A Note on Synchronization Risk and Delayed Arbitrage

Hideaki Sakawa
Graduate School of Economics, Osaka University

Naoki Watanabel
Osaka school of International Public Policy

Abstract

This note reexamines Abreu and Brunnermeier's (2003) analysis of a bubble that persists towards synchronization risk. We find that a certain condition that usually does not hold is required for the existence of synchronization risk.

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1 Introduction

The purpose of this note is to reexamine the model presented in Abreu and Brunnermeier (2003), hereafter referred to as AB (2003). AB (2003) consider a market where arbitrageurs face uncertainty about when their peers will exploit a common arbitrage opportunity. In such a market, arbitrageurs face synchronization risk, and they delay usage of the arbitrage opportunity. The price-correction time is also associated with the pressure of the arbitrageurs' selling. AB (2003)[1] conclude that the bubble persists because of the delay in arbitrage. Their model suggests that a stochastic bubble persists due to arbitrage delay. As rational arbitrageurs are sequentially informed about the bubble bursting, they cannot synchronize to sell at the same time. Accordingly, the bubble persists owing to the delay in arbitrage delay[2].

Although AB (2003) incorporate a continuous-time model in their analysis, we check whether their model also exists in a discrete-time setting. We provide the following negative result: a certain condition that usually does not hold is required for the existence of synchronization risk. In Section 2, we introduce the discrete-time AB (2003) model and induce the existence condition of synchronization risk. Section 3 provides a conclusion.

2 Model

2.1 The discrete-time approximation model of AB (2003)

We represent discrete time as $0 = t_0 < t_1 < ... < t_n$, and a length of one trading period as $\Delta(\Delta = t_{i+1} - t_i, i \in [0, n))$. We consider a market where m risk-neutral arbitrageurs exist (m is a positive number). There is only one stock in the market. Each arbitrageur holds $\frac{1}{m}$ units of stock. We define a sequence of stock prices as ψ_i, and a sequence of fundamental prices as v_i. The interest rate on a risk-free asset is r.

Prior to time t_b, the price of the stock is equal to its fundamental price: $\psi_i = v_i (i \leqq b)$, and the interest rate is $g(> r)$. In other words, the fundamental price is higher than the risk-free rate before time t_b.

At time t_b, a shock occurs and the stock's fundamental growth rate adjusts to equal the risk-free interest rate. As a result, a bubble emerges after time t_b. This model treats time t_b as a random variable.

[1] Chamley (2004) surveys their paper.

[2] Blanchard (1979) and Blanchard and Watson (1982) analyze the mechanisms of bubble persistence and crash. Their model concludes that the crash stochastically occurs under a no-arbitrage condition. However, their model does not consider the mechanisms underlying a stochastic bubble.

After time t_b, the interest rate on the fundamental prices becomes r. Accordingly, it differs from the interest rate on the stock: g. Therefore, the sequence of fundamental and stock prices becomes:

$$\psi_i = (1+g)^{(i-b)\Delta} > v_i = (1+r)^{(i-b)\Delta} \tag{1}$$

Each risk-neutral arbitrageur detects mispricing in the turn after time t_b. All arbitrageurs become aware of the mispricing in each period t_i from t_b to $t_{b+m}(= t_b + \eta)$[3]. A new arbitrageur becomes aware of the mispricing during each Δ period. This model defines the arbitrageur who finds the mispricing at time t_j as the arbitrageur j. However, each arbitrageur j cannot know how many arbitrageurs have received the price information before or after themselves. Accordingly, the mispricing information does not become common knowledge. The first type of informed arbitrageur is arbitrageur $b+1$, and the last type of informed arbitrageur is arbitrageur $b+m$. In other words, arbitrageur j receives the mispricing information at time $t_j(= t_{b+l}, l \in [1, m])$.

