E C O N O M I C S   B U L L E T I N

Value allocations in economies with coalition structure

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Abstract

We embody a notion of stability for coalition structures by Hart and Kurz (1983) into the framework of general equilibrium, by generalizing the classical value allocation notion (Shapley, 1969) to situations where: (a) agents organize themselves voluntarily into coalition structures; (b) the process of coalition formation is treated as endogenous. To this end we introduce the definition of stable coalition structure value allocation and provide, under standard hypotheses, a preliminary existence result for the three--player case in an exchange economy.

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1 Introduction

In a previous work (Centrone and Meo, 2008), we faced the issue of endogenous coalition formation in the framework of games without side payments: precisely, we defined a new solution concept, the stable Coalition Structure (CS) valuation, based on Owen's extension (1977) of the Shapley value for games with side payments to situations where players are organized in coalitions (the Coalition Structure value), and on a notion of stability by Hart and Kurz (1983) based on the strong Nash equilibria of a "game among coalitions" and on the CS value. We provided an existence result for the three-player case, where the restrictive setting is caused by the non-existence of stable coalition structures in the general case.
The aim and main contribution of this note is to introduce the previous elements into the framework of a general equilibrium model by defining a solution concept for an exchange economy which seeks (a) to evaluate each agent's position in the market by taking into account how his own bargaining opportunities and gains attainable through cooperation vary within different coalition structures and (b) to select a collectively stable outcome. To this end we propose the notion of stable CS value allocation, which parallels and extends the classical notion of value allocation (Shapley, 1969) to account for endogenous formation of coalitions, with the CS value substituting for the Shapley value and the CS stable valuation for the classical Shapley one. As a byproduct of our main Theorem 1 we get an existence result for stable CS value allocations (Corollary 1) in an exchange economy with three players. We point out that Krasa, Temimi and Yannelis (2003) were the first to introduce the CS value allocations in the context of differential information economies; the extra twist of our approach is represented by the fact that, through the stable CS value allocation, agents are able to compare their prospects in the various coalition structures and choose the most advantageous one.

2 Basic notions

We will adopt the following notation. Given $n \in \mathbb{N}$, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space and $\mathbb{R}^n_+$ its nonnegative orthant. We will use the symbol $\geq$ to order vectors in $\mathbb{R}^n$ with the standard interpretation: $x \geq y \iff x_i \geq y_i$, for every $i = 1, \ldots, n$. For a set $S \subseteq \{1, \ldots, n\}$, π_S denotes the projection defined from $\mathbb{R}^n$ onto $\mathbb{R}^{|S|}$ which maps each $x \in \mathbb{R}^n$ onto the

element x_S with coordinates indexed by elements of S.
Let $N = \{1, \dots, n\}$ be a non-empty finite set of players. Each subset $S \subset N$ is referred to as a **coalition**. We denote the collection of coalitions, i.e., the set of all subsets of N, by $\mathcal{P}(N)$.
A **game with side payments** in characteristic form is a pair (N, ν) where $\nu : \mathcal{P}(N) \longrightarrow \mathbb{R}$ is a real-valued function such that $\nu(\emptyset) = 0$, called the characteristic function of the game.
The real number $\nu(S)$ may be interpreted as the total "utility" that members of S can divide among themselves when they agree to a contract about cooperation and act as a unit.
A **game without side payments** is a pair (N, V), where V is a correspondence from $\mathcal{P}(N)$ into $\mathbb{R}^n$ satisfying the following properties:

1. $V(S) \neq \emptyset$, for all $S \in \mathcal{P}(N)$;

2. if $x, y \in \mathbb{R}^n, S \in \mathcal{P}(N)$ and $x_i \geq y_i$ for all $i \in S,$ then $x \in V(S)$ implies that $y \in V(S)$ (comprehensiveness);

3. $V(N)$ is convex;

4. $V(S)$ is closed, for all $S \in \mathcal{P}(N)$;

5. $\pi_S(V(S)) \times \pi_T(V(T)) \subset \pi_{S \cup T}(V(S \cup T))$ for all disjoint S and T in $\mathcal{P}(N)$ (superadditivity).

