E C O N O M I C S   B U L L E T I N

Consistency and the core for fuzzy non-transferable-utility games

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Abstract

Different from the works of Hwang (2007), we provide two extensions of the reduced games introduced by Moulin (1985) and Voorneveld and van den Nouweland (1998) on fuzzy non-transferable-utility (NTU) games, respectively. Based on the reduced games, we provide an axiomatization of the core and show that the technique of the proof in Tadenuma (1992) can not be applied to the core in the context of fuzzy NTU games.

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1 Introduction

In the axiomatic characterization of solutions of standard coalition TU games, *consistency* is a crucial property which has been applied comprehensively. If a solution is not consistent, then a subgroup of agents might not respect the original compromise but revise the payoff distribution within the subgroup. The fundamental property of solutions has been investigated in various classes of problems by applying reduced games always. The core is, perhaps, the most intuitive solution concept in game theory. Relating to the core of standard coalition games, there are two different types of imaginary reduced standard coalition games in the literature, the "*max-reduced game*" (Davis-Maschler, 1965) and the "*complement-reduced game*" (Moulin, 1985). Based on the max-reduced games, Peleg (1986) characterized the core on the domain of standard coalition games whose core is non-empty. Subsequently, Serrano and Volij (1998) characterized the core on the domain of all standard coalition games. Based on the complement-reduced games, Tadenuma (1992) characterized the core on the domain of standard coalition games whose core is non-empty. Related results may be found in Peleg (1985), Voorneveld and Nouweland (1998), and so on.

The theory of fuzzy games started with work of Aubin (1974, 1981) where the notions of a fuzzy game and the core of a fuzzy game are introduced. Hwang (2007) extended the core and the max-reduced game to fuzzy NTU games. Inspired by Serrano and Volij (1998), Hwang (2007) offered axiomatizations of the core of fuzzy NTU games.

Different from the works of Hwang (2007), we provide an extension of the reduced game introduced by Voorneveld and van den Nouweland (1998) to fuzzy NTU games. Based on this extended reduction, an axiomatization of the core is proposed. On the other hand, we extend to the fuzzy NTU games case the complement-reduced game. Based on this extended complement-reduction, we show that the core is "not" the only solution satisfying non-emptiness, individual rationality, and related consistency on the domain of fuzzy NTU games with a nonempty core.

2 Preliminaries

Let U be the universe of players. If $N \subseteq U$ is a set of players, then a **fuzzy coalition** is a vector $\alpha \in [0,1]^N$. The i-th coordinate α_i of α is called the participation level of player i in the fuzzy coalition α. For all $T \subseteq N$, let $|T|$ be the number of elements in T. Instead of $[0,1]^T$, we will write F^T for the set of fuzzy coalitions. A player-coalition $T \subseteq N$ corresponds in a canonical way to the fuzzy coalition $e^T(N) \in F^N$, which is the vector with $e^T_i(N) = 1$ if $i \in T$, and $e^T_i(N) = 0$ if $i \in N \setminus T$. The fuzzy coalition $e^T(N)$ corresponds to the situation where the players in T fully cooperate (i.e. with participation level 1) and the players outsides T are not involved at all (i.e. they have participation level 0). Denote the zero vector in $\mathbb{R}^N$ by 0^N. The fuzzy coalition 0^N corresponds to the empty player-coalition. Note that if no confusion can arise $e^T(N)$ will be denoted by e^T.

To state the core of a fuzzy NTU game, some more notations will be needed. Let $\alpha \in F^N$, $A(\alpha, N) = \{i \in N \mid \alpha_i > 0, \alpha \in F^N\}$ is the set of players who participate in α. Let $x, y \in \mathbb{R}^N$. $x \geq y$ if $x_i \geq y_i$ for all $i \in N$; $x > y$ if $x \geq y$ and $x \neq y$; $x \gg y$ if $x_i > y_i$ for all $i \in N$. We denote $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N \mid x \geq 0^N\}$. Let $A \subseteq \mathbb{R}^N$. A is **comprehensive** if $x \in A$ and $x \geq y$ imply $y \in A$. The **boundary** of A is denoted by ∂A, and the **interior** of A is denoted by $intA$. If $x \in \mathbb{R}^N$ then $x + A = \{x + a \mid a \in A\}$.