Arbitrageurs sell their own stock for arbitrage profit as soon as they learn of the mispricing. As arbitrageurs receive this mispricing information sequentially, they cannot sell simultaneously. While the selling pressure is strictly smaller than $\kappa(< 1)$, the bubble persists. We consider the equilibrium where all arbitrageurs choose the same strategy 'the symmetric equilibrium'. This model defines the bubble term as equation (2)[4].

$$\beta(i-b) = \frac{\psi_i - v_i}{\psi_i} = 1 - (\frac{1+r}{1+g})^{(i-b)\Delta} \tag{2}$$

We assume that the bubble collapses within $\overline{\tau}\Delta$ periods[5]. Even if the selling pressure never exceeds κ, the bubble collapses in the $t_{b+\overline{\tau}}$ period. As a result, we can calculate the maximum of the bubble term as $\overline{\beta} = 1 - \beta(\overline{\tau})$.

It is assumed that arbitrageur j sells out the stock for $\tau^1\Delta$ periods after learning the mispricing information. In any period $t_{j+\tau^1}$, arbitrageurs h ($t_h \leqq t_{j+\tau^1}$) sell out their stock, and arbitrageurs k ($t_k \geqq t_{j+\tau^1+1}$) hold their stock; only one arbitrageur sells out of the stock in each period.

[3] The information that bubble bursts is spreading in the η periods. It takes Δ to be informed of one arbitrageur. So, the relation: $\eta = m\Delta$ exists. Under the assumption that the first arbitrageur receives the information at the t_b, all arbitrageurs are informed at the $t_{b+m}(t_{b+\frac{\eta}{\Delta}})$.

[4] The same definition is applied by AB (2003).

[5] $\overline{\tau}$ is exogenously given.

Each arbitrageur j considers that the bubble persists at time t_j. Arbitrageurs choose the following trigger strategy in the equilibrium.

Proposition 1 *Trigger Strategy*
In the equilibrium, arbitrageur j sells out his or her own stock at time $t_{j+\tau^1}$ after noticing the shock. Arbitrageur j does not buy back the stock. Arbitrageur \$j\$ takes the optimal strategy in delaying selling out the stock. In other words, arbitrageur j's optimal strategy is to hold the stock for $\tau^1\Delta(= t_{j+\tau^1} - t_j)$ periods after first noticing the mispricing.

When arbitrageurs undertake the trigger strategy, they begin to sell out the stock after time $t_{b+1+\tau^1}$. We define time $t_{b+\tau^*}$ as the time when the selling pressure crosses κ. The following equation holds.

$$\frac{(\tau^* - \tau^1)\Delta}{\eta} = \kappa \tag{3}$$

During the periods that arbitrageur j thinks that the capital gain of holding the stock exceeds the expected loss from the crash, he or she continues to hold the stock.

Arbitrageur j guesses the crash probability at time $t_{j+\tau}$[6]. When the bubble bursts at time $t_j + \tau$, the following equations are established.

$$(b + \tau^*)\Delta = (j + \tau)\Delta, \quad (b = j + \tau - \tau^*) \tag{4}$$

Corollary 1 *Arbitrageur j believes in a crash probability at time $t_{j+\tau}$. This belief is represented by the conditional probability density function: $\phi(t_{j+\tau}|j)$. Arbitrageur j thinks that the crash probability will become higher. Therefore, $\phi(t_{j+\tau}|j)$ is a monotonically increasing function of τ. All arbitrageurs think that bubble persists when they themselves find the mispricing. So, $\phi(t_j|j)$ becomes 0.*

In addition, we define a distribution function of the crash probability as $\Phi(t_{j+\tau}|j)$, and the transaction cost in this market as $C(< 1)$. Arbitrageur j is faced with the problem of maximizing expected profit as below.

$$Max\sum_{s=0}^{\tau-1}\{(\tfrac{1}{1+r})^{(j+s-b)\Delta}(1-\beta(j+s-b))(1+g)^{(j+s-b)\Delta}\phi(t_{j+\tau}|j)\}$$

