E C O N O M I C S   B U L L E T I N

# External conjectural variations in symmetric oligopoly equilibrium

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## *Abstract*

In this paper, we introduce the concept of conjectural variations across sectors called "external conjectural variations" in a pure exchange economy. Three results are obtained. First, the symmetric oligopoly equilibrium coincides with the Cournot equilibrium when the conjectural variations are zero. Second, the symmetric oligopoly equilibrium coincides with the competitive equilibrium when the conjectural variations take the value of the competitive market equilibrium price. Third, the optimal strategies of agents between sectors are complements (substitutes) when the conjectural variation is negative (positive).


This note is a second piece of a more general research devoted to the micro-foundations of cooperation failures and coordination failures.
**Citation:** JULIEN, Ludovic Alexandre, (2007) "External conjectural variations in symmetric oligopoly equilibrium." *Economics Bulletin,* Vol. 4, No. 44 pp. 1-6
**Submitted:** June 5, 2007. **Accepted:** November 27, 2007.
**URL:** http://economicsbulletin.vanderbilt.edu/2007/volume4/EB-07D40007A.pdf

## 1. Introduction

In imperfectly competitive economies each agent when making a decision generally does note the effect of her/his action on the market (Bowley (1924)). The conjectural approach takes into account the perceptions by individuals of their market environment and intends to study price formation without an auctioneer by attempting a general equilibrium analysis of imperfect competition (Gale (1978), Hahn (1977)). These conjectures indicate the way any producer in a given sector thinks the other producers supply choice will vary when s/he modifies her/his own supply choice[1]. The role played by (consistent) conjectures has been mainly developed in the context of production economies under partial equilibrium analysis (Bresnahan (1981), Figuières *et alii* (2004) or Perry (1982)).

It is possible to bring conjectural variations in oligopoly equilibrium for pure exchange economies (Julien (2006)). We thus consider the framework of strategic multilateral exchange initially developed by Codognato-Gabszewicz (1991), (1993), and more recently pursued by Gabszewicz and Michel (1997) and Gabszewicz (2002)[2]. The introduction of conjectural variations enables to study the effects of conjectures on prices and indirect utility at the symmetric oligopoly equilibrium (Julien (2006)). The price and the utility level generally increase with the value of the conjectural variations for agents who form it, but decrease with the conjecture formed by others. Moreover, the conjectural variations mimic the results which would be obtained in an environment where the oligopoly equilibria coincide with the competitive equilibrium[3].

In this note, we propose to introduce the concept of *external* conjectural variations which differs from the usual concept of (*internal*) conjectural variations. Both concepts capture the perceptions by individuals of their market environment, but they do not involve the same side of the market. The conjectural variations usually refer to the way any given agent expects the other agents who supply the same good will react to a change in her/his own strategy, whereas the external one refers to the way any given agent expects all the other agents who supply another good will react to a change in her/his own strategy. The latter concept is perhaps more appropriate in a general equilibrium framework than the former, which mainly emphasizes strategic interactions within a sector. In order to simplify, we

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[1] Consider *n* firms in a given sector. If $x_i$ represents the supply of firm *i* and $\sum_{-i\neq i} x_{-i}$ the supply of all the other firms, then the conjectural variations $\gamma$ is defined as $\partial \sum_{-i\neq i} x_{-i} / \partial x_i = \gamma$. Certain values are of particular interest (Perry (1982)). When $\gamma = -1$, each producer acts as a price taker and the equilibrium is competitive: each firm expects the other firms in their sector to absorb exactly its supply expansion by a corresponding supply reduction. When $\gamma = 0$, each firm ignores the consequence of its action on the others' actions: this is the Cournot equilibrium. Finally, when $\gamma = n-1$, the equilibrium is collusive: firms behave so as to maximize joint profit.

[2] These authors study the relationship between market power and the number of agents. Different concepts of oligopoly equilibria can be developed depending on the way strategic behavior is introduced.

[3] In large economies, the competitive equilibrium can be obtained by an asymptotic identification or through a replication procedure.

only focus on conjectural variations across sectors[4]. In particular, we show that the symmetric oligopoly equilibrium coincides with the symmetric Cournot equilibrium for a zero value of the conjectural variations. Moreover, we determine the conditions under which the symmetric oligopoly equilibrium coincides with the competitive equilibrium. Finally, we characterize the effect on any optimal strategy in one sector of a change in the optimal strategy of any agent located in the other sector.

The paper is organized as follows. In section 2, the basic economy is described and the concept of conjectural variation across sectors is introduced. In section 3, we study the relation between the values taken by conjectures and the type of equilibrium, and then present the several results obtained.