For each $S \in \mathcal{P}(N)$, $V(S)$ can be interpreted as the set of feasible payoff vectors which players in that coalition can assure for themselves by acting cooperatively.
A game without side payments (N, V) is **compactly generated** if there exists a compact set $H \subseteq \mathbb{R}^n$ such that:

$$V(N) = \{x \in \mathbb{R}^n : \exists\, y \in H \text{ such that } y \geq x\}.$$

The Coalition Structure (CS) value, first introduced by Owen (1977) and further developed by Hart and Kurz (1983), generalizes the Shapley value to situations where players can voluntarily organize themselves into coalitions.
The simplest form of organization is considered, that of a partition over the players set: given a game (N, ν), a **coalition structure** is a finite partition $\mathcal{B} = \{B_1, \dots, B_m\}$ of N.
The CS value has the same properties of the Shapley value (in particular, it is

efficient and individually rational) and is given for each player, by some averaging of the player's expected contributions to the coalitions he is a member of. This expected contribution takes into consideration the coalition structure $\mathcal{B}$ in the following way.

An order on N is **consistent with** $\mathcal{B}$ if, for every $k = 1, \dots, m$ and every $i, j \in B_k$, all elements of N between i and j also belong to B_k. There are $(m!b_1!b_2! \dots b_m!)^{-1}$ such consistent orders (each assumed to be equally likely), where $b_k = |B_k|$, $k = 1, \dots, m$.

In analogy with the Shapley value, the CS value of a game (N, ν) given the coalition structure $\mathcal{B}$ is the unique payoff vector which assigns to player i the quantity:

$$\phi_i(\nu, \mathcal{B}) = \frac{1}{m!b_1! \cdots b_m!} \sum_{\pi_{\mathcal{B}}} \left[\nu(P(\pi_{\mathcal{B}}, i) \cup \{i\}) - \nu(P(\pi_{\mathcal{B}}, i)) \right],$$

where the sum is taken over all orders $\pi_{\mathcal{B}}$ consistent with the coalition structure $\mathcal{B}$ and, for each $\pi_{\mathcal{B}}$, $P(\pi_{\mathcal{B}}, i)$ denotes the coalition of players preceding player i in the order $\pi_{\mathcal{B}}$.

Since the quantity $\phi_i(\nu, \mathcal{B})$ can be interpreted as the expected utility of player i in participating in the game ν when players are organized in coalitions according to $\mathcal{B}$, each player is then able to compare his prospects in the various coalition structures. Based on this idea, Hart and Kurz (1983) have introduced a non-cooperative game of coalition formation whose solutions identify in a complete endogenous way the "stable" coalition structures; they present two models of stability, each based on the strong Nash equilibria of an appropriate game and differing in the reaction of the other players when a coalition breaks apart.

In Model Δ, players announce a coalition they would like to belong to; the players which announce the same list will result in a coalition.

Model Δ: The noncooperative game Δ is described by:

1. The *set of players* is $N = \{1, \dots, n\}$;
2. For each $i \in N$, the set of *strategies* of i is $\Sigma_i = \{S \subset N : i \in S\}$;
3. For each n-tuple of strategies $\sigma = (S_1, \dots, S_n) \in \Sigma_1 \times \dots \times \Sigma_n$ and each $i \in N$, the *payoff* to i is $\phi_i(\nu, \mathcal{B}_\sigma^{(\delta)})$, where $\mathcal{B}_\sigma^{(\delta)}$ is the coalition structure which forms according to the criterion explained above, that is: $\mathcal{B}_\sigma^\delta = \{T \subset N : i, j \in T \quad \text{if and only if} \quad S_i = S_j\}$.

In Model Γ, players announce a coalition they would like to belong to. A coalition forms if and only if all members make the same proposal.

Model Γ: The noncooperative game Γ is described by:

1. The *set of players* is $N = \{1, \ldots, n\}$;
2. For each $i \in N$, the set of *strategies* of i is $\Sigma_i = \{S \subset N : i \in S\}$;
3. For each n-tuple of strategies $\sigma = (S_1, \ldots, S_n) \in \Sigma_1 \times \cdots \times \Sigma_n$ and each $i \in N$, the *payoff* to i is $\phi_i(\nu, \mathcal{B}_\sigma^{(\gamma)})$, where $\mathcal{B}_\sigma^{(\gamma)}$ is the coalition structure which forms according to the criterion explained above, that is $\mathcal{B}_\sigma^{(\gamma)} = \{T_i^\sigma : i \in N\}$, where:
$$T_i^\sigma = \begin{cases} S_i, & \text{if } S_j = S_i \text{ for all } j \in S_i, \\ \{i\}, & \text{otherwise} \end{cases}$$

Considered a coalition structure $\mathcal{B}$ and a player $i \in N$, let $S_i^{\mathcal{B}}$ be the element of $\mathcal{B}$ to which i belongs and set $\sigma_{\mathcal{B}} = (S_i^{\mathcal{B}})_{i \in N}$. If the players choose $\sigma_{\mathcal{B}}$, then in both Δ and Γ the resulting coalition structure is $\mathcal{B}$.