Definition 1 *A* **fuzzy NTU game** *is a pair* (N, V)*, where* N *is a non-empty and finite set of players and* V *is a characteristic function that assigns to each fuzzy coalition* $\alpha = (\alpha_i)_{i \in N} \in F^N \setminus \{0^N\}$ *a subset* $V(\alpha)$ *of* $\mathbb{R}^{A(\alpha,N)}$*, such that*

1. $V(\alpha)$ *is non-empty, closed and comprehensive,*
2. $V(\alpha) \cap (x + \mathbb{R}_+^{A(\alpha,N)})$ *is bounded for each* $x \in \mathbb{R}^{A(\alpha,N)}$*,*
3. *if* $x, y \in \partial V(e^N)$ *and* $x \geq y$*, then* $x = y$*. (non-levelness)*

Denote the class of all fuzzy NTU games by $\mathcal{FG}$. Let $(N, V) \in \mathcal{FG}$. A **payoff vector** of (N, V) is a vector $x = (x_i)_{i \in N} \in \mathbb{R}^N$. Then

- A payoff vector x of $(N, V) \in \mathcal{FG}$ is **efficient (EFF)** if $x \in \partial V(e^N)$.
- A payoff vector x of $(N, V) \in \mathcal{FG}$ is **individually rational (IR)** if for all $i \in N$ and for all $j \in (0, 1]$, $jx_i \notin intV(je^{\{i\}})$.

Moreover, x is an **imputation** of (N, V) if it is EFF and IR. The set of imputations of (N, V) is denoted by

$$I(N, V) = \{ \ x \in \mathbb{R}^N \mid x \text{ is an imputation of } (N, V) \ \}.$$

Definition 2 *The* **core** $C(N, V)$ *of* $(N, V) \in \mathcal{FG}$ *consists of all* $x \in \partial V(e^N)$ *that satisfy for all* $\alpha \in F^N \setminus \{0^N\}$*,* $(\alpha_i x_i)_{i \in A(\alpha,N)} \notin intV(\alpha)$*.*

3 Axioms, V-N Reduction and Axiomatization

In this section, we characterize the core based on the extension of the reduced game introduced by Voorneveld and van den Nouweland (1998).

A **solution on** $\mathcal{FG}$ is a function σ which associates with each $(N, V) \in \mathcal{FG}$ a subset $\sigma(N, V)$ of $V(e^N)$. Let σ be a solution on $\mathcal{FG}$. σ satisfies **Non-emptiness (NE)** if for all $(N, V) \in \mathcal{FG}$, $\sigma(N, V) \neq \emptyset$. σ satisfies **Efficiency (EFF)** if for all $(N, V) \in \mathcal{FG}$, $\sigma(N, V) \subseteq \partial V(e^N)$. σ satisfies **Individual rationality (IR)** if for all $(N, V) \in \mathcal{FG}$, $\sigma(N, V) \subseteq I(N, V)$. σ satisfies **One-person rationality (OPR)** if for all $(N, V) \in \mathcal{FG}$ with $|N| = 1$, $\sigma(N, V) = I(N, V)$.

Hwang (2007) extended to the fuzzy NTU games case the reduced game introduced by Davis and Maschler (1965) as follows.

Definition 3 [1] *Let* $(N, V) \in \mathcal{FG}$*,* $x \in \mathbb{R}^N$ *and* $S \subseteq N$*,* $S \neq \emptyset$*. The* **DM-reduced game with respect to** S **and** x *is the game* $(S, V^{DM}_{S,x})$ *defined by for all* $\alpha \in F^S \setminus \{0^S\}$*,*

$$\begin{array}{ll} V^{DM}_{S,x}(\alpha) = \{y \in \mathbb{R}^S \mid (y, x_{N \setminus S}) \in V(e^N)\} & , \ \textit{if } \alpha = e^S(S) \\ V^{DM}_{S,x}(\alpha) = \bigcup_{\beta \in F^{N \setminus S}} \{y \in \mathbb{R}^{A(\alpha,S)} \mid \Big(y, (\beta_i x_i)_{i \in A(\beta, N \setminus S)}\Big) \in V(\alpha, \beta)\} & , \ \textit{otherwise}\ . \end{array}$$