## 2. The basic economy

Consider a pure exchange economy with two consumption goods (1 and 2) and $m+n$ consumers. Preferences are represented by the following utility function:

$$U_i = x_{i1}^{\alpha} x_{i2}^{1-\alpha} \quad , \; 0 < \alpha < 1 \quad , \forall i \, . \tag{1}$$

The structure of the initial endowments is assumed to be the same as in Gabszewicz-Michel (1997):

$$\omega_i = \left( \frac{1}{m}, 0 \right) \quad , i = 1,2,...,m$$

$$\omega_i = \left( 0, \frac{1}{n} \right) \quad , i = m+1,...,m+n \, . \tag{2}$$

It is assumed that good 2 is taken as the *numéraire*, so $p$ is the price of good 1 as expressed in units of good 2.

We consider that each agent behaves strategically. Each agent $i$ manipulates the price by contracting her/his supply, i.e. the quantity of good 1 or 2 s/he offers. We denote $s_{i1}$ the pure strategy of agents $i = 1,2,...,m$, with $s_{i1} \in [0,1/m]$, and $s_{i2}$ the pure strategy of agents $i = m+1,...,m+n$, with $s_{i1} \in [0,1/n]$.

Finally, let us assume the agents who have an endowment in good 1 form a conjectural variation about the combined strategic supplies response of all other consumers who have an endowment of good 2 to a unit of change in their own strategic supply. This effect characterize some *external* conjectural variations, which is denoted $\nu_1$:

$$\frac{\partial \sum_{i=m+1}^{m+n} s_{i2}}{\partial s_{i1}} = \nu_1 \quad , \text{ where } \nu_1 \in [-1, n] \tag{3}$$

Equivalently, we define the conjectural variations for consumers who initially own quantities of good 2:

$$\frac{\partial \sum_{i=1}^{m} s_{i1}}{\partial s_{i2}} = \nu_2 \; , \text{ where } \nu_2 \in [-1, m] \tag{4}$$

We assume $\nu_1$ and $\nu_2$ to be the same for each consumer and independent of both the supply of the other agents and the number of other agents. We moreover

[4] In a recent research, we develop a model which gathers both concepts.

consider that the conjectures are consistent[5]. These conjectures indicate the way that each consumer thinks the other consumers supply choice will vary after s/he varies her/his own supply choice. They are not exactly similar to those formed within each sector because they involve strategic interactions across sectors. Nevertheless, we verify through a simple example that most of the results we obtained with the usual concept of conjectural variations in pure exchange economies under general oligopoly equilibrium hold with this concept. In particular, it can be shown that for a certain value of the conjectural variations, the symmetric oligopoly equilibrium coincides with the Cournot equilibrium.

## 3. Symmetric oligopoly equilibrium and conjectural variations

A *symmetric oligopoly equilibrium* is defined as a $(m+n)$-tuple of strategies $(\tilde{s}_{11},...,\tilde{s}_{m1},\tilde{s}_{m+12},...,\tilde{s}_{m+n2})$, with $\tilde{s}_{i1}\in[0,1/m]$ for $i=1,2,...,m$ and $\tilde{s}_{i2}\in[0,1/n]$ for $i=m+1,...,m+n$, and an allocation $(\tilde{x}_1,...,\tilde{x}_m,\tilde{x}_{m+1},...,\tilde{x}_{m+n})\in IR_+^{2(m+n)}$ such that (i) $\tilde{x}_i=x_i(\tilde{s}_{i1},\tilde{s}_{-i})$ and $U_i(x_i(\tilde{s}_{i1},\tilde{s}_{-i}))\geq U_i(x_i(s_{i1},\tilde{s}_{-i}))$ for $i=1,2,...,m$ and (ii) $\tilde{x}_i=x_i(\tilde{s}_{i2},\tilde{s}_{-i})$ and $U_i(x_i(\tilde{s}_{i2},\tilde{s}_{-i}))\geq U_i(x_i(s_{i2},\tilde{s}_{-i}))$ for $i=m+1,...,m+n$. We here assume that the symmetric oligopoly equilibrium exists and is unique.

The market clearing condition implies that the price must be $p=\dfrac{\sum_{i=m+1}^{i=m+n}s_{i2}}{\sum_{i=1}^{i=m}s_{i1}}=\dfrac{s_2}{s_1}$. Consequently, the non-cooperative equilibrium is associated with the resolution of the simultaneous strategic programs:

$$\underset{\{\tilde{s}_{i1}\}}{Arg\max}\left(\frac{1}{m}-s_{i1}\right)^{\alpha}\left(\frac{s_2}{s_1}s_{i1}\right)^{1-\alpha}\quad,\ i=1,2,...,m \tag{5}$$

$$\underset{\{\tilde{s}_{i2}\}}{Arg\max}\left(\frac{s_1}{s_2}s_{i2}\right)^{\alpha}\left(\frac{1}{n}-s_{i2}\right)^{1-\alpha}\quad,\ i=m+1,...,m+n\,. \tag{6}$$

