ECONOMICS BULLETIN

A new axiomatization of the core on fuzzy NTU games

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Abstract

In this note we show that on the domain of fuzzy NTU games whose core is non-empty, the core is the only solution satisfying non-emptiness, individual rationality and the reduced game property.

The authors are very grateful to Associate Editor for valuable comments which much improve the paper.
Citation: Hwang, Yan-An and Yu-Hsien Liao, (2007) "A new axiomatization of the core on fuzzy NTU games." *Economics Bulletin*, Vol. 3, No. 37 pp. 1-6
Submitted: July 25, 2007. **Accepted:** August 30, 2007.
URL: http://economicsbulletin.vanderbilt.edu/2007/volume3/EB-07C70025A.pdf

1 Introduction

The theory of fuzzy games started with work of Aubin (1974, 1981) where the notions of a fuzzy game and the core of a fuzzy game are introduced. There are two crucial factors in the framework of fuzzy games, the players and their participation levels. Hwang (2007) extends the notion of the reduced game (Davis and Maschler, 1965) to fuzzy games by only reducing the number of the players. Also, inspired by Serrano and Volij (1998), he offers an axiomatization of the core in the context of fuzzy games. Here, we turn to a different definition of the reduced game by reducing both the number of the players and the participation levels. We shall introduce such a reduced game to fuzzy NTU games and define the related reduced game property and its converse. Also, we offer an extension of Peleg (1985)'s axiomatization of the core to fuzzy NTU games.

2 Preliminaries

Let U be the universe of players.[1] If $N \subseteq U$ is a set of players, then a **fuzzy coalition** is a vector $\alpha \in [0,1]^N$. The i-th coordinate α_i of α is called the participation level of player i in the fuzzy coalition α. For all $T \subseteq N$, let $|T|$ be the number of elements in T. Instead of $[0,1]^T$, we will write F^T for the set of fuzzy coalitions. A player-coalition $T \subseteq N$ corresponds in a canonical way to the fuzzy coalition $e^T(N) \in F^N$, which is the vector with $e_i^T(N) = 1$ if $i \in T$, and $e_i^T(N) = 0$ if $i \in N \setminus T$. The fuzzy coalition $e^T(N)$ corresponds to the situation where the players in T fully cooperate (i.e. with participation level 1) and the players outsides T are not involved at all (i.e. they have participation level 0). Denote the zero vector in $\mathbb{R}^N$ by 0^N. The fuzzy coalition 0^N corresponds to the empty player-coalition. Note that if no confusion can arise $e^T(N)$ will be denoted by e^T.

Let $\alpha \in F^N$, $A(\alpha, N) = \{i \in N \mid \alpha_i > 0, \alpha \in F^N\}$ is the set of players who participate in α. Let $x, y \in \mathbb{R}^N$. $x \geq y$ if $x_i \geq y_i$ for all $i \in N$; $x > y$ if $x \geq y$ and $x \neq y$; $x \gg y$ if $x_i > y_i$ for all $i \in N$. We denote $\mathbb{R}^N_+ = \{x \in \mathbb{R}^N \mid x \geq 0^N\}$. Let $A \subseteq \mathbb{R}^N$. A is **comprehensive** if $x \in A$ and $x \geq y$ imply $y \in A$. The **boundary** of A is denoted by ∂A, and the **interior** of A is denoted by $intA$. If $x \in \mathbb{R}^N$ then $x + A = \{x + a \mid a \in A\}$.

Definition 1 *A* **fuzzy NTU game** *is a pair (N, V), where N is a non-empty and finite set of players and V is a characteristic function that*

[1] Assume that U is infinite.

assigns to each fuzzy coalition $\alpha = (\alpha_i)_{i\in N} \in F^N \setminus \{0^N\}$ *a subset* $V(\alpha)$ *of* $\mathbb{R}^{A(\alpha,N)}$*, such that*

$$V(\alpha) \text{ is non-empty, closed and comprehensive,} \tag{1}$$

$$V(\alpha) \cap (x + \mathbb{R}_+^{A(\alpha,N)}) \text{ is bounded for every } x \in \mathbb{R}^{A(\alpha,N)}, \tag{2}$$

$$\text{if } x, y \in \partial V(\alpha) \text{ and } x \geq y, \text{ then } x = y.\text{(non-levelness)} \tag{3}$$

The core of a fuzzy NTU game (N, V) is as follows.