[6] AB (2003) specifies the crash probability in a continuous-time setting. The next subsection specifies this.

$$+(\frac{1}{1+r})^{(j+\tau-b)\Delta}(1+g)^{(j+\tau-b)\Delta}(1-\Phi(t_{j+\tau}|j))-C \quad (5)$$

We define $D(\tau, j)$ as the difference between the increase of the first term and the decrease of the second term from time $t_{\tau-1}$ to time t_τ.

$$D(\tau, j) = (\tfrac{1+g}{1+r})^{(j+\tau-1-b)\Delta}\{(1-\beta(j+\tau-1-b))\phi(t_{j+\tau}|j)\}$$

$$-\{(\frac{1+g}{1+r})^{(j+\tau-b-1)\Delta}(1-\Phi(t_{j+\tau-1}|j))-(\frac{1+g}{1+r})^{(j+\tau-b)\Delta}(1-\Phi(t_{j+\tau}|j)\} \quad (6)$$

This difference equation: $D(\tau, j)$ is a nonincreasing function of τ[7]. The optimal strategy of arbitrageur j is to sell out when he or she estimates that the expected profit of selling out with the mispricing is as large as the expected loss of the bubble bursting. The arbitrage profit is zero when taking the optimal strategy: $\tau = \tau^1$. Therefore, no arbitrage condition (7) exists by substituting the right-hand side of equation (6) into zero.

$$\frac{\phi(t_{j+\tau}|j)}{1-\Phi(t_{j+\tau}|j)} = \frac{(1+g)^\Delta-(1+r)^\Delta}{\beta(j+\tau^1-b)(1+g)^\Delta} \quad (7)$$

The left hand side of the above equation is defined as the hazard rate: $h(t_{j+\tau}|j)$. In a symmetric equilibrium, all arbitrageurs take the optimal strategy: $\tau = \tau^1$. All arbitrageurs delay to arbitrage for $\tau^1\Delta$ periods. Considering a no-arbitrage condition, an equilibrium condition is given as follows;

Proposition 2 *Perfect Bayesian Nash Equilibrium: Each arbitrageur j holds his or her own stock upon noticing the mispricing. In an endogenous crash equilibrium*[8]*, the bubble bursts when an arbitrageur $j(t_j = t_{b+\tau^*-\tau^1})$ sells out the stock at the time when the no-arbitrage condition (7) is satisfied. Under the condition that $\frac{(1+g)^\Delta-(1+r)^\Delta}{\overline{\beta}} < 1$, this equilibrium is uniquely determined.*

$$j = b + m\kappa \quad (8)$$

In equilibrium, the bubble bursts when arbitrageur $b+m\kappa$ learns of the mispricing.

[7] The induction is referred to in the appendix.

[8] An endogenous crash means that the bubble bursts before the exogenous crash time: $t_{b+\overline{\tau}}$.

2.2 Crash probability's specification

We consider the prior distribution as a discrete approximation of an exponential distribution function[9]. Arbitrageur j's prior distribution function of $t_b : \Phi(t_b)$ is given as follows:

$$\Phi(t_b) = 1 - p^b, 0 < p < 1 \tag{9}$$

Then the prior density function $\phi(t_b)$ is calculated by using the definition of the distribution function.

$$\phi(t_b) = \Phi(t_{b+1}) - \Phi(t_b) = (1-p)p^b \tag{10}$$

Arbitrageur j's posterior distribution function of the bursting time t_b is calculated as follows. Equation (10) represents the prior belief, and it shows the state where there is no information about time t_b. Arbitrageur j estimates that the bubble bursts during the interval: $t_{j-m} \leqq t_b \leqq t_{j-1}$ after receiving information about the burst at the t_j. Therefore, the posterior distribution function: $\Phi(t_b|j)$ follows the truncated distribution function with the interval: $t_{j-m} \leqq t_b \leqq t_{j-1}$. $\Phi(t_b|j)$ is calculated by using the definition of the conditional probability.

