E c o n o m i c s B u l l e t i n

Habit formation and multiplicity of balanced growth path: a comment

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Abstract

Chen (Journal of Money, Credit and Banking [2007] 25-48) studies the AK model where the utility function includes multiplicative habits in consumption and argues that the model has multiple balanced growth paths with different growth rates. However, one of his calculations is incorrect. I show that in his model, the optimal growth rate is uniquely determined.

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1. Introduction

Recent paper by Chen (2007) studies the AK model where the utility function includes multiplicative habits in consumption and argues that the model has multiple balanced growth paths (BGP). However, one of his calculations is incorrect. We show that in his model, the optimal growth rate along the BGP is actually uniquely determined.

The note is organized as follows. Section 2 describes the model. Section 3 shows that the optimal growth rate is unique. The conclusions are summarized in Section 4. Additional calculations are in Appendix.

2. Model in Chen (2007)

This section briefly describes the set-up in Chen (2007). In his model, the representative consumer maximizes her utility $U = \int_0^\infty e^{-\rho t} u(c_t, S_t)\, dt$ subject to

$$\begin{aligned} \dot{k}_t &= Ak_t - c_t - \delta_k k_t, & (1) \\ \dot{S}_t &= B(c_t)^\mu (S_t)^{1-\mu} - \delta_S S_t,\ \mu \in (0,1], & (2) \end{aligned}$$

where c_t is the consumption, δ_k is the capital depreciation rate, δ_k is the habit depreciation rate, k_t is the capital stock, A is the productivity, S_t is the habit, and ρ is the discount rate. Parameters A, ρ, δ_k, B, and δ_S are positive. Initial capital and habit (k_0, S_0) are given. Equation (1) describes capital accumulation and equation (2) determines the habit accumulation.

To avoid confusions, we write equation (N) in Chen (2007) as (N*). Equations (1) and (2) corresponds to (4*) and (2*) in Chen (2007), respectively.

The instantaneous utility function is given by $u(c,S) = \{(c/S^\gamma)^{1-\sigma} - 1\}/(1-\sigma)$, where $\sigma > 1$ is the coefficient of the relative risk aversion and the parameter γ satisfies $0 < \gamma < 1$. Chen (2007) introduces a following current-value Hamiltonian.

$$H(c,k,S,\lambda_k,\lambda_S) = \frac{(c/S^\gamma)^{1-\sigma} - 1}{1-\sigma} + \lambda_k (Ak - c - \delta_k k) - \lambda_S \left(Bc^\mu S^{1-\mu} - \delta_S S\right), \tag{3}$$

where λ_k and λ_S are the multipliers on (1) and (2), respectively. Note that the sign on the two multipliers λ_k and λ_S in (3) are opposite in Chen (2007). From the Hamiltonian, he derives the following first order conditions.

$$\begin{aligned} \frac{\partial H}{\partial c_t} &= 0 : \left(\frac{c_t}{S_t^\gamma}\right)^{1-\sigma} \frac{1}{c_t} = \lambda_k - \lambda_S B\mu \frac{S_t^{1-\mu}}{c_t^{1-\mu}}, & (4) \\ \dot{\lambda}_k - \rho\lambda_k + \frac{\partial H}{\partial k_t} &= 0 : A - \delta_k = \rho - \frac{\dot{\lambda}_k}{\lambda_k}, & (5) \\ -\dot{\lambda}_S + \rho\lambda_S + \frac{\partial H}{\partial S_t} &= 0 : -\left(\frac{c_t}{S_t^\gamma}\right)^{1-\sigma} \frac{\gamma}{S_t\lambda_S} + \delta_S - B(1-\mu)\frac{c_t^\mu}{S_t^\mu} = \frac{\dot{\lambda}_S}{\lambda_S} - \rho. & (6) \end{aligned}$$

The conditions (4), (5) and (6) are exactly the same as equation (5a*), (5b*) and (5c*) in Chen (2007), respectively. Chen (2007) introduces new variables $x_t = c_t/S_t$ and $\lambda = \lambda_S/\lambda_k$ and we will also use these two variables below.

3. Unique Balanced Growth Path

In this section, we show that the BGP is uniquely determined.