Definition 2 *The* **core** $C(N, V)$ *of* (N, V) *consists of all* $x \in \partial V(e^N)$ *that satisfy for all* $\alpha \in F^N \setminus \{0^N\}$, $(\alpha_i x_i)_{i\in A(\alpha,N)} \notin intV(\alpha)$.

We denote

$$\Gamma_c = \{(N, V) \mid C(N, V) \neq \emptyset\}.$$

3 Axioms and Reduced Games

A **solution on** Γ_c is a function σ which associates with each $(N, V) \in \Gamma_c$ a subset $\sigma(N, V)$ of $V(e^N)$. Let $(N, V) \in \Gamma_c$. Then, a payoff vector x of $(N, V) \in \Gamma_c$ is **efficient (EFF)** if $x \in \partial V(e^N)$; a payoff vector x of $(N, V) \in \Gamma_c$ is **individually rational (IR)** if for all $i \in N$ and for all $j \in (0, 1]$, $jx_i \notin intV(je^{\{i\}})$. We will make use of the following axioms:

Let σ be a solution on Γ_c. σ satisfies **non-emptiness (NE)** if for all $(N, V) \in \Gamma_c$, $\sigma(N, V) \neq \emptyset$. σ satisfies **efficiency (EFF)** if for all $(N, V) \in \Gamma_c$ and for all $x \in \sigma(N, V)$, x is EFF. σ satisfies **individual rationality (IR)** if for all $(N, V) \in \Gamma_c$ and for all $x \in \sigma(N, V)$, x is IR.

We extend to the fuzzy NTU games case the reduced game introduced by Davis and Maschler (1965). Given $x \in \mathbb{R}^N$ and $S \subseteq N$, we denote $x_S \in \mathbb{R}^S$ to be the restriction of x to S.

Definition 3 *Let* $(N, V) \in \Gamma_c$*,* $x \in \mathbb{R}^N$*,* $S \subseteq N$*,* $S \neq \emptyset$ *and* $\gamma \in (0, 1]^{N\setminus S}$*. The* **reduced game with respect to** S**,** x **and** γ*,* $(S, V_{x,S,\gamma})$*, is defined by for all* $\alpha \in F^S \setminus \{0^S\}$*,*

$$V_{x,S,\gamma}(\alpha) = \{y \in \mathbb{R}^S \mid (y, x_{N\setminus S}) \in V(e^N)\} \quad , \text{ if } \alpha = e^S(S)$$
$$V_{x,S,\gamma}(\alpha) = \bigcup_{Q\subseteq N\setminus S} \{y \in \mathbb{R}^{A(\alpha,S)} \mid \Big(y, (\gamma_i x_i)_{i\in Q}\Big) \in V(\alpha, \gamma_Q, 0^{(N\setminus S)\setminus Q})\} \quad , \text{ otherwise .}$$

Note that the condition $\gamma \in (0, 1]^{N\setminus S}$ means that when renegotiating the payoff distribution within S, the members of $N \setminus S$ will continue to cooperate with the members of S. All members in $N \setminus S$ take nonzero levels based on the participation vector γ to cooperate.

The reduced game property and its converse are defined as follows:

- **Reduced game property (RGP)**: If $(N, v) \in \Gamma_c$, $S \subseteq N$ with $S \neq \emptyset$, $\gamma \in (0,1]^{N\setminus S}$ and $x \in \sigma(N, v)$, then $(S, V_{x,S,\gamma}) \in \Gamma_c$ and $x_S \in \sigma(S, V_{x,S,\gamma})$.

- **Converse reduced game property (CRGP)**: If $(N, v) \in \Gamma_c$ with $|N| \geq 3$, $x \in V(e^N)$, and for all $S \subset N$ with $|S| = 2$ and for all $\gamma \in (0,1]^{N\setminus S}$ such that $(S, V_{x,S,\gamma}) \in \Gamma_c$ and $x_S \in \sigma(S, V_{x,S,\gamma})$, then $x \in \sigma(N, V)$.

4 Axiomatization

In this section we shall use NE, IR, and RGP to characterize the core.

