.*

Proof. Let $(N, m, v) \in MC$ and $z \in M^N \setminus \{0_N\}$. Obviously, $z = (m_S, z_{S^c})$ for some $S \subseteq N$ where $z_i \neq m_i$ for all $i \in S^c$. For all $i \in N \setminus S(z^*)$ and for all $0 < j \leq z_i$,

$$\begin{aligned} \Theta_{i,j}(N, z, v_z^\Theta) &= \sum_{y \leq z, y_i \geq j} \frac{a^y(v_z^\Theta)}{\|y\|} \\ &= \sum_{\substack{y \leq z, y_i \geq j \\ |S(y_{S^c}, 0_S)| = 0}} \frac{a^y(v_z^\Theta)}{\|y\|} \quad \text{(By definition of } v_z^\Theta) \\ &= \sum_{\substack{y \leq z, y_i \geq j \\ |S(y_{S^c}, 0_S)| = 0}} \frac{1}{\|(y_S, 0_{S^c})\|} \cdot \sum_{t \leq m_{S^c}} \frac{\|(y_S, 0_{S^c})\|}{\|(y_S, 0_{S^c})\| + \|(t, 0_S)\|} a^{(y_S, t)}(v) \\ &\qquad \text{(By Lemma 1)} \\ &= \sum_{x \in M^N, x_i \geq j} \frac{a^x(v)}{\|x\|} \\ &= \Theta_{i,j}(N, m, v). \end{aligned}$$

Hence the solution Θ satisfies LCON. ■

Remark 3 *Hwang and Liao (2008) characterized the D&P value by means of consistency. By Remark 2 and Theorem 5, it's easy to check that consistency could be replaced by level-consistency in those axiomatizations.*

In fact, as some axioms are alterd to fit solutions form focusing on "dividend", the executions for characterizations among solutions on multi-choice games are similar. The dividend forms not only offer interpretations for solutions but also provide motivations for axioms of solutions.

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