[1]From now on we restrict our attention to bounded fuzzy NTU games, defined as those games (N, V) such that, there exists a real number M_v such that for all $\alpha \in F^N \setminus \{0^N\}$ and for all $x \in V(\alpha)$, $x(\alpha) \leq M_v$. We use it here in order to guarantee that, in the Definition 3, $V^{DM}_{S,x}(\alpha)$ is well-defined.

Consistency [2] requires that if x is prescribed by σ for a game (N, V), then the projection of x to S should be prescribed by σ for the reduced game with respect to S and x for all S. Thus, the projection of x to S should be consistent with the expectations of the members of S as reflected by their reduced game.

- **DM-consistency (DMCON)**: If $(N, V) \in \mathcal{FG}$, $S \subseteq N$, $S \neq \emptyset$, and $x \in \sigma(N, V)$, then $(S, V_{S,x}^{DM}) \in \mathcal{FG}$ and $x_S \in \sigma(S, V_{S,x}^{DM})$.

Converse consistency requires that if the projection of an efficient payoff vector x to every proper S is consistent with the expectations of the members of S as reflected by their reduced game then x itself should be recommended for whole game.

- **Converse DM-consistency (CDMCON)**: If $(N, V) \in \mathcal{FG}$ with $|N| \geq 2$, $x \in \partial V(e^N)$, and for all $S \subset N$, $0 < |S| < |N|$, $(S, V_{S,x}^{DM}) \in \mathcal{FG}$ and $x_S \in \sigma(S, V_{S,x}^{DM})$, then $x \in \sigma(N, V)$.

- **Weak converse DM-consistency (WCDMCON)**: If $(N, V) \in \mathcal{FG}$ with $|N| \geq 2$, $x \in I(N, V)$, and for all $S \subset N$, $0 < |S| < |N|$, $(S, V_{S,x}^{DM}) \in \mathcal{FG}$ and $x_S \in \sigma(S, V_{S,x}^{DM})$, then $x \in \sigma(N, V)$.[3]

Hwang (2007) characterized the core by means of IR, OPR, DMCON and WCDMCON. Next, we extend to the fuzzy NTU games case the reduced game introduced by Voorneveld and van den Nouweland (1998).

Definition 4 *Let $(N, V) \in \mathcal{FG}$, $x \in \mathbb{R}^N$ and $S \subseteq N$, $S \neq \emptyset$. The* **V-N reduced game with respect to** S **and** x *is the game $(S, V_{S,x}^{VN})$ defined by for all $\alpha \in F^S \setminus \{0^S\}$,*

$$\begin{array}{ll} V_{S,x}^{VN}(\alpha) = \{y \in \mathbb{R}^S \mid (y, x_{N \setminus S}) \in V(e^N)\} & , \textit{ if } \alpha = e^S(S) \\ V_{S,x}^{VN}(\alpha) = \bigcup\limits_{\substack{\beta \in F^{N \setminus S} \\ \beta \neq 0^{N \setminus S}}} \{y \in \mathbb{R}^{A(\alpha,S)} \mid \big(y, (\beta_i x_i)_{i \in A(\beta, N \setminus S)}\big) \in V(\alpha, \beta)\} & , \textit{ otherwise} . \end{array}$$

"V-N reduced game" instead of "DM-reduced game", we introduce the properties of **V-N consistency (VNCON)**, **converse V-N consistency (CVNCON)** and **weak converse V-N consistency (WCVNCON)**.