Lemma 1 *Let (N, V) be a fuzzy NTU game, $x \in V(e^N)$, $S \subseteq N$ with $S \neq \emptyset$, and $\gamma \in (0,1]^{N\setminus S}$. Then the reduced game $(S, V_{x,S,\gamma})$ is a fuzzy NTU game.*

Proof. It can easily be deduced from the proof of Lemma 3.3 in Peleg (1985). ■

Lemma 2 *Let $(N, V$ be a fuzzy NTU game, $x \in V(e^N)$, $S \subseteq N$ with $S \neq \emptyset$, and $\gamma \in (0,1]^{N\setminus S}$. Then x is EFF in (N, V) if and only if x_S is EFF in the reduced game $(S, V_{x,S,\gamma})$.*

Proof. It can easily be deduced from the proof of Lemma 4.4 in Peleg (1985). ■

Lemma 3 *The core satisfies RGP.*

Proof. It can easily be deduced from the proof of Lemma 4.5 in Peleg (1985). ■

Lemma 4 *The core satisfies CRGP.*

Proof. Let $(N, V) \in \Gamma_c$ with $|N| \geq 3$ and let $x \in V(e^N)$. Suppose that for all $S \subset N$ with $|S| = 2$ and for all $\gamma \in (0,1]^{N\setminus S}$, $(S, V_{x,S,\gamma}) \in \Gamma_c$ and $x_S \in C(S, V_{x,S,\gamma})$. We will show that $x \in C(N, V)$. Since $x_S \in C(S, V_{x,S,\gamma})$, x_S is EFF in $(S, V_{x,S,\gamma})$. Hence x is EFF in (N, V) by Lemma 2. It remains to show that for all $\alpha \in F^N \setminus \{0^N, e^N\}$, $(\alpha_i x_i)_{i \in A(\alpha,N)} \notin intV(\alpha)$. Assume, on the contrary, that there exists $\alpha \in F^N \setminus \{0^N, e^N\}$ such that $(\alpha_i x_i)_{i \in A(\alpha,N)} \in intV(\alpha)$. Two cases can be distinguished:
Case 1: $A(\alpha, N) = N$:
Choose $k \in A(\alpha, N)$ with $\alpha_k \neq 1$ (This can be done since $\alpha \neq e^N$).

Let $j \in N$, $j \neq k$, and let $S = \{k, j\}$, $\gamma = \alpha_{N \setminus S}$. Since $(\alpha_i x_i)_{i \in A(\alpha,N)} \in intV(\alpha)$ and

$$V_{x,S,\gamma}(\alpha_k, \alpha_j) = \bigcup_{Q \subseteq N \setminus S} \{y \in \mathbb{R}^{\{k,j\}} \mid \Big(y, (\gamma_i x_i)_{i \in Q}\Big) \in V(\alpha_{\{k,j\}}, \gamma_Q, 0^{(N \setminus S) \setminus Q})\},$$

by taking $Q = N \setminus S$, we have that $(\alpha_k x_k, \alpha_j x_j) \in intV_{x,S,\gamma}(\alpha_k, \alpha_j)$.
Case 2: $A(\alpha, N) \neq N$:
Choose $k \in A(\alpha, N)$. Let $j \notin A(\alpha, N)$, and let $S = \{k, j\}$. For convenience, let $T = N \setminus (A(\alpha, N) \cup \{j\})$. Taking $\gamma = \big(\alpha_{A(\alpha,N) \setminus \{k\}}, e^T(T)\big)$. Since $\alpha_j = 0$, by the same arguments as case 1 except taking $Q = A(\alpha, N) \setminus \{k\}$, we can derive that $\alpha_k x_k \in intV_{x,S,\gamma}(\alpha_k, 0)$.