$$\Phi(t_b|j) = \frac{\sum_{j=i-m}^{b-1} \phi(t_j)}{\Phi(t_{j-1}) - \Phi(t_{j-m})} \tag{11}$$

Arbitrageurs who discover the mispricing later estimate a higher crash probability. The posterior probability function is inducted as follows.

$$\phi(t_b|j) = \Phi(t_{b+1}|j) - \Phi(t_b|j) \tag{12}$$

We assume that arbitrageur j estimates that the bursting time is $t_{b+\zeta}$. As the bubble bursts at time $t_{j+\tau}$ when he or she sells out the stock, the equation below is established.

$$(b+\zeta)\Delta = (j+\tau)\Delta, \quad (b = j + \tau - \zeta) \tag{13}$$

By equation (8), the variable ζ that represents the bursting time becomes:

$$\zeta = m\kappa + \tau$$

[9] AB (2003) define the prior distribution as an exponential distribution function.

By substituting the equation above into (12), the hazard rate: $h(t_{j+\tau}|j)$ at time $t_{j+\tau}$ when arbitrageur j sells out is calculated. By substituting the above equation into (7) and (8), the necessary conditions of the symmetric Bayesian Nash equilibrium become:

$$h(t_{j+\tau}|j) = \frac{\phi(t_{j+\tau}|j)}{1 - \Phi(t_{j+\tau}|j)} = \frac{(1+g)^{\Delta} - (1+r)^{\Delta}}{\beta(\tau^1 + m\kappa)(1+g)^{\Delta}} \tag{14}$$

The optimal trading time: $t_{j+\tau^1}$ is the solution of equation (14). The bubble bursts when arbitrageur $\tau^1 + m\kappa$ sells out the stock at time $b + \tau^1 + m\kappa$ in equilibrium. Arbitrageur $\tau^1 + m\kappa$'s optimal strategy becomes the solution of the logarithm transformation of (15). For the sake of simplicity, we specify the crash probability as the case where p is equal to $\frac{1}{2}$.

$$(\tau^1 + m\kappa)\Delta = \frac{\ln(1+g) - \ln[(1+g)^{\Delta} - (2 - 2^{2-m\kappa})\{(1+g)^{\Delta} - (1+r)^{\Delta}\}]}{\ln(1+g) - \ln(1+r)} \tag{15}$$

In an endogenous crash, selling pressure crosses $\kappa(< 1)$ at $t_{b+\tau^*}$. Therefore, the equation below exists by equation (6) and (10).

$$\tau^* \Delta = (\tau^1 + m\kappa)\Delta \tag{16}$$

By substituting (16) into (15), we solve the $t_{b+\tau^*}$ as follows:

$$\tau^* \Delta = \frac{\ln(1+g) - \ln[(1+g)^{\Delta} - (2 - 2^{2-m\kappa})\{(1+g)^{\Delta} - (1+r)^{\Delta}\}]}{\ln(1+g) - \ln(1+r)} \tag{17}$$

The comparative static of equation (17) shows that either the bigger $m(\eta)$ or κ, the longer the length of the bubble's persistence interval. This is the same implication that AB (2003) prove in the continuous-time setting.

2.3 Numerical Example

In Section 2.2, we specify the arbitrageurs' optimal strategy τ^1 as in equation (17) using a discrete approximation of the exponential distribution function. Since the optimality of τ^1 is proved in Proposition 1 under the condition of positive $D(1, j)$, there is a possibility of positive τ^1. We check the domain of the nonpositive τ^1 to satisfy the existence of arbitrage delay.

These relations are expressed as a numerical example in Figure 1. In this calculation, the numerical value is given as follows: $g = 0.05$, $r = 0.03$,

$\kappa = 0.5$. The horizontal axis is the trading time interval (Δ), and the vertical axis represents the number of arbitrageurs (m). A blackened domain means a negative τ^1.