Lemma 1

1. *The core satisfies V-N consistency.*
2. *The core satisfies weak converse V-N consistency.*

Proof. The proofs are immediate analogues of Lemmas 8, 9, 10, 11 and 12 in Hwang (2007), hence we omit it. ∎

Lemma 2 *Let σ be a solution on $\mathcal{FG}$. If σ satisfies IR and VNCON, then it also satisfies EFF.*

[2] The axiom was originally introduced by Harsanyi (1959) under the name of bilateral equilibrium. For discussion of this axiom, please see Thomson (2005).

[3] The following axiom is a weakening of the previous axiom, since it requires that x be individually rational as well.

Proof. It can easily be deduced from the proof of Lemma 5.4 in Peleg (1985). ■

Inspired by Serrano and Volij (1998), we provide an axiomatization of the core on fuzzy NTU games.

Theorem 1 *A solution σ on $\mathcal{FG}$ satisfies OPR, IR, VNCON and WCVNCON if and only if for all $(N, V) \in \mathcal{FG}$, $\sigma(N, V) = C(N, V)$.*

Proof. By Lemma1, the core satisfies VNCON and WCVNCON. And clearly, the core satisfies OPR and IR.

To prove uniqueness, assume that a solution σ satisfies OPR, IR, VNCON and WCVNCON. By Lemma 2, σ satisfies EFF. Let$(N, V) \in \mathcal{FG}$. The proof proceeds by induction on the number $|N|$. If $|N| = 1$ then by OPR of σ, $\sigma(N, V) = I(N, V) = C(N, V)$. Assume that $\sigma(N, V) = C(N, V)$ if $|N| < k$, $k \geq 2$.
The case $|N| = k$:
First we prove that $\sigma(N, V) \subseteq C(N, V)$. Let $x \in \sigma(N, V)$. Since σ satisfies IR and EFF, $x \in I(N, V)$. By VNCON of σ, for all $S \subset N$ with $0 < |S| < |N|$, $x_S \in \sigma(S, V_{S,x}^{VN})$. By the induction hypothesis, for all $S \subset N$ with $0 < |S| < |N|$, $x_S \in \sigma(S, V_{S,x}^{VN}) = C(S, V_{S,x}^{VN})$. By WCVNCON of the core, $x \in C(N, V)$. The opposite inclusion may be shown analogously by interchanging the roles of σ and C. Hence $\sigma(N, V) = C(N, V)$. ■

The following examples show that each of the axioms used in Theorem 1 is logically independent of the others. [4]

Example 1 *Let $\sigma(N, V) = \emptyset$ for all $(N, V) \in \mathcal{FG}$. Then σ satisfies IR, VNCON and WCVNCON, but it violates OPR.*

Example 2 *Define the solution σ on $\mathcal{FG}$ by*

$$\sigma(N, V) = \begin{cases} I(N, V) & , \text{ if } |N| = 1 \\ \partial V(e^N) & , \text{ otherwise.} \end{cases}$$

Then σ satisfies OPR, VNCON and WCVNCON, but it violates IR.

Example 3 *Let $\sigma(N, V) = I(N, V)$ for all $(N, V) \in \mathcal{FG}$. Then σ satisfies OPR, IR and WCVNCON, but it violates VNCON.*

Example 4 *Define the solution σ on $\mathcal{FG}$ by*

$$\sigma(N, V) = \begin{cases} I(N, V) & , \text{ if } |N| = 1 \\ \emptyset & , \text{ otherwise.} \end{cases}$$

Then σ satisfies OPR, IR and VNCON, but it violates WCVNCON.

4 Moulin Reduction

In this section, we extend to the fuzzy NTU games case the reduced game introduced by Moulin (1985). Given $(N, V) \in \mathcal{FG}$, $S \subseteq N$ with $S \neq \emptyset$ and a payoff vector x. The

[4] In order to show the logical independence of the used axioms $|U| \geq 2$ is needed.

M-reduced game with respect to S **and** x is the game $(S, V^M_{S,x})$ defined as: for all $\alpha \in F^S$,

$$V^M_{S,x}(\alpha) = \{y \in \mathbb{R}^{S(\alpha)} | \Big(y, x_{N\setminus S}\Big) \in V(\alpha, e^N_{N\setminus S})\}.$$

"M-reduced game" instead of "DM-reduced game", we introduce the properties of **M-consistency (MCON)**, **converse M-consistency (CMCON)** and **weak converse M-consistency (WCMCON)**.