Hence, by cases 1 and 2, $x_S \notin C(S, V_{x,S,\gamma})$, the desired contradiction has been obtained. ■

Lemma 5 *Let σ be a solution on Γ_c. If σ satisfies IR and RGP then it also satisfies EFF.*

Proof. It can easily be deduced from the proof of Lemma 5.4 in Peleg (1985). ■

Lemma 6 *If a solution σ on Γ_c satisfies IR and RGP, then for all $(N, V) \in \Gamma_c$, $\sigma(N, V) \subseteq C(N, V)$.*

Proof. Let $(N, V) \in \Gamma_c$. If $|N| = 1$, then by IR of σ and C, $\sigma(N, v) \subseteq C(N, v)$. If $|N| = 2$, by Lemma 5, σ satisfies EFF. Let $x \in \sigma(N, V)$. It remains to show that for all $\alpha \in F^N \setminus \{0^N, e^N\}$, $(\alpha_i x_i)_{i \in A(\alpha,N)} \notin intV(\alpha)$. Let $\alpha \in F^N \setminus \{0^N, e^N\}$. Two cases can be distinguished:
Case 1: $|A(\alpha, N)| = 1$:
We have done by IR of σ.
Case 2: $|A(\alpha, N)| = 2$:
Assume that $N = \{k, j\}$, and $k \in A(\alpha, N)$ with $\alpha_k \neq 1$ (This can be done since $\alpha \neq e^N$). Let $\gamma = \alpha_j \neq 0$. Consider the reduced game $(\{k\}, V_{x,\{k\},\gamma})$. By RGP of σ, $x_k \in \sigma(\{k\}, V_{x,\{k\},\gamma})$. Thus, $x_k \in C(\{k\}, V_{x,\{k\},\gamma})$. So, $\alpha_k x_k \notin intV_{x,\{k\},\gamma}(\alpha_k)$. Since

$$V_{x,\{k\},\gamma}(\alpha_k) = \bigcup_{Q \subseteq \{j\}} \{y \in \mathbb{R}^{\{k\}} \mid \Big(y, (\gamma_i x_i)_{i \in Q}\Big) \in V(\alpha_{\{k\}}, \gamma_Q, 0^{\{j\} \setminus Q})\},$$

by taking $Q = \{j\}$, we have that $(\alpha_k x_k, \alpha_j x_j) \notin intV(\alpha_k, \alpha_j)$.
Hence, by cases 1 and 2, $\sigma(N, v) \subseteq C(N, v)$. It remains to consider the case $|N| \geq 3$. Let $x \in \sigma(N, V)$. Since σ satisfies RGP, for all $S \subseteq N$ with $|S| = 2$ and for all $\gamma \in (0, 1]^{N \setminus S}$, $x_S \in \sigma(S, V_{x,S,\gamma})$. Hence, $x_S \in C(S, V_{x,S,\gamma})$. So, $x \in C(N, V)$ by CRGP of the core. ■

Lemma 7 *Let* $(N,V) \in \Gamma_c$ *and* $x \in C(N,V)$. *There exists* $(N',V') \in \Gamma_c$, *where* $N \subset N'$ *and* $|N'| = |N|+1$ *such that* $\{(x,0)\} = C(N',V')$ *and for all* $\gamma \in (0,1)$, $(N,(V')_{(x,0),N,\gamma}) = (N,V)$.

Proof. Let $(N,V) \in \Gamma_c$ and $x \in C(N,V)$. Let $N' = N \cup \{p\}$ where $p \in U \setminus N$. We define a game (N',V') by the following rules:[2]

1. For all $0 < u \leq 1$, let $V'\big(ue^{\{p\}}(N')\big) = \{z \in \mathbb{R}^{\{p\}} \mid z \leq 0\}$.
2. For all $t \in F^{N'} \setminus \{0^{N'}, e^{N'}(N')\}$ with $t_p = 1$, let
$$V'(t) = \{z \in \mathbb{R}^{A(t,N')} \mid \sum_{i \in A(t,N')} z_i \leq \sum_{i \in A(t,N') \setminus \{p\}} x_i\}.$$
3. For all $t \in F^{N'} \setminus \{0^{N'}\}$ with $t_p = 0$, let $V'(t) = V(t_N)$.
4. For other t, let $a^{A(t,N')} \in \mathbb{R}^{A(t,N')}$ be such that $a_p^{A(t,N')} = 1$ and $a_i^{A(t,N')} = -1$ for all $i \in A(t,N') \setminus \{p\}$. Then let
$$V'(t) = \big(V(t_N) \times \{0\}\big) + \{ua^{A(t,N')} \in \mathbb{R}^{A(t,N')} \mid u \in \mathbb{R}\}.^3$$

