E C O N O M I C S   B U L L E T I N

Non-linear unit root testing in the presence of heavy-tailed innovation processes

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Abstract

The literature concerning the impact of heavy-tailed innovations upon unit root tests is extended via analysis of the finite-sample distribution and size of the non-linear unit of Kapetanios et al. (2003) in the presence of alternative finite and infinite variance innovation processes. Simulation results obtained show the test to exhibit a degree of oversizing far in excess of that previously noted for the linear Dickey-Fuller (1979) test.

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1 Introduction

Typically, the finite-sample distributions and properties of unit root tests are derived using identically and independently distributed standard Normal, or Gaussian, innovation processes. However, in recent research Patterson and Heravi (2003) have examined the impact of fat or heavy-tailed distributions upon the behaviour of unit root tests. This research is to be welcomed as there a large literature suggesting that many time series processes (particularly financial series time series) are better characterised by non-normal, heavy-tailed distributions (see, *inter alia*, Granger and Orr, 1972; Loretan and Phillips 1994; Deo 2000; Resnick 2006). In response to this, Patterson and Heravi (2003) analysed the finite-sample properties of the Dickey-Fuller (1979) and weighted symmetric Dickey-Fuller test of Park and Fuller (1995) in the presence of innovation processes following the $t(v)$ distribution for $v = \{1, 2, 3, 4\}$. An attractive feature of the use of a range of degrees of freedom varying from from 1 to 4 is that it allows consideration of innovation processes with both infinite ($v = 1, 2$) and finite variances ($v = 3, 4$). The results of Monte Carlo analysis undertaken by Patterson and Heravi (2003) showed that while the Dickey-Fuller test exhibited moderate or slight oversizing for the alternative non-normal disturbances considered, the weighted symmetric Dickey-Fuller test was found to suffer from more substantial size distortion in the form of undersizing. It is therefore apparent that the impact of heavy-tailed distributions has differing effects upon these *linear* unit root tests both in terms of the extent of any resulting size distortion and its form. In the present paper this literature is extended to consider the impact of the heavy-tailed distributions considered by Patterson and Heravi (2003) upon the *non-linear* unit root test of Kapetanios *et al.* (KSS) (2003). The results obtained from the following Monte Carlo analysis show the non-linear KSS test to suffer more severe size distortion than either of the linear tests considered by Patterson and Heravi (2003), with substantial oversizing noted.

This paper proceeds as follows. In section [2], the non-linear KSS test is presented. Section [3] provides finite-sample critical values for the KSS test in the presence of innovation processes following the standard normal and alternative heavy-tailed distributions. In section [4] the further simulation analysis is undertaken to examine the finite-sample size of the KSS test in the presence of heavy-tailed distributions when commonly employed critical values obtained from use of innovations following the standard normal distribution are employed. Section [5] concludes.

2 Non-linear unit root testing

KSS recently extended the literature on unit root testing via the introduction of a exponential smooth transition autoregressive (ESTAR) alternative. KSS consider the following ESTAR process, noting the expected low power of the linear ADF test when applied to such a series:

$$\Delta y_t = \gamma y_{t-1} \left\{1 - \exp\left(-\theta y_{t-1}^2\right)\right\} + \varepsilon_t \tag{1}$$

The analysis of KSS focusses upon the parameter θ, with the relevant null and alternative hypotheses given as $\mathrm{H}_0 : \theta = 0$ and $\mathrm{H}_1 : \theta > 0$. As γ is unidentified under the null, $\theta = 0$ cannot be tested directly. Consequently, KSS draw upon the work of Luukkonen *et al.* (1988) with a first-order Taylor series expansion employed to the ESTAR model under the null H_0: $\theta = 0$ to derive a t-type test statistic. The relevant testing equation is then as below:

$$\Delta y_t = \psi y_{t-1}^3 + \sum_{i=1}^{p} \kappa_i \Delta y_{t-i} + \nu_t \tag{2}$$

with the unit root hypothesis tested via the statistic t_{NL} given as the t-type test of $\psi = 0$. A noticeable feature of (2) is the absence of any deterministic terms. To allow application of the test with intercept or intercept and trend terms included, these deterministic terms are removed via preliminary regression with the demeaned or detrended version of y_t employed in (2). These alternative specifications of the t_{NL} test are referred to here as the intercept and trend specifications.