Lemma 3 *The core satisfies M-consistency.*

Proof. The proof of this lemma is similar to Lemmas 9, 10 and 11 in Hwang (2007), hence we omit it. ∎

The following example shows that the core violates (W)CMCON.

Example 5 *Define a game* (N, V) *by* $N = \{i, k\}$ *and*

$$V(p,q) = \begin{cases} \{(r,s)|p+q \leq 4\} & , \textit{ if } (p,q) = (1,1) \\ \{t|t \leq p+1\} & , \textit{ if } p \in (0,1],\ q = 0 \\ \{t|t \leq q+1\} & , \textit{ if } q \in (0,1],\ p = 0 \\ \{(r,s)|r+s \leq p+q+1\} & , \textit{ otherwise.} \end{cases}$$

Let $x \in I(N,V)$ *with* $x_i = \frac{3}{2}, x_k = \frac{5}{2}$*. Cleraly,* $V^M_{\{i\},x}(1) = \{t|(t,\frac{5}{2}) \in V(1,1)\}$*,* $V^M_{\{k\},x}(1) = \{t|(\frac{3}{2},t) \in V(1,1)\}$*,* $V^M_{\{i\},x}(p) = \{t|(t,\frac{5}{2}) \in V(p,1)\}$ *for all* $p \in (0,1)$ *and* $V^M_{\{k\},x}(q) = \{t|(\frac{3}{2},t) \in V(1,q)\}$*. It is easy to derive that* $x_i \in C(\{i\}, V^M_{\{i\},x})$ *and* $x_k \in C(\{k\}, V^M_{\{k\},x})$*. But, for all* $p \in (0,1]$*,* $p \cdot x_i \in \mathbf{int}V(p,0)$*. That is,* $x \notin C(N,v)$*. The core is not (W)CMCON.*

We say that the fuzzy NTU game (N,V) is *balanced* if $C(N,V) \neq \emptyset$. Let $\mathcal{FG}^c$ denote the set of all balanced fuzzy NTU games. To end up this section, we show that on $\mathcal{FG}^c$, there are solutions other than the core satisfying non-emptiness, individual rationality and consistency.

Lemma 4 *On* $\mathcal{FG}^c$*, there are solutions other than the core satisfying non-emptiness, individual rationality and M-consistency.*

Proof. Clearly, we have known that on $\mathcal{FG}^c$, the core satisfies the three properties. The remaining proof is by way of an example. Define a solution C_{mw} on $\mathcal{FG}$ by for all $(N,V) \in \mathcal{FG}$,

$$\begin{aligned} & C_{mw}(N,V) \\ = \ & \{x \in I(N,V) | (\alpha_i x_i)_{i \in A(\alpha,N)} \notin \mathbf{int}V(\alpha) \text{ for all } \alpha \in F^N,\ \alpha_i = 1 \text{ for some } i \in N\}. \end{aligned}$$

Since C_{mw} is a solution on $\mathcal{FG}$, it is also a solution on $\mathcal{FG}^c$. Clearly, it satisfies NE and IR on $\mathcal{FG}^c$. To verify that C_{mw} satisfies MCON on $\mathcal{FG}^c$, it can easily be deduced from the proof of Lemma 1, hence we omit it. Similar to Lemma 1, it is also easy to derive that C_{mw} satisfies WCMCON but it violates CMCON.

Furthermore, similar to Theorem 1, the solution C_{mw} is the only solution satisfying IR, OPR, MCON and WCMCON on $\mathcal{FG}$. ∎

Remark 1 *Based on Lemma 4, the core is "not" the only solution satisfying NE, IR and MCON on the domain of balanced fuzzy NTU games. This means that the technique of the proof in Tadenuma (1992) can not be applied to the core in the context of fuzzy NTU games. The reason is that the core does not satisfy (weak) converse M-consistency.*

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