E c o n o m i c s B u l l e t i n

A Note on Jackson's Theorems in Bayesian Implementation

Ismail Saglam
TOBB University of Economics and Technology

Abstract

This note shows that in an incomplete information situation the closure condition will be satisfied by all social choice sets if and only if the set of states of the society which all agents believeoccur with positive probability satisfies the `connection' condition.It then follows from Jackson's [1] fundamental theorems that whenever `connection' is satisfied and there are at least three agents in the society, for the implementability of social choice sets in Bayesian equilibrium the incentive compatibility and Bayesian monotonicity conditions are both necessary and sufficient in economic environments. It also follows that the incentive compatibility and monotonicity-no-veto conditions are sufficient in noneconomic environments.

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1. INTRODUCTION

In his seminal paper, Jackson [1] examined the problem of implementing collections of social choice functions in situations where agents have incomplete information about the state of the society. His work has very important features;he characterized conditions for implementability not only in economic environments but also in noneconomic environments both of which admit situations with externalities. The economic environments he considered is much more general than exchange economies, as the former cover any environment in which agents cannot be simultaneously satisfied. Moreover, the existence of a worst outcome from the viewpoints of all agents in the society is not needed for his theorems characterizing implementable social choice sets. Regarding the distribution of information among the members of a society, he allowed for situations where agents possess exclusive information. Besides, the set of states which all agents in the society believe occur with positive probability is not necessarily required to coincide with the set of possible states of the society.

His first theorem showed that a collection of social choice functions in an economic environment with at least three agents is Bayesian implementable if and only if closure (C), incentive compatibility (IC), and Bayesian monotonicity (BM) conditions are satisfied. As he stated, this result closed the gap between the necessary and sufficient conditions[1] of Palfrey and Srivastava [4], who examined[2] implementation for exchange economies in which agents may have exclusive information.

The second theorem of Jackson [1] showed that closure, incentive compatibility, and a condition that combines Bayesian monotonicity and no-veto power (which he calls (MNV)) are sufficient for implementation in noneconomic environments with three or more agents.

The closure condition in the implementation literature requires that the social choice set be closed under concatenation of common knowledge events. In this paper, we examine the situations in which the closure condition is satisfied by any collection of social choice rules in both economic and noneconomic environments. To this end, we define a new condition for all environments, which we call 'connection'. An environment is said to satisfy connection if between any two probable states, there exists a string of probable states such that from any one of them to the other, there is always one agent who cannot make the distinction between the two. We first prove that if there is always one agent who cannot make the distinction between any two probable states, then the only event which is common knowledge can be the set of all probable states (Lemma 1). Then, it easily follows that all social choice sets satisfy closure if and only if the environment, whether economic or noneconomic, satisfies connection (Proposition 1). We also show

[1] Palfrey and Srivastava [4] showed that a collection of social choice rules is implementable in Bayesian (Nash) equilibrium if it satisfies the Bayesian monotonicity and incentive compatibility conditions. Moreover, they showed that Bayesian monotonicity and a stronger incentive compatibility condition (ε-IC) are sufficient for implementation.

[2] See also Palfrey and Srivastava [2], and Postlewaite and Schmeidler [5] for Bayesian implementation results in exchange economies where there are at least three agents and the information is nonexclusive.

that this new condition for environments becomes less restrictive as the number of possible information states of agents or the number of agents increases; so we should expect the connection condition to be satisfied in the limit (Proposition 2).

Apparently, the connection condition allows us to restate Jackson's [1] implementation results. We simply argue that when the environment with at least three agents satisfies connection, the designer should pay attention to only (IC) and (BM) in economic environments (Corollaries 3-5) and (IC) and (MNV) in noneconomic environments for Bayesian implementation (Corollary 6).

The paper is organized as follows: Section 2 reintroduces the environment and preliminary definitions of Jackson [1]. Section 3 presents Jackson's [1] results in Bayesian implementation. Finally, Section 4 gathers the results of this paper and some concluding remarks.

2. BASIC STRUCTURES (JACKSON [1])

Environments

There are a finite number, N, of agents. Let S^i describe the finite number of possible information sets of agent i. A state is a vector $s = (s^1, ..., s^N)$ and the set of states is $S = \prod_{i=1}^{N} S^i$.