We can see that there is a positive τ^1 in the smaller trading time interval region. As a result, there is a very high probability of positive τ^1 in the continuous-time model of AB (2003). However, in a discrete-time setting, we find that the threshold of the positive τ^1 domain is under some value $\eta(= m\Delta)$[10]. In other words, synchronization risk exists when the information about the burst is conveyed to all arbitrageurs in a shorter horizon.

3 Conclusion

In this paper, we make two main points about AB (2003). First, this paper shows that AB (2003) is consistent with a discrete-time setting, and the bursting time is specified as in equation (17). This implication concerning the synchronization risk applies to a discrete-time setting.

Second, we find that synchronization risk exists when information about the burst is extended to all arbitrageurs in a shorter time using a numerical example. Therefore, it is also an important future task to check whether this model's implications are plausible in a positive analysis.

[10]The shape of the threshold line shows that m is inversely proportional to Δ.

4 Appendix

4.1 Proof of Proposition 1

We show that in equilibrium, the arbitrageur's optimal strategy is a trigger strategy. Equation (6) means that the trigger strategy satisfies the necessary condition of maximizing the problem. We now need to check the sufficient condition of maximizing.

Proof. The sufficient equation is equivalent to the condition that (6) is a nonincreasing function when $D(1, j) > 0$ exists.

$D(\tau, j) = (\frac{1+g}{1+r})^{j+\tau-b-1}(1 - \beta(j + \tau - b - 1)\phi(t_{j+\tau}|j)$

$-\{(\frac{1+g}{1+r})^{j+\tau-b-1}(1 - \Phi(t_{j+\tau-1}|j)) - (\frac{1+g}{1+r})^{j+\tau-b}(1 - \Phi(t_{j+\tau}|j)\}$

The distribution function satisfies $\Phi(t_{j+\tau}|j) = \Phi(t_{j+\tau-1}|j) + \phi(t_{j+\tau}|j)$, and we substitute it into $D(\tau, j)$:

$D(\tau, j) = (\frac{1+g}{1+r})^{j+\tau-b}\{\frac{1+g}{1+r}(-\beta(j + \tau - b)\phi(t_{j+\tau}|j)) + \frac{g-r}{1+r}(1 - \Phi(t_{j+\tau-1}|j)\}$

$D(\tau, j) = \frac{1}{1+r}(\frac{1+g}{1+r})^{j+\tau-b}\{-(1 + g)\beta(j + \tau - b)\phi(t_{j+\tau}|j)$

$+(g - r)(1 - \Phi(t_{j+\tau-1}|j))\}$

The sign of $D(\tau, j)$'s increment becomes as follows:

$\{\beta(j + \tau + 1 - b)\phi(t_{j+\tau+1}|j) - \beta(j + \tau - b)\phi(t_{j+\tau}|j)\} > 0$ exists.

Therefore, the sign of $D(\tau, j)$'s first term increment becomes negative.

$(1 - \Phi(t_{j+\tau}|j)) - (1 - \Phi(t_{j+\tau-1}|j)) = \Phi(t_{j+\tau-1}|j)) - \Phi(t_{j+\tau}|j)) < 0$

So, the sign of $D(\tau, j)$'s second term increment becomes negative.

As the equations above show that the sign of $D(\tau, j)$'s increment is negative, (6) is a nonincreasing function. ■

4.2 Proof of Proposition 2

We show the symmetric equilibrium's uniqueness as follows. Equation (7) exists if the variable τ indicates the optimal strategy of equilibrium. The uniqueness is proven by the following two steps.

Proof. Step 1: The left-hand side of equation (7) is a monotonically increasing function of τ, the right-hand side is a monotonically decreasing function.