The *(m+n)* conditions of optimality $\partial U_i/\partial s_{i1}=0$ for $i=1,2,...,m$ and $\partial U_i/\partial s_{i2}=0$ for $i=m+1,...,m+n$, yield the two types of reaction functions:

$$\begin{aligned}&m\xi_1(s_{i1})^3+\left\{\frac{\alpha}{1-\alpha}mns_{i2}+\nu_1\left[m(m-1)s_{-i1}-1\right]\right\}(s_{i1})^2\\&\qquad+(m-1)\left(\frac{mn}{1-\alpha}s_{i2}-\nu_1\right)s_{-i1}(s_{i1})-n(m-1)s_{-i1}s_{i2}=0\end{aligned} \tag{7}$$

$$\begin{aligned}&n\nu_2(s_{i2})^3+\left\{\frac{mn}{1-\alpha}s_{i1}+\nu_2\left[n(n-1)s_{-i2}-1\right]\right\}(s_{i2})^2\\&\qquad+(n-1)\left(\frac{mn}{\alpha}s_{i1}-\nu_2\right)s_{-i2}(s_{i2})-m(n-1)s_{-i2}s_{i1}=0\end{aligned} \tag{8}$$

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[5] See Bresnahan (1981).

**Result 1.** *When $v_1 = v_2 = 0$, the symmetric oligopoly equilibrium coincides with to the symmetric Cournot equilibrium.*

*Proof.* We compute the symmetric oligopoly equilibrium for $v_1 = v_2 = 0$ and compare it with the results obtained for the symmetric Cournot equilibrium.

When $v_1 = 0$ and $v_2 = 0$, and with $\tilde{s}_{ij} = \tilde{s}_{-ij} \quad \forall j = 1,2$, (7) and (8) become respectively $[m-(1-\alpha)]mn\tilde{s}_{i2}\tilde{s}_{i1} - (1-\alpha)(m-1)n\tilde{s}_{i2} = 0$ for $i = 1,2,...,m$ and $(n-\alpha)mn\tilde{s}_{i1}\tilde{s}_{i2} - \alpha(n-1)m\tilde{s}_{i1} = 0$ for $i = m+1,...,m+n$. This yields $\tilde{s}_{i1} = (1-\alpha)(m-1)/\{m[m-(1-\alpha)]\}$ and $\tilde{s}_{i1} = \alpha(n-1)/n(n-\alpha)$. Moreover, the equilibrium market price is $\tilde{p} = \left(\frac{\alpha}{1-\alpha}\right)\left(\frac{n-1}{n-\alpha}\right)\left(\frac{m-(1-\alpha)}{m-1}\right)$. The individual allocations are thus $(\tilde{x}_{i1}, \tilde{x}_{i2}) = \left(\frac{\alpha}{m-(1-\alpha)}, \frac{\alpha(n-1)}{m(n-\alpha)}\right)$ for $i = 1,2,...,m$ and $(\tilde{x}_{i1}, \tilde{x}_{i2}) = \left(\frac{(1-\alpha)(m-1)}{n[m-(1-\alpha)]}, \frac{1-\alpha}{n-\alpha}\right)$ for $i = m+1,...,m+n$. Finally, the utility levels reached are respectively $\tilde{U}_i = \frac{\alpha}{m}\left[\frac{m}{m-(1-\alpha)}\right]^{\alpha}\left(\frac{n-1}{n-\alpha}\right)^{1-\alpha}$ for $i = 1,2,...,m$ and $\tilde{U}_i = \left(\frac{1-\alpha}{n}\right)\left[\frac{m-1}{m-(1-\alpha)}\right]^{\alpha}\left(\frac{n}{n-\alpha}\right)^{1-\alpha}$ for $i = m+1,...,m+n$.

Consider now the programs given by (5) and (6). We can easily verify that the *(m+n)* conditions of optimality $\partial U_i / \partial s_{i1} = 0$ for $i = 1,2,...,m$, and $\partial U_i / \partial s_{i2} = 0$ for $i = m+1,...,m+n$, lead to the preceding expressions for the optimal strategies prices, allocations and utility levels. This completes the proof.

**Result 2.** *If $v_1 = \gamma$ (resp. $v_2 = \varepsilon$) and $\tilde{s}_{i2} = \alpha / n$ (resp. $\tilde{s}_{i1} = (1-\alpha)/n$), then $\gamma = \alpha/(1-\alpha)$ (resp. $\varepsilon = \alpha/(1-\alpha)$): the symmetric oligopoly equilibrium coincides with the competitive equilibrium.*

*Proof.* We first determine the competitive allocation. Second, we compute the symmetric oligopoly equilibrium for $v_1 = \gamma$ and $v_2 = \varepsilon$ and we compare it with the results found previously.