3.1. Problem in calculations

In this subsection, we point out one problem of calculations in Chen (2007). In page 31 (just after equation (6a*)), he differentiated the first order condition (5a*), which is (4) in our paper to get

$$\begin{aligned} 0 \;=\; & [\dot{\lambda}_k/\lambda_k + \gamma(1-\sigma)\dot{S}/S + \sigma\dot{c}/c] + \\ & B\mu\left(\lambda/x^{1-\mu}\right)[\dot{\lambda}_S/\lambda_S + [\gamma(1-\sigma)+(1-\mu)]\,\dot{S}/S - (1-\sigma-\mu)\dot{c}/c]. \end{aligned} \tag{7}$$

Here we show that (7) has to be

$$\begin{aligned} 0 \;=\; & -[\dot{\lambda}_k/\lambda_k + \gamma(1-\sigma)\dot{S}/S + \sigma\dot{c}/c] + \\ & B\mu\left(\lambda/x^{1-\mu}\right)[\dot{\lambda}_S/\lambda_S + [\gamma(1-\sigma)+(1-\mu)]\,\dot{S}/S - (1-\sigma-\mu)\dot{c}/c], \end{aligned} \tag{8}$$

instead. In (7), negative sign (–) on $[\dot{\lambda}_k/\lambda_k + \gamma(1-\sigma)\dot{S}/S + \sigma\dot{c}/c]$ is missing. The first order condition (4) can be written as

$$\left(\frac{c_t}{S_t^{\gamma}}\right)^{1-\sigma}\frac{1}{\lambda_k c_t} + \lambda B\mu x_t^{-(1-\mu)} = 1. \tag{9}$$

If two functions f_t and g_t satisfy $f_t + g_t = 1$, we have $(1-g_t)\left(\dot{f}_t/f_t\right) + g_t(\dot{g}_t/g_t) = 0$ and then $(\dot{f}_t/f_t) + g_t(\dot{g}_t/g_t - \dot{f}_t/f_t) = 0$. Now let $f_t = \left(c_t S_t^{-\gamma}\right)^{1-\sigma}/(\lambda_k c_t)$ and $g_t = \lambda B\mu x_t^{-(1-\mu)}$. These two functions satisfy the following three equations.

$$\begin{aligned} \dot{f}_t/f_t &= -[\dot{\lambda}_k/\lambda_k + \gamma(1-\sigma)\dot{S}_t/S_t + \sigma\dot{c}_t/c_t], \\ \dot{g}_t/g_t &= \dot{\lambda}_S/\lambda_S - \dot{\lambda}_k/\lambda_k - (1-\mu)(\dot{c}_t/c_t - \dot{S}_t/S_t), \\ f_t + g_t &= 1. \end{aligned}$$

Hence we get

$$\begin{aligned} 0 \;=\; & -[\dot{\lambda}_k/\lambda_k + \gamma(1-\sigma)\dot{S}_t/S_t + \sigma\dot{c}_t/c_t] \\ & +\lambda B\mu x_t^{-(1-\mu)}[\dot{\lambda}_S/\lambda_S + \gamma(1-\sigma)\dot{S}_t/S_t + \sigma\dot{c}_t/c_t - (1-\mu)(\dot{c}_t/c_t - \dot{S}_t/S_t)], \end{aligned}$$

which is the same as (8).

3.2. Modification of equation (7a) in Chen (2007)

In this subsection, we derive the correct version of (7a*) in Chen (2007). We closely follow the argument by Chen (2007, p. 31–32) although we use (8) instead of (7). He argues that there exists two (x, λ) that satisfy (7a*) and (7b*). He derives equation (7a*) and (7b*) by using the following steps.

[Derivation of (7a*)]

1. Use (5a*) and (5c*) to get (6a*). (See Chen(2007, p. 31).)
2. Differentiate (5a*) with respect to time to get one equation which is between (6a*) and (6b*). Chen (2007) does not name the equation and here we call the equation as (6X*). (See Chen(2007, p. 31).)
3. Substitute (2*), (5b*) and (6a*) into (6X*) to get (6b*). (See Chen(2007, p. 31).)
4. Take differences between (6b*) and (2*) to get (7a*). (See Chen(2007, p. 32).)

[Derivation of (7b*)]

1. Take differences between (6a*) and (5b*) to get (7b*). (See Chen(2007, p. 32).)

However, as we show in subsection 3.1, (6X*) is incorrect. Hence (7a*) is also incorrect, while (7b*) is still correct. Here we derive a correct version of (7a*). Later we prove that there is unique (x, λ) that satisfies the correct version of (7a*) and (7b*).

First, we follow Step 1 to derive (6a*). We use (4) and (6) to get

$$\dot{\lambda}_S/\lambda_S = \delta_S + \rho - \gamma\,(x/\lambda) - B(1-\mu+\mu\gamma)x^{\mu}, \tag{10}$$

which is exactly the same as (6a*) in Chen (2007).