Definition 1. *The coalition structure $\mathcal{B}$ is $\delta -$* **stable** *($\gamma -$* **stable***) in the game (N, ν) if $\sigma_{\mathcal{B}}$ is a strong equilibrium of Δ (Γ, respectively); that is, if there exists no nonempty $T \subset N$ and no $\widehat{\sigma}_i \in \Sigma_i$ for all $i \in T$, such that $\phi_i(\nu, \widehat{\mathcal{B}}) > \phi_i(\nu, \mathcal{B})$ for all $i \in T$, where $\widehat{\mathcal{B}}$ corresponds to $((\widehat{\sigma}_i)_{i \in T}, (\sigma_j^{\mathcal{B}})_{j \in N \setminus T})$.*

Remark 1. In general the existence of stable coalition structures is not guaranteed due to the fact that strong equilibria fail to exist in a broad class of situations; nonetheless Hart and Kurz (1984) have proven that all three players games have coalition structures that are both δ-stable and γ-stable.

3 Stable CS value allocations for pure exchange economies

The aim of this section is to introduce an "optimal" mechanism of resource allocation for an exchange economy, based on a cooperative solution concept. We recall the following definition (Centrone and Meo, 2008).

Definition 2. *Given a game without side payments (N, V), a* ***stable coalition structure valuation*** *(stable CS valuation, henceforth) is a pair (λ^*, ξ^*) such that:*

1. *$\lambda^* \in \Sigma$;*

2. *ξ^* is feasible for the grand coalition, that is $\xi^* \in V(N)$;*

3. *$\lambda_i^* \xi_i^* = \phi_i \left(\nu_{\lambda^*}, \mathcal{B}^s_{\lambda^*}\right), \forall\, i \in N$,*

where ν_{λ^*} is the game with side payments derived from (N, V) by allowing transfers of utilities at the rates λ^*, i.e.:

$$\nu_{\lambda^*}(S) = \begin{cases} \sup\left\{\sum_{i\in S} \lambda_i^* \xi_i \,:\, \xi \in V(S)\right\} & \text{if } S \neq \emptyset \\ \\ 0 & \text{if } S = \emptyset \end{cases} \tag{1}$$

and $\mathcal{B}^s_{\lambda^*}$ and $\phi_i(\nu_{\lambda^*}, \mathcal{B}^s_{\lambda^*})$ are, respectively, a stable coalition structure for the game ν_{λ^*} and the CS value for player i corresponding to $\mathcal{B}^s_{\lambda^*}$.

In Centrone and Meo (2008) the following theorem has been proved:

Theorem 1. *Let (N, V) be a compactly generated game without side payments, with $n = 3$. Then a stable CS valuation exists for this game.*

Consider now a pure exchange economy E consisting of a set $N = \{1, \ldots, n\}$ of economic agents and l commodities.
We assume that each agent is able to consume any commodity bundle in $\mathbb{R}^l_+$, that is, $\mathbb{R}^l_+$ represents the consumption set of each agent $i \in N$. Each agent $i \in N$ is characterized by the pair (e_i, u_i) where $e_i \in \mathbb{R}^l_+$ is his initial endowment of physical resources and u_i is the utility function representing his preferences or tastes over the various consumption bundles in his consumption set, that is:

$$u_i : \mathbb{R}^l_+ \longrightarrow \mathbb{R}.$$

Throughout the rest we will assume that $\sum_{i=1}^n e_i > 0$.
We denote a list of n consumption vectors of $\mathbb{R}^l_+$ by $\boldsymbol{x} = (x_1, \ldots, x_n)$ and call it an **allocation** of the economy E.
An allocation $\boldsymbol{x} = (x_1, \ldots, x_n)$ is **feasible** if it is a non-wasteful redistribution of the total endowment $\sum_{i=1}^n e_i$ among the n agents in the economy, that is:

$$\sum_{i=1}^n x_i = \sum_{i=1}^n e_i.$$

For a coalition $S \in \mathcal{P}(N)$, the allocation $\boldsymbol{x} = (x_1, \ldots, x_n)$ is called **S-feasible** if $\sum_{i\in S} x_i = \sum_{i\in S} e_i$.

Given an exchange economy E, the game without side payments (N, V_u) associated with it is defined in a very natural way as follows:

$$V_u : \mathcal{P}(N) \longrightarrow \mathbb{R}^n$$

$$\begin{aligned} V_u(S) =& \{\xi \in \mathbb{R}^n : \text{ there exists an S-feasible allocation } \boldsymbol{x} = (x_1, \ldots, x_n) \\ & \text{such that } \xi_i \leq u_i(x_i), \forall\, i \in S\}, \quad \forall\, S \in \mathcal{P}(N). \end{aligned} \tag{2}$$

The interpretation is clear: for each coalition S, we consider the utility levels that may arise, given each agent's utility function, subject to the constraint that no coalition can consume more than its total initial endowment.
Based on this strict link between general equilibrium and cooperative game theory, we propose the following definition.