When the behavior of each agent is competitive, the individual plans come from a non-strategic maximization of the utility subject to the budget constraint. The competitive equilibrium price, the competitive supply of each good and the associated allocation for each type of agents are respectively: $p^* = \alpha/(1-\alpha)$, $s_{i1}^* = (1-\alpha)/m$ and $(x_{i1}^*, x_{i2}^*) = (\alpha/m, \alpha/m)$ for $i = 1,2,...,m$ and $s_{i2}^* = \alpha/n$ and $(x_{i1}^*, x_{i2}^*) = ((1-\alpha)/n, (1-\alpha)/n)$ for $i = m+1,...,m+n$. The utility levels are $U_i^* = \alpha/m$ for $i = 1,2,...,m$ and $U_i^* = (1-\alpha)/n$ for $i = m+1,...,m+n$.

We now consider $v_1 = \gamma$ and $\tilde{s}_{h2} = \alpha/n$. Substituting these values in (7) gives $(1-\alpha)^3\gamma - \alpha(1-\alpha)^2 - (1-\alpha)^2\gamma = 0$. This leads to $\gamma = \alpha/(1-\alpha)$. Symmetrically, consider $v_2 = \varepsilon$ and $\tilde{s}_{h1} = (1-\alpha)/n$ . Substituting these values in (8) gives $\varepsilon = \alpha/(1-\alpha)$. Thus $\gamma = \varepsilon = p^*$. This completes the proof.

We remark that the value taken by the external conjectural variations that sustains the competitive equilibrium is not $v_1 = v_2 = -1$.

**Result 3.** *When* $v_1 > 0$ *(* $v_1 < 0$ *), we have* $\partial s_{i1}/\partial s_{i2} < 0$ *(* $\partial s_{i1}/\partial s_{i2} > 0$ *). Equivalently, when* $v_2 > 0$ *(*$v_2 < 0$*), we have* $\partial s_{i2}/\partial s_{i1} < 0$ *(*$\partial s_{i2}/\partial s_{i1} > 0$*).*

*Proof.* In order to simplify, consider the case where $\alpha = 1/2$ and $m = n = 2$. These two restrictions imply $\tilde{s}_{i2} \in [0,1/2]$ and $v_1 \in [-1,2]$. Under these conditions (7) becomes $2v_1 s_{i1}^2 + [-v_1 + 6s_{i2}]s_{i1} - s_{i2} = 0$ . The discriminant is $\Delta = v_1^2 + 36s_{i2}^2 - 4v_1 s_{i2}$ , with $\Delta > 0$ . The optimal strategy is $s_{i1} = \frac{1}{4} - \frac{3}{2}\left(\frac{s_{i2}}{v_1}\right) + \frac{1}{4}\sqrt{1 - \frac{4}{v_1}s_{i2} + \left(\frac{6}{v_1}s_{i2}\right)^2}$ . Differentiating it with respect to $s_{i2}$ gives $\frac{\partial s_{i1}}{\partial s_{i2}} = \frac{1}{\Omega}\left\{-\frac{3}{2}\left(\frac{1}{v_1}\right)\sqrt{1 - \frac{4}{v_1}s_{i2} + \left(\frac{6}{v_1}s_{i2}\right)^2} - \frac{1}{2}\left(\frac{1}{v_1}\right) + \left(\frac{3}{v_1}\right)^2 s_{i2}\right\}$ , where $\Omega \equiv \sqrt{1 - \frac{4}{v_1}s_{i2} + \left(\frac{6}{v_1}s_{i2}\right)^2}$ . It is immediate that $\frac{\partial s_{i1}}{\partial s_{i2}} > 0$ when $v_1 < 0$. Suppose now $v_1 > 0$. If, for instance, $\tilde{s}_{i2} = 0$, then $\frac{\partial s_{i1}}{\partial s_{i2}} = -\frac{2}{v_1} < 0$. Moreover, if $\tilde{s}_{i2} = 1/2$, then $\frac{\partial s_{i1}}{\partial s_{i2}} = -\frac{3}{2} - \frac{\sqrt{2}}{32} < 0$ for $v_1 = 1$ and $\frac{\partial s_{i1}}{\partial s_{i2}} = -\frac{35}{48} < 0$ for $v_1 = 2$ respectively. The argument is similar for (8). This completes the proof.

We remark that the optimal supplies are strategic complements for negative values of the conjectural variations and strategic substitutes for positive values. The fact that reactions curves may be upward-sloping explains the presence of strategic complementarities.

## 4. Concluding remarks

The role of conjectural variations can be transposed across various sectors in general oligopoly equilibrium for pure exchange economies. As it stands for conjectural variations in a given sector, the oligopoly equilibrium can be viewed as a case where the conjectural variation across sector takes a zero value. Moreover, the existence of strategic complementarities may be captured by the external conjectural variations.

The preceding model should be developed more deeply in order to take into account the two types of conjectural variations.

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