Next, as we showed in Subsection 3.1, we can derive the correct version of equation (6X*) in Chen (2007) by following Step 2. Let

$$\Psi = \sigma + (1-\sigma-\mu)B\mu\left(\lambda/x^{1-\mu}\right).$$

Then the correct version of equation (6X*), (8) can be rewritten as

$$\begin{aligned}\Psi\dot{c}/c \;&=\; -\dot{\lambda}_k/\lambda_k + B\mu\left(\lambda/x^{1-\mu}\right)\dot{\lambda}_S/\lambda_S \\ &\quad + [B\mu\left(\lambda/x^{1-\mu}\right)\{\gamma(1-\sigma)+(1-\mu)\} - \gamma(1-\sigma)]\dot{S}/S.\end{aligned} \tag{11}$$

Third, we derive the correct version of (6b*) by following Step 3. Note that we can rewrite the habit accumulation (2) as

$$\dot{S}/S = B\left(\frac{c}{S}\right)^{\mu} - \delta_S = Bx^{\mu} - \delta_S. \tag{12}$$

We substitute (2) (or equivalently (12)), (5) and (10) into (11) to obtain

$$\begin{aligned}\Psi\dot{c}/c &= (A-\delta_k-\rho) \\ &\quad +B\mu\left(\lambda/x^{1-\mu}\right)\{\delta_S+\rho-\gamma\,(x/\lambda)-B(1-\mu+\mu\gamma)x^{\mu}\} \\ &\quad +\left[B\mu\left(\lambda/x^{1-\mu}\right)\{\gamma(1-\sigma)+(1-\mu)\}-\gamma(1-\sigma)\right](Bx^{\mu}-\delta_S).\end{aligned} \tag{13}$$

Equation (13) is the correct version of equation (6b*). [1]

Finally, we derive the correct version of (7a*) in Chen (2007) by following Step 4. Note that $\dot{x}/x=\dot{c}/c-\dot{S}/S$ by definition. We take a difference of (13) and (12) to obtain

$$\begin{aligned}\Psi\dot{x}/x &= \Psi\left(\dot{c}/c-\dot{S}/S\right) \\ &= (A-\delta_k-\rho)+B\mu\left(\lambda/x^{1-\mu}\right)\{\delta_S+\rho-\gamma\,(x/\lambda)-B(1-\mu+\mu\gamma)x^{\mu}\} \\ &\quad +[B\mu\left(\lambda/x^{1-\mu}\right)(\gamma(1-\sigma)+(1-\mu))-\gamma(1-\sigma)](Bx^{\mu}-\delta_S) \\ &\quad -\Psi(Bx^{\mu}-\delta_S),\end{aligned} \tag{14}$$

which corresponds to (7a*).

Now we derive (7b*). We can take a difference of (10) and (5) to obtain

$$\dot{\lambda}/\lambda=\dot{\lambda}_S/\lambda_S-\dot{\lambda}_k/\lambda_k=\delta_S-\gamma\,(x/\lambda)-B(1-\mu+\mu\gamma)x^{\mu}+A-\delta_k, \tag{15}$$

which is exactly the same as (7b*) in Chen (2007).

3.3. Uniqueness

In this subsection, we show that the growth rate along the balanced growth path is actually uniquely determined. Along the BGP, $\dot{\lambda}/\lambda=\dot{x}/x=0$. Here we show that when $\dot{\lambda}/\lambda=\dot{x}/x=0$, there exists unique (x,λ) that satisfies (14) and (15). Note that equations (14) and (15) correspond to (7a*) and (7b*), respectively. Substituting $\dot{\lambda}/\lambda=0$ into (15) yields

$$\delta_S-\gamma\,(x/\lambda)-B(1-\mu+\mu\gamma)x^{\mu}=-A+\delta_k. \tag{16}$$

We now substitute (16), $\Psi=\sigma+(1-\sigma-\mu)B\mu\left(\lambda/x^{1-\mu}\right)$ and $\dot{x}/x=0$ into (14) to obtain

$$\begin{aligned}0 &= (A-\delta_k-\rho)\left(1-B\mu\left(\lambda/x^{1-\mu}\right)\right) \\ &\quad +\left(1-B\mu\left(\lambda/x^{1-\mu}\right)\right)(\gamma(1-\sigma)+\sigma)(Bx^{\mu}-\delta_S).\end{aligned} \tag{17}$$

In Appendix, we describe how we get (17) in detail. Using (17), we have

$$0=\left(1-B\mu\left(\lambda/x^{1-\mu}\right)\right)\cdot\{(A-\delta_k-\rho)-(Bx^{\mu}-\delta_S)(\gamma(1-\sigma)+\sigma)\} \tag{18}$$

Notice that the first order condition (4) implies

$$\left(\frac{c}{S^{\gamma}}\right)^{1-\sigma}\frac{1}{c} \;=\; \lambda_k-\lambda_S B\mu\frac{1}{x^{1-\mu}}=\lambda_k\left(1-B\mu\left(\lambda/x^{1-\mu}\right)\right).$$

[1] If we divide both sides of (13) by Ψ, we get an equation that is more close to (6b*).