Let A denote the set of feasible allocations. A social choice function is a function from states to allocations. The set of all social choice functions is $X = \{x | x : S \to A\}$.

Each agent has a probability measure q^i defined on S. It is assumed that if $q^i(s) > 0$ for some i and $s \in S$, then $q^j(s) > 0$ for all $j \neq i$. Let T denote the set of states which all agents believe occur with positive probability, that is $T = \{s \in S | q^i(s) > 0,\ \forall i\}$.

Π^i are partitions of T defined by q^i. For a given information set $s^i \in S^i$, $\pi^i(s^i) = \{t \in S | t^i = s^i \text{ and } q^i(t) > 0\} \in \Pi^i$ denotes the set of states which i believes may be the true state. It is assumed that $\pi^i(s^i) \neq \emptyset$ for all i and $s^i \in S^i$. Let Π denote the finest partition which is coarser than each Π^i. For a given state $s \in S$, let $\pi(s)$ be the element of Π which contains s.

Each agent has preferences $U^i : A \times S \to \Re_+$ over social choice functions which have a conditional expected utility representation. Given $x, y \in X$ and $s^i \in S^i$, agent i's weak preference relation R^i is such that

$$xR^i(s^i)y \Leftrightarrow \sum_{s \in \pi^i(s^i)} q^i(s)U^i[x(s), s] \geq \sum_{s \in \pi^i(s^i)} q^i(s)U^i[y(s), s].$$

Preferences are complete and transitive. The strict preference and indifference relations associated with R^i are P^i and I^i, respectively.

An environment is a collection $[N, S, A, \{q^i\}, \{U^i\}]$, whose structure is assumed to be common knowledge among the agents.

Definitions

DEFINITION 1. A *social choice set* is a subset of X.

DEFINITION 2. The social choice functions x and y are *equivalent* if $x(s) = y(s)$ for all $s \in T$. The social choice sets F and $\hat{F}$ are *equivalent* if for each $x \in F$ there exists $\hat{x} \in \hat{F}$ which is equivalent to x, and for each $\hat{x} \in \hat{F}$ there exists $x \in F$ which is equivalent to $\hat{x}$.

DEFINITION 3. Let $x/_C z$ be a splicing of two social choice functions x and z along a set $C \in S$. The social choice function $x/_C z$ is defined by $[x/_C z](s) = x(s)\ \forall s \in C$, and $[x/_C z](s) = z(s)$ otherwise. An environment satisfies (E) if for any $z \in X$ and $s \in S$, there exist i and j $(i \neq j)$, $x \in X$ and $y \in X$ such that x and y are constant, $x/_C zP^i(s^i)z$ and $y/_C zP^j(s^j)z$ for all $C \subset S$ such that $s \in C$. Environments satisfying (E) are said to be *economic*.

DEFINITION 4. Let B and D be any disjoint sets of states such that $B \cup D = T$ and for any $\pi \in \Pi$ either $\pi \subset B$ or $\pi \subset D$. A social choice set F satisfies *closure* (C) if for any $x \in F$ and $y \in F$ there exists $z \in F$ such that $z(s) = x(s)\ \forall s \in B$ and $z(s) = y(s)$ $\forall s \in D$.

DEFINITION 5. Given i, $x \in X$, and $t^i \in S^i$, define x_{t^i} by $x_{t^i}(s) = x(s^{-i}, t^i)$, $s \in S$. A social choice set F satisfies *incentive compatibility* (IC) if for all $x \in F$, i, and $t^i \in S^i$,

$$xR^i(s^i)x_{t^i} \quad \forall s^i \in S^i.$$

DEFINITION 6. A *deception* for i is a mapping $\alpha^i : S^i \to S^i$. Let $\alpha = (\alpha^1, ..., \alpha^N)$ and $\alpha(s) = [\alpha^1(s^1), ..., \alpha^N(s^N)]$. Let $x{\circ}\alpha$ represent the social choice function which results in $x[\alpha(s)]$ for each $s \in S$.