Definition 3. *Given the exchange economy E, a* ***stable CS value allocation*** *is an allocation $\boldsymbol{x}^* = (x_1^*, \ldots, x_n^*)$ such that:*

i. *$\boldsymbol{x}^*$ is feasible;*

ii. *there exists $\lambda^* \in \Sigma$ such that $(\lambda^*, u_i(x_i^*))$ is a stable CS valuation of the game without side payments (N, V_u) associated with the economy E and defined by (2).*

Thus, $\boldsymbol{x}^*$ is a stable CS value allocation if it is feasible and:

$$\lambda_i^* u_i(x_i^*) = \phi_i(\nu_{u_{\lambda^*}}, \mathcal{B}_{\lambda^*}^s),\ \forall\, i \in N,$$

where $\nu_{u_{\lambda^*}}$ is the game with side payments associated with (N, V_u).
The interpretation is the following: the stable CS value allocation yields to each agent in the economy a "utility level" which is equal to the sum of the agent's expected marginal contributions to all coalitions he is a member of, when all the agents are organized in a stable coalition structure.

Remark 2. If the stable coalition structure we find in correspondence of the utility weights λ^* is the trivial one, i.e. $\mathcal{B}_{\lambda^*}^s = \{1\,|\,2\,|\,3\}$, then it is easily seen that the notion of stable CS value allocation coincides with the traditional notion of value allocation (Shapley, 1969).

We are now able to prove the following existence result:

Corollary 1. *Let E be an exchange economy with three agents.*
If the utility functions u_i are concave and continuous for each $i \in N$, then

1. *a stable CS value allocation* $\boldsymbol{x}^* = (x_1^*, x_2^*, x_3^*)$ *exists;*

2. *if* $\lambda^* = (\lambda_1^*, \lambda_2^*, \lambda_3^*)$ *are weights associated with* $\boldsymbol{x}^*$*, then* $\boldsymbol{x}^*$ *is constrained Pareto-optimal, that is* $\sum_{i \in N} \lambda_i^* u_i(x_i^*) = \nu_\lambda(N)$ *;*

3. $\boldsymbol{x}^*$ *is constrained individually rational, that is* $\lambda_i u_i(x_i) \geq \lambda_i u_i(e_i)$ *for every* $i \in N$*.*

Proof. The assumptions of continuity and concavity of the utility functions guarantee that (N, V_u) is a game without side payments.
The existence of a stable CS value allocation for the exchange economy E follows from Theorem 1 after noticing that the continuity of the u_i's assures that the subset of $\mathbb{R}^3$ defined by:

$$H = \{(u_1(x_1), u_2(x_2), u_3(x_3)) : \boldsymbol{x} = (x_1, x_2, x_3) \text{ is a feasible allocation for } E\}$$

is compact and therefore (N, V_u) is compactly generated.
The constrained Pareto optimality and the constrained individual rationality of the stable CS value allocations are respectively straightforward consequences of the efficiency and individual rationality of the CS value. □

4 Conclusions and future work

This short note wants to represent a first preliminary step towards the generalization of the value allocation notion (Shapley, 1969) to a context where an endogenous process of coalition formation is introduced into a pure bargaining situation among economic agents. Such a step is realized primarily by the merging of our previously introduced notion of stable CS valuation (Centrone and Meo, 2008) with the classical definition of value allocation: we introduce a new cooperative solution concept for a pure exchange economy model, the stable CS value allocation, for which we provide an existence result in the three–agent setting.
Many open issues still remain to be addressed in order to try to extend our result. The most important one concerns the existence of stable coalition structures for games with more than three players in order to obtain existence results for stable CS value allocations in economic models more general than the one we have considered here. Since all the trouble with the existence result stem from the fact that Hart and Kurz characterize stable coalition structure by means of strong equilibria, which fails to exist in a broad class of situations,

a possible way to obviate the problem is by using a Nash equilibrium notion and thus reinterpreting the non-cooperative coalition formation game substituting fuzzy coalitions for the traditional ones (the existence of Nash equilibria is indeed guaranteed in mixed strategies which call for such a substitution). Moreover, it would be of interest to investigate if one can provide non-cooperative foundations for stable CS value allocation in order to both explain the dynamics of reaching equilibrium outcomes and select among different stable coalition structures. This will be the subject of future work.

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