Next, we prove that $(x,0) \in C(N',V')$. Let $y = (x,0)$. By rule 1, for all t with $t = ue^{\{p\}}(N')$ where $0 < u \leq 1$, $t_p y_p = u \cdot 0 = 0 \notin intV'(t)$. By rules 2 and 3, $(t_i y_i)_{i \in A(t,N')} \notin intV'(t)$ for all $t \in F^{N'} \setminus \{0^{N'}, e^{N'}(N')\}$ with $t_p = 1$ or $t_p = 0$. It remains to show that for all t in rule 4, $(t_i y_i)_{i \in A(t,N')} \notin intV'(t)$. Suppose not, then there exists t in rule 4 such that $(t_i y_i)_{i \in A(t,N')} \in intV'(t)$. That is, there exist $(t_i z_i)_{i \in A(t,N')} \in V'(t)$ such that $t_i z_i > t_i y_i$ for all $i \in A(t,N')$. Since $t_p \neq 0$ and $y_p = 0$, $z_p > 0$. Hence, it follows from rule 4 that $(t_i z_i)_{i \in A(t,N')} = \big((t_i w_i)_{i \in A(t,N') \setminus \{p\}}, 0\big) + ua^{A(t,N')}$ for some $(t_i w_i)_{i \in A(t,N') \setminus \{p\}} \in V(t_N)$ and some $u > 0$. But then, $(t_i w_i)_{i \in A(t,N') \setminus \{p\}} > (t_i z_i)_{i \in A(t,N') \setminus \{p\}} > (t_i y_i)_{i \in A(t,N') \setminus \{p\}} = (t_i x_i)_{i \in A(t,N') \setminus \{p\}}$ and hence $x \notin C(N,V)$. This is a contradiction. So, $y \in C(N',V')$.

To verify the uniqueness, let $z \in C(N',V')$. By rule 1, $z_p \geq 0$. By rule 2, $z_i + z_p \geq x_i$ for all $i \in N$. By rule 4, there exists $w \in V(e^N(N))$ such that $z = (w,0) + \big((-z_p)e^N(N), z_p\big)$. Hence, $w_i = z_i + z_p \geq x_i$ for all $i \in N$. Since $x \in \partial V(e^N(N))$, $w_i = z_i + z_p = x_i$ for all $i \in N$. Now, if $z_p > 0$ then $z_i < x_i$ for all $i \in N$. Since $x \in V(e^N(N))$, $z_N \in intV(e^N(N)) = intV'(e^N(N')$ by rule 3. This contradicts to that $z \in C(N',V')$. Hence, $\{(x,0)\} = \{y\} = C(N',V')$. It remains to show

[2] In contrast with the proof of Peleg (1985), the design of (N',V') in Lemma 7 can not be applied to Lemma 6.2 of Peleg.

[3] The definition of $V'(t)$ in rule 4 is adapted from Peleg (1985, p.210).

that for all $\gamma \in (0,1)$, $(N, (V')_{y,N,\gamma}) = (N, V)$. Let $\gamma \in (0,1)$. Clearly, for all t in rule (4), $\{z \in \mathbb{R}^{A(t,N')\setminus\{p\}} \mid (z,0) \in V'(t)\} = V(t_N)$. Combining this with rule (3), we have $(N, (V')_{y,N,\gamma}) = (N, V)$ by the definition of $(V')_{y,N,\gamma}$. ■

Theorem 1 *On Γ_c, the core is the only solution satisfying NE, IR and RGP.*

Proof. It can easily be deduced from the proof of Theorem 5.5 in Peleg (1985). ■

The following examples show that each of the axioms used in Theorem 1 is logically independent of the others. These are corresponding to Examples 7.3, 7.4, and 7.5 in Peleg (1985), respectively.

Example 1 *Let $\sigma(N, V) = \emptyset$ for all $(N, V) \in \Gamma_c$. Then σ satisfies IR and RGP, but it violates NE.*

Example 2 *Let $\sigma(N, V) = \partial V(e^N)$ for all $(N, V) \in \Gamma_c$. Then σ satisfies NE and RGP, but it violates IR.*

Example 3 *Let $\sigma(N, V) = \{x \in \partial V(e^N) \mid x \text{ is IR } \}$ for all $(N, V) \in \Gamma_c$. Then σ satisfies NE and IR, but it violates RGP.*

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