Aggregate Accuracy under Majority Rule with Heterogeneous Cost Functions

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Abstract

We investigate an election model with costly accuracy improvement by allowing heterogeneity in the cost functions. We find that the aggregate accuracy in large elections is characterized by the average value of the inverse of the second derivative at zero information.

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1. Introduction

Consider the following situation. A large society is using majority rule to try to choose the "correct" alternative of two choices. However, each member of the society, say i, must invest the costs of $C_i(q_i)$ so that he or she can vote for the correct alternative with probability q_i. The members' utility from the correct choice by society is normalized to 1, and the incorrect one to 0.

There are two opposite effects of the size of the society on the accuracy of society's choice. On the one hand, society can utilize the law of large numbers.[1] On the other hand, large size gives each member only a negligible incentive to improve his or her accuracy.[2] If so, how does the aggregate accuracy, namely the probability that the majority of members vote for the correct alternative, depend on the parameters of the cost functions?

In his pioneering work, Martinelli (2004) considered cases with homogeneous cost functions (i.e., $C_i = C$ for all i), and found that, if $C'(1/2) = 0$, then the probability depends only on $C''(1/2)$. We allow heterogeneity here, and find that the generalized key parameter is not the average of $C_i''(1/2)$ but the inverse of the average of $1/C_i''(1/2)$.

2. The Model

2.1 Settings

For $n \in N$, the following normal form game is considered. There are $2n+1$ players, i.e., "voters". The strategy of each voter $i = 1, ..., 2n+1$ is the accuracy of his or her vote, $q_{i,n} \in [1/2, 1]$. The corresponding payoff is given by

$$\Pr\left(\sum_{j=1}^{2n+1} x_j(q_{j,n}) \geq n+1\right) - C_i(q_{i,n}), \tag{1}$$

where each $x_j(q_{j,n})$ is independently drawn from its corresponding distribution as

$$x_j(q_{j,n}) = \begin{cases} 1 & \text{with probability} \quad q_{j,n} \\ 0 & \text{with probability} \quad 1-q_{j,n} \end{cases}.$$

[1] At the extreme, this effect makes the accuracy converge to 1. This line dates back to Condorcet (1785). For general results with exogenous accuracies in this line, see, e.g., Berend and Paroush (1998).

[2] At the extreme, this effect makes all members give up any investment (i.e., $q_i = 1/2$ for all i), even with a finite size of their society, which results in its choice being as if it were decided by a toss of a fair coin (i.e., being correct only with probability 1/2). This line dates back to Downs (1957). For models with a binary choice of accuracies in this line, see, e.g., Mukhopadhaya (2003).

The first term in (1) represents the expected utility from the chosen alternative, which, by normalization, is equal to the probability that the correct alternative is chosen.[3] The second term represents the costs that yield the accuracy.

For simplicity, we focus on cases with a finite number of cost functions, $\{C_k\}_{k=1,...,K}$; for all k, $m_{k,n}$ voters have C_k, and $\sum_k m_{k,n} = 2n+1$. We assume that, for all k, $m_{k,n}/(2n+1) \to \alpha_k^* > 0$ as $n \to \infty$. For all k, $C_k(q)$ is strictly increasing, strictly convex, and twice differentiable in q. We also have $C_k(1/2) = 0$, $C_k(1) > 1$, and $C_k'(1/2) = 0$.

2.2 Equilibrium and Aggregate Accuracy

The equilibrium concept is that of pure Nash. The first-order conditions imply that[4]

$$\Pr\left(\sum_{j\neq i} x_j(q_{j,n}) = n\right) = C_i'(q_{i,n}) \text{ for } i = 1, ..., 2n+1. \tag{2}$$

By the assumptions about cost functions (e.g., $C_k(1) > 1$ and $C_k'(1/2) = 0$ for all k assure the interiority), (a) (2) is the sufficient and necessary condition for $\{q_{i,n}\}_i$ to constitute an equilibrium, and (b) there exists at least one strategy profile $\{q_{i,n}\}_i$ satisfying (2) by Kakutani's fixed point theorem.[5]

The *aggregate accuracy*, namely the probability that the correct alternative is chosen in the equilibrium, is given by

$$\Pr\left(\sum_{i=1}^{2n+1} x_i(q_{i,n}) \geq n+1\right).$$

3. The Result

We yield the following asymptotic property.

Theorem. *Consider any equilibrium sequence* $\{\{q_{i,n}\}_i\}_n$. *Then,*

$$\Pr\left(\sum_{i=1}^{2n+1} x_i(q_{i,n}) \geq n+1\right) \to \Phi(c^*) \text{ as } n \to \infty, \tag{3}$$

[3] *Strategic* voting is not explicitly considered here. For the importance of distinguishing between strategic and *sincere* voting, see the seminal work of Austen-Smith and Banks (1996).

[4] Note that voters i can affect the "outcome" only when their vote is pivotal, i.e., $\sum_{j\neq i} x_j(q_j) = n$.

[5] There may be multiple equilibria.

where $c^* \in [0,\infty]$ *solves*

$$4\frac{\phi(c^*)}{c^*} = \frac{1}{\sum_k \alpha_k^* \frac{1}{C_k''(1/2)}}, \tag{4}$$

where Φ *is the standard normal distribution function and* ϕ *the corresponding probability density function.*[6]

Thus, in a large society (i.e., asymptotically), the aggregate accuracy depends only on the inverse of the *average* of $1/C_i''(1/2)$.