3 Finite-sample distribution and heavy-tailed innovations

To examine the finite-sample critical values of the t_{NL} test in the presence of heavy-tailed innovation processes, the following data generation process (DGP) is considered:

$$y_t = y_{t-1} + \eta_t \qquad t = 1, ..., T \tag{3}$$

where the following alternative cases are considered for the innovation process η_t:

$$\begin{array}{ll} \text{Case I:} & \eta_t \sim t(1) \\ \text{Case II:} & \eta_t \sim t(2) \\ \text{Case III:} & \eta_t \sim t(3) \\ \text{Case IV:} & \eta_t \sim t(4) \\ \text{Case V:} & \eta_t \sim N(0,1) \end{array} \tag{4}$$

As noted previously, Cases I and II provide examples of innovations with infinite variances, while Cases III and IV allow examination of the impact of heavy-tailed, but finite variance, innovations. Case V, where the innovation process follows the standard normal distribution, represents the situation typically employed in practice to derive critical values for unit root tests. All experiments are performed over 50,000 simulations using the EViews 6.0 programming facility. Three sample sizes are considered $(T = 100, 250, 500)$, with critical values for the t_{NL} test in both its intercept and trend specifications derived at the following levels of significance: $\alpha = \{0.01, 0.025, 0.05, 0.1, 0.5, 0.9, 0.95\}$. The results obtained from these experiments are presented in Tables One and Two. From inspection of the results it is apparent that heavy-tailed distributions have a substantial impact on the finite-sample distribution of the t_{NL} test in both its intercept and trend specifications, with $t(1)$ or Cauchy innovations having the greatest impact. As the degrees of freedom of the t-distributed innovations increase, the observed leftward movement of the distribution of the t_{NL} statistic is reduced, with values similar to those from application of standard normal innovations being observed.

Table One: Critical values for the intercept model

	α	Innovation distribution $t\,(1)$	$t\,(2)$	$t\,(3)$	$t\,(4)$	$N\,(0,1)$
$T = 100$	0.950	0.058	0.044	−0.013	−0.036	−0.045
	0.900	−0.075	−0.335	−0.462	−0.508	−0.588
	0.500	−1.471	−1.627	−1.674	−1.690	−1.714
	0.100	−3.202	−2.926	−2.778	−2.713	−2.622
	0.050	−4.140	−3.508	−3.200	−3.070	−2.903
	0.025	−5.459	−4.180	−3.649	−3.437	−3.154
	0.010	−8.199	−5.216	−4.360	−3.980	−3.469
$T = 250$	0.950	0.071	0.039	−0.033	−0.056	−0.061
	0.900	−0.067	−0.364	−0.516	−0.562	−0.565
	0.500	−1.473	−1.650	−1.702	−1.719	−1.735
	0.100	−3.221	−2.901	−2.750	−2.694	−2.640
	0.050	−4.127	−3.415	−3.142	−3.023	−2.912
	0.025	−5.411	−4.029	−3.507	−3.341	−3.160
	0.010	−7.850	−5.061	−4.099	−3.758	−3.475
$T = 500$	0.950	0.066	0.021	−0.057	−0.069	−0.074
	0.900	−0.069	−0.393	−0.545	−0.575	−0.589
	0.500	−1.467	−1.660	−1.717	−1.737	−1.744
	0.100	−3.192	−2.876	−2.730	−2.685	−2.653
	0.050	−4.129	−3.367	−3.078	−2.998	−2.931
	0.025	−5.343	−3.941	−3.426	−3.285	−3.186
	0.010	−7.697	−4.197	−3.951	−3.691	−3.460

Table Two: Critical values for the trend specification

	α	Innovation distribution $t\,(1)$	$t\,(2)$	$t\,(3)$	$t\,(4)$	$N\,(0,1)$
$T = 100$	0.950	−0.399	−0.677	−0.784	−0.832	−0.886
	0.900	−0.934	−1.105	−1.188	−1.221	−1.268
	0.500	−2.382	−2.194	−2.172	−2.172	−2.170
	0.100	−4.099	−3.719	−3.404	−3.272	−3.086
	0.050	−5.020	−4.361	−3.909	−3.686	−3.375
	0.025	−6.390	−5.059	−4.449	−4.098	−3.624
	0.010	−9.508	−6.308	−5.213	−4.717	−3.942
$T = 250$	0.950	−0.378	−0.680	−0.811	−0.853	−0.843
	0.900	−0.912	−1.116	−1.212	−1.242	−1.272
	0.500	−2.382	−2.205	−2.188	−2.191	−2.203
	0.100	−4.072	−3.645	−3.339	−3.219	−3.126
	0.050	−4.969	−4.271	−3.766	−3.568	−3.405
	0.025	−6.253	−4.952	−4.208	−3.913	−3.650
	0.010	−8.891	−6.077	−4.847	−4.352	−3.943
$T = 500$	0.950	−0.325	−0.668	−0.817	−0.845	−0.852
	0.900	−0.883	−1.111	−1.221	−1.252	−1.267
	0.500	−2.376	−2.199	−2.193	−2.195	−2.194
	0.100	−4.045	−3.593	−3.270	−3.173	−3.114
	0.050	−4.907	−4.173	−3.664	−3.505	−3.392
	0.025	−6.172	−4.810	−4.056	−3.833	−3.638
	0.010	−8.678	−5.882	−4.666	−4.237	−3.927