Step 2: $\frac{\phi(t_j|j)}{1-\Phi(t_j|j)} < \frac{(1+g)^\Delta-(1+r)^\Delta}{\beta(j-b)(1+g)^\Delta}, \frac{\phi(t_{b+\overline{\tau}}|j)}{1-\Phi(t_{b+\overline{\tau}}|i)} > \frac{(1+g)^\Delta-(1+r)^\Delta}{\beta(j+\overline{\tau}-b)(1+g)^\Delta}$

The left-hand and right-hand sides of equation (7) cross one point when Steps 1 and 2 above are both satisfied. Therefore, an unique equilibrium: $\tau = \tau^1$ is established.

(The Proof of Step 1)

By corollary 1, $\phi(t_{j+\tau}|j)$ is a monotonically increasing function of τ. By the property of a distribution function, $\phi(t_{j+\tau}|j)$ is a monotonically increasing function of τ when $\phi(t_{j+\tau}|j)$ is a monotonically increasing function. We need to check that the left-hand side of equation (7) is a monotonically increasing function by the sign of the increment in the time $t_{j+\tau+1}$, and $t_{j+\tau}$.

$sign\{\frac{\phi(t_{j+\tau+1}|j)}{1-\Phi(t_{j+\tau+1}|j)} - \frac{\phi(t_{j+\tau}|j)}{1-\Phi(t_{j+\tau}|j)}\}$
$= sign\{\frac{\phi(t_{j+\tau+1}|j)\{1-\Phi(t_{j+\tau}|j)\}-\phi(t_{j+\tau}|j)\{1-\Phi(t_{j+\tau+1}|j)\}}{\{1-\Phi(t_{j+\tau+1}|j)\}\{1-\Phi(t_{j+\tau}|j)\}}\}$

Since the sign of the denominator is positive, we only check whether the sign of the numerator is positive.

$\phi(t_{j+\tau+1}|j)\{1-\Phi(t_{j+\tau}|j)\} - \phi(t_{j+\tau}|j)\{1-\Phi(t_{j+\tau+1}|j)\}$
$= \phi(t_{j+\tau+1}|j)\{1-\Phi(t_{j+\tau}|j)\} - \phi(t_{j+\tau}|j)\{1-\Phi(t_{j+\tau}|j) - \phi(t_{j+\tau+1}|j)$
$= \phi(t_{j+\tau+1}|j) - \phi(t_{j+\tau}|j)\}\{1-\Phi(t_{j+\tau}|j) + \phi^2(t_{j+\tau}|j) > 0$

We check that the left-hand side is a monotonically increasing function. We next check that the right-hand side is a monotonically decreasing function.

Because $\beta(i+\tau-b)$ is a monotonically decreasing function of τ:

$\frac{(1+g)^\Delta-(1+r)^\Delta}{\beta((i+\tau+1-b))(1+g)^\Delta} - \frac{(1+g)^\Delta-(1+r)^\Delta}{\beta((i+\tau-b))(1+g)^\Delta}$
$= \frac{\{(1+g)^\Delta-(1+r)^\Delta\}\{\beta((i+\tau-b))-\beta((i+\tau+1-b))\}}{(1+g)^\Delta\beta((i+\tau+1-b))\beta((i+\tau-b))} < 0$

The above equation shows that the right-hand side is a monotonically decreasing function.

(Proof of Step 2)

$\frac{\phi(t_j|j)}{1-\Phi(t_j|j)} = 0 < \frac{(1+g)^\Delta-(1+r)^\Delta}{\beta(j-b)(1+g)^\Delta}, \frac{\phi(t_{b+\overline{\tau}}|j)}{1-\Phi(t_{b+\overline{\tau}}|i)} = 1$
$\frac{(1+g)^\Delta-(1+r)^\Delta}{\beta(j+\overline{\tau}-b)(1+g)^\Delta} < \frac{(1+g)^\Delta-(1+r)^\Delta}{\overline{\beta}} < 1 = \frac{\phi(t_{b+\overline{\tau}}|j)}{1-\Phi(t_{b+\overline{\tau}}|i)}$ ■

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Figure 1: A numerical example