DEFINITION 7. Consider $x \in F$ and a deception α. A social choice set f satisfies *Bayesian monotonicity* (BM) if whenever there is no social choice function in F which is equivalent to $x{\circ}\alpha$, there exists $i, s^i \in S^i$ and $y \in X$ such that

$$y{\circ}\alpha\, P^i(s^i)\, x{\circ}\alpha, \quad \text{while} \quad x\, R^i(t^i)\, y_{\alpha^i(s^i)} \quad \forall t^i \in S^i.$$

DEFINITION 8. A social choice function $z \in X$ satisfies the *no-veto hypothesis* (NVH) at $s \in T$ if there exists i such that $zR^j(s^j)b^j/_S z$ for all $j \neq i$.

DEFINITION 9. Consider the social choice set F, a deception α, and for each $x \in F$ and i consider a set $B^i_x \subset S^i$. Let $B_x = B^1_x \times ... \times B^N_x$. Suppose that there exists z such that for each $x \in F$ and $s \in B_x$, $z(s) = x{\circ}\alpha(s)$. Furthermore, suppose that z satisfies (NVH) for each $s \in T - (U_{x \in F} B_x)$. F satisfies *monotonicity-no-veto* (MNV) if whenever there is no social choice function in F which is equivalent to z, there exists i, $y \in X$,

$x \in F$, and $s^i \in B^i_x$ such that

$$y_{\circ}\alpha/_{B_x} z \, P^i(s) \, z, \quad \text{while} \quad x \, R^i(t^i) \, y_{\alpha^i(s^i)} \; \forall t^i \in S^i.$$

DEFINITION 10. An environment is said to have a "0" *outcome* if there exists a $0 \in A$ such that $U^i(0, s) = 0$ for all i and $s \in T$, and for each $s \in T$ and $a \neq 0$ there exists i such that $U^i(a, s) > 0$. In such an environment, given $x \in X$, let x^0 denote the allocation such that $x^0(s) = x(s)$ for $s \in T$ and $x^0(s) = 0$ otherwise. Given a social choice set F, let F^0 be the social choice set which is equivalent to F and such that $x = x^0$ for all $x \in F^0$.

Implementation

A mechanism is a pair consisting of an action space $M = \prod_{i=1}^{N} M^i$ and a function $g : M \to A$.

A strategy for agent i is a mapping $\sigma^i : S^i \to M^i$. Let $\sigma = [\sigma^1, ..., \sigma^N]$ and $\sigma(s) = (\sigma^1(s^1), ..., \sigma^N(s^N))$ and $g(\sigma)$ be the allocation which results when σ is played.

A vector of strategies σ is a Bayesian (Nash) equilibrium if $g(\sigma) R^i(s^i) \; g(\sigma^{-i}, \tilde{\sigma}^i)$ for all i, s^i, and $\tilde{\sigma}^i$.

A mechanism (M, g) implements a social choice set F if:

(i) for any $x \in F$ there exists an equilibrium σ with $g[\sigma(s)] = x(s)$ for all $s \in T$, and

(ii) for any equilibrium σ there exists $x \in F$ with $g[\sigma(s)] = x(s)$ for all $s \in T$.

A social choice set F is implementable if there exists a mechanism (M, g) which implements F.

3. IMPLEMENTATION RESULTS OF JACKSON [1]

THEOREM 1. (Jackson [1]) *In an environment which satisfies (E) and $N \geq 3$, a social choice set F is implementable if and only if there exists a social choice set $\hat{F}$ which is equivalent to F and satisfies (C), (IC), and (BM).*

COROLLARY 1. (Jackson [1]) *In an environment which satisfies (E), $S = T$, and $N \geq 3$, a social choice set F is implementable if and only if it satisfies (C), (IC), and (BM).*

COROLLARY 2. [1]) *In an environment which satisfies (E) and $N \geq 3$, and has a 0 outcome, a social choice set F is implementable if and only if F^0 satisfies (C), (IC), and (BM).*

THEOREM 2. (Jackson [1]) *If $N \geq 3$, social choice set F which satisfies (C), (IC), and (MNV), is implementable.*

4. RESULTS

Definition 11. A set of states $T \subseteq S$ satisfies the *connection* (CO) condition if for all $s_a \in T$ and $s_b \in T$ there exists a string of states $s_a \equiv s_0, s_1, ..., s_r \equiv s_b$ such that for all $k \in \{0, ..., r-1\}$ there exists an agent $i(k)$ satisfying $s_k^{i(k)} = s_{k+1}^{i(k)}$.