4 Finite-sample size and heavy-tailed innovations

The results of the above analysis show heavy-tailed distributions to influence the finite-sample critical values of the t_{NL} test, with a leftward shift in the distribution of the statistic observed. It can therefore be concluded that practitioners employing critical values obtained from experimentation using normally distributed innovations may experience spurious rejection when examining heavy-tailed data. To gauge the extent of the spurious rejection or oversizing that may occur in these circumstances, Monte Carlo experimentation is undertaken. Using the above DGP of (1)-(2), the t_{NL} test is applied to data generated using standard normal and heavy-tailed innovations. The empirical sizes of the test under the alternative DGPs are then noted at the 1%, 5% and 10% levels of significance when the commonly employed critical values obtained from consideration of normally distributed innovations are employed. As previously, all experiments are performed over 50,000 replications with three alternative sample sizes considered: $T = \{100, 250, 500\}$.

Considering the results for the intercept and trend specifications reported in Tables Three and Four, it can be seen that the degree of oversizing of the t_{NL} can be substantial if commonly employed critical values are used when analysing heavy-tailed data. As expected, the degree of oversizing is greater when the degrees of freedom of the $t-$distributed innovations are lower, with maximum distortion noted for the innovations following the Cauchy distribution. Interestingly, substantial oversizing persists ever for a relatively large sample size ($T = 500$). To illustrate these findings, consider the empirical size of the trend specification at the 10% level of significance. When the innovations follow the $t\,(1)$ distribution an empirical size of 29% is noted for $T = 100$, while the corresponding value for $T = 500$ is only slightly lower at 27.30%. However, the corresponding figures for $t\,(4)$ innovations are 13.66% and 11.03% respectively. Oversizing for the trend specification can therefore be seen to vary from 10.3% to 190% depending upon the degrees of freedom of the t-distributed errors and the sample size employed. Analogous results are observed for the intercept model.

Table Three: Empirical sizes for the intercept specification

	Nominal size(%)	Innovation distribution $t(1)$	$t(2)$	$t(3)$	$t(4)$	$N(0,1)$
$T = 100$	10	16.89	14.96	12.90	11.74	10.00
	5	13.02	10.30	8.06	6.93	5.00
	1	8.08	5.23	3.26	2.36	1.00
$T = 250$	10	16.60	14.10	12.01	11.12	10.00
	5	12.99	9.86	7.44	6.30	5.00
	1	8.01	4.62	2.64	1.86	1.00
$T = 500$	10	16.25	13.78	11.56	10.75	10.00
	5	12.57	9.18	6.68	5.84	5.00
	1	8.03	4.47	2.34	1.65	1.00

Table Four: Empirical sizes for the trend specification

	Nominal size (%)	Innovation distribution $t(1)$	$t(2)$	$t(3)$	$t(4)$	$N(0,1)$
$T = 100$	10	29.00	20.13	15.71	13.66	10.00
	5	21.74	14.71	10.43	8.39	5.00
	1	11.65	7.79	4.77	3.26	1.00
$T = 250$	10	28.37	18.81	14.19	12.18	10.00
	5	21.16	13.42	9.20	7.16	5.00
	1	11.42	7.38	3.90	2.42	1.00
$T = 500$	10	27.30	17.78	12.82	11.03	10.00
	5	20.38	12.59	7.84	6.25	5.00
	1	11.01	6.54	3.09	1.93	1.00

5 Conclusion

In this paper a finite-sample analysis of the non-linear unit root test of KSS in the presence of heavy-tailed innovations has been undertaken. Results presented previously by Patterson and Heravi (2003) have shown the Dickey-Fuller (1979) test to exhibit moderate oversizing in the presence of heavy-tailed innovations, while the weighted symmetric Dickey-Fuller test of Park and Fuller (1995) found to be substantially undersized. The results obtained from the current analysis add to the above results for linear unit root tests, showing the non-linear test of KSS to suffer far greater oversizing than the Dickey-Fuller test in the presence of identical heavy-tailed innovations. Given the prevalence of heavy-tailed data in economics and finance and the increasing popularity of application of non-linear unit root testing to these data (see, *inter alia*, Liew *et al.*, 2004), the present results suggest practitioners should exercise care when presented with rejection of the unit root hypothesis in these circumstances.

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