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The core of a housing market with externalities

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Abstract

It is known that the core of a housing market always exists and contains a unique matching when agents have independent preferences. We show that when preferences of agents are interdependent, there are housing markets with an empty core as well as housing markets with a core containing more than one matching.

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1 Introduction

The implications of externalities along with the possibility of an empty core in several matching markets have successfully been explored. Sasaki and Toda (1996), Hafalir (2006), and Mumcu and Saglam (2006) study *interdependent* preferences in one-to-one matching problems, while Echenique and Yenmez (2007) in the assignment of students to colleges. Klaus and Klijn (2005) deal with the same issue in a job market with couples. In this paper, we aim to extend the existing matching literature under externalities to housing markets.

In the classical housing market literature, each individual's preferences over the set of houses are *independent* of the realized matching in the market; i.e., from the viewpoint of every agent, the ordering of any two matchings is equivalent to the ordering of the corresponding houses at these two matchings. Under independent preferences, it is well known by the pioneering work of Roth and Postlewaite (1977) that the core of each housing market always exists and contains a unique matching. We show that these results are no longer true when there are externalities, i.e., there are housing markets with an empty core as well as housing markets with a core containing more than one matching.

The organization of the paper is as follows: Section 2 introduces the model. Section 3 presents our results, and Section 4 concludes.

2 The Model

We denote a *housing market* by a four-tuple $\mathcal{H} = (A, H, P, \mu)$.[1] Here, $A = \{a_1, a_2, \ldots, a_n\}$ is a finite set of agents and $H = \{h_1, h_2, \ldots, h_n\}$ is a finite set of houses. A (house) matching is a bijection from A to H. Let $\mathcal{M}$ be the set of all matchings. The third component $P = \{P_{a_1}, P_{a_2}, \ldots, P_{a_n}\}$ of a housing market is a list of preference relations of agents, where for each $a \in A$, P_a is a strict preference relation over the set $\mathcal{M}$, and the final component μ is a matching denoting the initial endowment. We denote any matching ν by a permutation $(\tilde{h}_1, \tilde{h}_2, \ldots, \tilde{h}_n)$ of houses such that $\nu(a_i) = \tilde{h}_i \in H$ for all $i \in \{1, 2, \ldots, n\}$. We denote the strict preference of any agent by a permutation $(\nu_1, \nu_2, \ldots, \nu_{|\mathcal{M}|})$ of matchings in $\mathcal{M}$.

[1]We borrow some of our notations and definitions from Abdülkadiroglu and Sönmez (1998).

Definition 1. Given a housing market (A, H, P, μ), a coalition $T \subseteq A$ blocks a matching ν_1 via another matching ν_2 if:

(i) $T = \{a \in A : \nu_1(a) \neq \nu_2(a)\}$,

(ii) $\nu_2(a) \in \{h \in H : h = \mu(a') \text{ for some } a' \in T\}$ for all $a \in T$,

(iii) $\nu_2 \, P_a \, \nu_1$ for all $a \in T$.

Definition 2. A matching is in the core $C(\mathcal{H})$ of the housing market $\mathcal{H}$ if and only if there is no coalition that blocks it via any other matching.

3 Results

Proposition 1. *There exists a housing market $\mathcal{H}$ such that $C(\mathcal{H}) = \emptyset$.*

Proof. Consider the housing market $\mathcal{H} = (A, H, P, \mu)$ with $A = \{a_1, a_2, a_3\}$ and $H = \{h_1, h_2, h_3\}$. The six possible matchings in this market are

$$\begin{aligned}
\nu_1 &= (h_1, h_2, h_3), \\
\nu_2 &= (h_1, h_3, h_2), \\
\nu_3 &= (h_2, h_1, h_3), \\
\nu_4 &= (h_2, h_3, h_1), \\
\nu_5 &= (h_3, h_1, h_2), \\
\nu_6 &= (h_3, h_2, h_1).
\end{aligned}$$

Let the preferences of agents be

$$\begin{aligned}
P_{a_1} &= (\nu_6, \nu_3, \nu_2, \nu_1, \nu_4, \nu_5), \\
P_{a_2} &= (\nu_3, \nu_2, \nu_6, \nu_1, \nu_4, \nu_5), \\
P_{a_3} &= (\nu_2, \nu_1, \nu_6, \nu_3, \nu_4, \nu_5),
\end{aligned}$$

and let $\mu = \nu_1$. Note that both ν_4 and ν_5 are blocked by the grand coalition A via ν_1, ν_1 is blocked by a_2 and a_3 via ν_2, ν_2 is blocked by a_1 and a_2 via ν_3, ν_3 is blocked by a_1 and a_3 via ν_6, and ν_6 is blocked by a_2 and a_3 via ν_2. Therefore, $C(\mathcal{H}) = \emptyset$. ∎

Proposition 2. *There exists a housing market $\mathcal{H}$ such that $|C(\mathcal{H})| > 1$.*

Proof. Consider the housing market $\mathcal{H} = (A, H, P, \mu)$ with $A = \{a_1, a_2, a_3\}$ and $H = \{a_1, a_2, a_3\}$. The six possible matchings are $\{\nu_1, \nu_2, \ldots, \nu_6\}$ as defined in the proof of Proposition 1. Let the preferences of agents be

$$\begin{aligned} P_{a_1} &= (\nu_1, \nu_3, \nu_2, \nu_4, \nu_5, \nu_6), \\ P_{a_2} &= (\nu_3, \nu_1, \nu_2, \nu_4, \nu_5, \nu_6), \\ P_{a_3} &= (\nu_3, \nu_1, \nu_2, \nu_4, \nu_5, \nu_6), \end{aligned}$$

and let $\mu = \nu_2$. Any of the matchings in the set $\{\nu_4, \nu_5, \nu_6\}$ is blocked by the grand coalition A via ν_2, whereas ν_2 is blocked by a_2 and a_3 via ν_1. But, none of ν_1 and ν_3 are blocked by any coalition. Thus, $C(\mathcal{H}) = \{\nu_1, \nu_3\}$. ∎

We should note that preferences are always independent when there are at most two agents in the market, in which case the core of each housing market always exists and has a unique element. The nonconventional results under externalities may only arise when there are more than two agents in the market. In fact, for any housing society involving at least three members, we can always find a preference profile under which the core of the related housing market either becomes empty or contains more than one matching, by simply extending the examples in the proofs of Propositions 1 and 2. We should also note that our results do not trivially follow from those in the one-to-one matching literature under externalities since the core concept for a housing market takes into consideration the initial endowments of agents, as well.

4 Conclusions

We have showed that when agents have interdependent preferences, there are housing markets with an empty core as well as housing markets with a nonempty core involving more than one matching. Apparently *Gale's top trading cycles algorithm* that determines the unique matching in the core of each housing market when agents have independent preferences, is no longer of any direct use when externalities are present. Further research may characterize preference restrictions to ensure a unique core matching for each housing market with externalities and develop an algorithm to determine the core.

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