Lemma 1. *An environment satisfies (CO) if*[3] *and only if* $\Pi = \{T\}$.

Proof. We will first show that $\Pi = \{T\}$ implies (CO). Take any environment such that $\Pi = \{T\}$. Suppose towards a contradiction that (CO) does not hold. Then there exist some $s_a, s_b \in T$ such that there exists no string of states $s_a \equiv s_0, s_1, ..., s_r \equiv s_b$ satisfying that for all $k \in \{0, ..., r-1\}$ there exists some agent $i(k)$ such that $s_k^{i(k)} = s_{k+1}^{i(k)}$. Now consider $\pi(s_a)$. We have $\pi(s_a) = T$ since $\Pi = \{T\}$. We also have $s_b \notin \pi(s_a)$ since (CO) does not hold, contradicting that $s_b \in T$. Therefore in any environment satisfying $\Pi = \{T\}$, (CO) must hold.

To show the sufficiency part, assume (CO) is satisfied. Take any $\tilde{s} \in T$ and $s \in T$. Since (CO) holds by assumption, there exists a string of states $\tilde{s} \equiv s_0, s_1, ..., s_r \equiv s$ such that for all $k \in \{0, ..., r-1\}$ there exists $i(k)$ such that $s_k^{i(k)} = s_{k+1}^{i(k)}$. Thus, $s \in \pi(\tilde{s})$. Since this is true for all $s \in T$, we have $T \subseteq \pi(\tilde{s})$. We also have $\pi(\tilde{s}) \subseteq T$ (by the suppositions that $\pi(s) \in \Pi$ and Π is a partition of T). It then follows that $\pi(\tilde{s}) = T$. Therefore, $\Pi = \{T\}$. □

Note that S satisfies the (CO) condition since for all $s_a \in S$ and $s_b \in S$ the string $s_a \equiv s_0, s_1, s_2 \equiv s_b$ with $s_1 = (s_a^{-i}, s_b^i)$ for some agent i connects s_a to s_b. (Note s_1 is an element of S as $S = S^1 \times ... \times S^N$). See Example 1 of Jackson [1] for an example of $T \subset S$ satisfying the (CO) condition.

Proposition 1. *All social choice sets satisfy closure if and only if the environment satisfies connection.*[4]

Proof. To show the 'if' part, take any environment which satisfies (CO). Then $\Pi = \{T\}$ by Lemma 1. Let K be defined as

$$K = \{(B, D) | B \cap D = \emptyset,\ (B \cup D) = T \text{ and } \forall \pi \in \Pi, \pi \in B \text{ or } \pi \in D\}.$$

It is obvious that whenever $\Pi = \{T\}$, Π has the single element $\pi = T$. Thus we have

$$K = \{(T, \emptyset), (\emptyset, T)\}.$$

[3]The 'if part' of the Lemma 1 as well as the need for the (CO) condition for an iff statement were proposed by Matthew Jackson, for which the author is grateful. The previous version of Lemma 1 was just an 'only if' statement which stated that a condition stronger than (CO) implies $\Pi = \{T\}$.

[4]Lemma 1 and the proof of Proposition 1 suggest that Proposition 1 can be restated by replacing the connection condition in the statement with $\Pi = \{T\}$. The need for Definition 11 will become clear while proving Proposition 2.

Any social choice set F then satisfies closure since for any $x \in F$ and $y \in F$, we have a social choice function $z \in F$ given by

$$z = \begin{cases} x & \text{if } B = T \\ y & \text{if } D = T \end{cases}$$

implying that $z(s) = x(s)\ \forall s \in B$ and $z(s) = y(s)\ \forall s \in D$.

To show the 'only if' part, suppose (CO) is not satisfied implying that there exist $\pi_1, \pi_2 \in \Pi$ such that $\pi_1 \neq \pi_2$. Then the social choice set $F = \{x(.), y(.)\}$ where $x(s) \neq y(s)$ for all $s \in S$ does not satisfy closure. □

We can now obtain a corollary for Theorem 1 of Jackson [1].