Since $0 < \left(cS^{-\gamma}\right)^{1-\sigma}/c$, $(1 - B\mu\left(\lambda/x^{1-\mu}\right)) > 0$. Therefore (18) implies

$$(A - \delta_k - \rho) - (Bx^{\mu} - \delta_S)(\gamma(1-\sigma) + \sigma) = 0,$$

which can be expressed as

$$Bx^{\mu} - \delta_S = \frac{A - \delta_k - \rho}{\gamma(1-\sigma) + \sigma}.$$

Hence x is uniquely determined. From (15), λ is also uniquely determined when x is fixed.

Finally let us show that the growth rate is also uniquely determined. Along the BGP, $\dot{S}/S = \dot{c}/\dot{c} = \dot{y}/y$. Therefore, the BGP growth rate $\dot{S}/S = (Bx^{\mu} - \delta_S)$ (See (12)) is also uniquely determined and it is equal to $(A - \delta_k - \rho)/\{\gamma(1-\sigma) + \sigma\}$. Note that the same growth rate is found in Carroll et al. (2000). [2] The parameter μ on the habit accumulation equation (2) does not affect the optimal growth rate, although it changes the value of (x, λ).

4. Conclusion

In this note, we show that the endogenous growth model by Chen (2007) has only one balanced growth path.

Appendix: derivations of equation (17)

Here we describe how (17) is derived in detail. We substitute equations

$$(16) : \delta_S - \gamma\left(x/\lambda\right) - B(1 - \mu + \mu\gamma)x^{\mu} = -A + \delta_k,$$

$\Psi = \sigma + (1 - \sigma - \mu)B\mu\left(\lambda/x^{1-\mu}\right)$ and $\dot{x}/x = 0$ into

$$\begin{aligned}(14) \quad : \quad & \Psi\dot{x}/x = (A - \delta_k - \rho) \\ & +B\mu\left(\lambda/x^{1-\mu}\right)\{\delta_S + \rho - \gamma\left(x/\lambda\right) - B(1 - \mu + \mu\gamma)x^{\mu}\} \\ & +[B\mu\left(\lambda/x^{1-\mu}\right)(\gamma(1-\sigma) + (1-\mu)) - \gamma(1-\sigma)](Bx^{\mu} - \delta_S) \\ & -\Psi(Bx^{\mu} - \delta_S),\end{aligned}$$

to obtain

$$\begin{aligned}0 \quad = \quad & (A - \delta_k - \rho) & (19) \\ & +B\mu\left(\lambda/x^{1-\mu}\right)(-A + \delta_k + \rho) \\ & +[B\mu\left(\lambda/x^{1-\mu}\right)(\gamma(1-\sigma) + (1-\mu)) - \gamma(1-\sigma)](Bx^{\mu} - \delta_S) \\ & -[\sigma + (1 - \sigma - \mu)B\mu\left(\lambda/x^{1-\mu}\right)](Bx^{\mu} - \delta_S). \\ & & (20)\end{aligned}$$

[2] See equation (10) in Carroll et al. (2000). Note that they use θ to express the discount rate, while Chen (2007) uses ρ.

Here we use (16) which implies $\delta_S + \rho - \gamma(x/\lambda) - B(1-\mu+\mu\gamma)x^{\mu} = -A + \delta_k + \rho$. Now we have

$$\begin{aligned}
& [B\mu\left(\lambda/x^{1-\mu}\right)(\gamma(1-\sigma)+(1-\mu))-\gamma(1-\sigma)] \\
& -[\sigma+(1-\sigma-\mu)B\mu\left(\lambda/x^{1-\mu}\right)] \\
= \ & B\mu\left(\lambda/x^{1-\mu}\right)(\gamma(1-\sigma)+1-\mu)-(1-\sigma-\mu)B\mu\left(\lambda/x^{1-\mu}\right)-\gamma(1-\sigma)-\sigma \\
= \ & -\left(1-B\mu\left(\lambda/x^{1-\mu}\right)\right)(\gamma(1-\sigma)+\sigma).
\end{aligned}$$

Hence equation (19) implies

$$\begin{aligned}
0 \ = \ & (A-\delta_k-\rho)\left(1-B\mu\left(\lambda/x^{1-\mu}\right)\right) \\
& -\left(1-B\mu\left(\lambda/x^{1-\mu}\right)\right)(\gamma(1-\sigma)+\sigma)(Bx^{\mu}-\delta_S),
\end{aligned}$$

which is the same as (17).

References

[1] Carroll, C., Overland, J., Weil, D. (2000) "Saving and Growth with Habit formation" *American Economic Review*, **90**, 341–355.

[2] Chen, B. (2007) "Multiple BGPs in a Growth Model with Habit Persistence" *Journal of Money, Credit and Banking*, **39**, 25-48.