COROLLARY 3. *In an environment which satisfies (E), (CO) and $N \geq 3$, a social choice set F is implementable if and only if there exists a social choice set $\hat{F}$ which is equivalent to F and satisfies (IC), and (BM).*

Since S satisfies the (CO) condition, we can restate the Corollary 1 of Jackson [1] as follows:

COROLLARY 4. *In an environment which satisfies (E), $T = S$, and $N \geq 3$, a social choice set F is implementable if and only if it satisfies (IC) and (BM).*

As a special case of Corollary 2 of Jackson [1], we obtain the following result.

COROLLARY 5. *In an environment which satisfies (E), (CO), $N \geq 3$, and has a 0 outcome, a social choice set F is implementable if and only if F^0 satisfies (IC), and (BM).*

The (CO) condition has an implication on Theorem 2 of Jackson [1], as well.

COROLLARY 6. *If $N \geq 3$, and (CO) holds, social choice set which satisfies (IC), and (MNV), is implementable.*

It may be of an interest to know how restrictive the (CO) condition may be in societies involving very large number of states or agents.

PROPOSITION 2. *Let $|S^i|$ denote the cardinality of the set of states S^i and be[5] equal to $p \geq 2$ for all i. Let $r^{co}(p, N)$ denote the ratio of the number of possible sets of states which do not satisfy the (CO) condition to the number of possible sets of states. Then*

$$\lim_{N \to \infty} r^{co}(p, N) = 0 \quad \textit{and} \quad \lim_{p \to \infty} r^{co}(p, N) = 0.$$

[5] Note when $|S^i| = 1$ for all i, that is when information is common knowledge, we have $|S| = 1$, and thus S satisfies the (CO) condition regardless what the number of agents is.

Proof. I will prove the claim by showing that an upper-bound for $r^{co}(p,N)$ goes to zero when N or p approaches to infinity. Take any $\hat{s} \in S$. Let $D^{\hat{s}}$ represent the set $\{s \in S | s^i \neq \hat{s}^i,\ \forall i\}$, at every element of which every agent can make the difference from $\hat{s}$. Note that $|D^{\hat{s}}| = (p-1)^N$ for all $\hat{s} \in S$. Let $E^{\hat{s}}$ denote the set $\{G \cup \{\hat{s}\} | G \subseteq D^{\hat{s}}$ and $G \neq \emptyset\}$, i.e., all nonempty subsets of $D^{\hat{s}}$ unioned with $\hat{s}$. We have $|E^{\hat{s}}| = 2^{(p-1)^N} - 1$. Now consider the set $H = \cup_{\hat{s} \in S} E^{\hat{s}}$. We have $|H| \leq \sum_{\hat{s} \in S} |E^{\hat{s}}| = [2^{(p-1)^N} - 1]p^N$. Note that for all $s_a \in S$ and $s_b \in S$ such that $s_a^i \neq s_b^i\ \forall i$, we have $\{s_a, s_b\} \in E^{s_a} \cap E^{s_b}$, thus $|H| < [2^{(p-1)^N} - 1]p^N$. The cardinality of the set of possible nonempty subsets of S is equal to $2^{p^N} - 1$. Hence, we have

$$r^{co}(p,N) \leq \frac{|H|}{2^{p^N} - 1} < \frac{2^{(p-1)^N} p^N}{2^{p^N} - 1}$$

for any p and N. Therefore, $\lim_{N \to \infty} r^{co}(p,N) = 0$ and $\lim_{p \to \infty} r^{co}(p,N) = 0$, which completes the proof. □

A stronger condition than (CO) is that between any two states of the society, there exists a state through which one agent can always serve as a link. Clearly, when (CO) fails, this stronger condition fails, too. As the number of possible states or the number of agents becomes infinitely large, the measure of the possible sets of states which do not satisfy this stronger condition become zero. Thus, we get the proof of Proposition 2.

Even though Proposition 2 does not cover situations where S^i may differ across agents, it, nevertheless, helps to make a conjecture that the fraction of possible sets of states which satisfy connection will be 'almost' one when the number of possible states or the number of agents in the society is sufficiently large. This observation strengthens the fundamental theorems of Jackson [1] as in 'large' environments the closure may have no bite at all in Bayesian implementation.

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