Proof: For simplicity, denote $S_n \equiv \sum_{i=1}^{2n+1} x_i(q_{i,n})$ and $S_n^i \equiv \sum_{j\neq i} x_j(q_{j,n})$, and let c_n be defined by

$$c_n \equiv \frac{\frac{E_n}{2n+1} - \frac{1}{2}}{\frac{\sqrt{V_n}}{2n+1}} \geq 0,$$

where $E_n \equiv E[S_n] = \sum_i q_{i,n}$ and $V_n \equiv \mathrm{var}(S_n) = \sum_i q_{i,n}(1-q_{i,n})$. Then, (2) yields

$$\frac{\sqrt{V_n}\Pr\left(S_n^i = n\right)}{\phi(c_n)} \frac{1}{\frac{V_n}{2n+1}} \frac{\phi(c_n)}{c_n} = \frac{C_i'(q_{i,n})}{\frac{E_n}{2n+1} - \frac{1}{2}}. \tag{5}$$

Denote $S_n^{i,j} \equiv \sum_{h\neq i,j} x_h(q_{h,n})$. Then, $\Pr(S_n^{i,j} = n-1) \leq \Pr(S_n^{i,j} = n)$, since $q_{h,n} \geq 1/2$ for all $h \neq i,j$. Thus,

$$\Pr(S_n^i = n) = q_{j,n}\Pr(S_n^{i,j} = n-1) + (1-q_{j,n})\Pr(S_n^{i,j} = n) \tag{6}$$

is nonincreasing in $q_{j,n}$. Therefore,

$$\Pr(S_n^i = n) \leq \Pr\left(\sum_{j\neq i} x_j(1/2) = n\right) = \binom{2n}{n}(1/4)^n, \tag{7}$$

which implies, by (2),

$$\lim_{n\to\infty} \max_i |q_{i,n} - 1/2| = 0. \tag{8}$$

Note that, by (6) for i and j,

$$\frac{\Pr(S_n^i = n)}{\Pr(S_n^j = n)} = \frac{\frac{1}{2} + \left(q_{j,n} - \frac{1}{2}\right)\frac{\Pr(S_n^{i,j}=n-1)-\Pr(S_n^{i,j}=n)}{\Pr(S_n^{i,j}=n-1)+\Pr(S_n^{i,j}=n)}}{\frac{1}{2} + \left(q_{i,n} - \frac{1}{2}\right)\frac{\Pr(S_n^{i,j}=n-1)-\Pr(S_n^{i,j}=n)}{\Pr(S_n^{i,j}=n-1)+\Pr(S_n^{i,j}=n)}}.$$

Thus, by (8),

$$\max_{i,j}\left|\frac{\Pr(S_n^i = n)}{\Pr(S_n^j = n)} - 1\right| \leq 4\max_i\left|q_{i,n} - \frac{1}{2}\right| \to 0 \text{ as } n \to \infty.$$

[6] We use conventions as $1/\infty = 0$ and $1/0 = \infty$.

By (2), it implies

$$\lim_{n\to\infty} \max_{i,j} \left| \frac{C_i'(q_{i,n})}{C_j'(q_{j,n})} - 1 \right| = 0.$$

Therefore, since $\alpha_k^* > 0$ for all k,

$$\frac{C_i'(q_{i,n})}{\frac{E_n}{2n+1} - \frac{1}{2}} = \frac{C_i'(q_{i,n})}{\frac{\sum_{j=1}^{2n+1}(q_{j,n}-1/2)}{2n+1}} = \frac{1}{\frac{\sum_{j=1}^{2n+1} \frac{C_j'(q_{j,n})}{C_i'(q_{i,n})} \frac{q_{j,n}-1/2}{C_j'(q_{j,n})}}{2n+1}} \to \frac{1}{\sum_k \alpha_k^* \frac{1}{C_k''(1/2)}} \text{ as } n \to \infty.$$

Suppose that, along some subsequence, $c_n \to \infty$. Then, by (7), we have

$$\begin{aligned} \sqrt{V_n} \Pr(S_n^i = n) &\le \sqrt{V_n} \binom{2n}{n} \left(\frac{1}{2}\right)^n \left(1 - \frac{1}{2}\right)^n \\ &= \sqrt{\frac{V_n}{2n+1}} \sqrt{\frac{2n+1}{\pi n}} \frac{\binom{2n}{n}}{\frac{4^n}{\sqrt{\pi n}}} \to \frac{1}{\sqrt{2\pi}} \text{ as } n \to \infty, \end{aligned}$$

where the first term converges to $\sqrt{1/4}$ as we will see in (ii) below, and the third term converges to 1 by Stirling's formula.[7] Thus, along such subsequences, the LHS of (5) converges to 0, which equals $4\phi(\infty)/\infty$.

Suppose that along some subsequence, $c_n \to c^* < \infty$. Since

$$\begin{aligned} &\text{(i)} \quad \lim_{n\to\infty} \frac{1}{2n+1} \sum_{i=1}^{2n+1} E[\exp(x_i(q_{i,n}))] \le \exp(1) < \infty, \\ &\text{(ii)} \quad \lim_{n\to\infty} \frac{1}{2n+1} \sum_{i=1}^{2n+1} \text{var}(x_i(q_{i,n})) = \frac{1}{4} > 0, \text{ and} \\ &\text{(iii)} \quad \lim_{n\to\infty} \frac{1}{2n+1} \sum_{i=1}^{2n+1} \min\{\Pr(x_i(q_{i,n}) = 0), \Pr(x_i(q_{i,n}) = 1)\} = \frac{1}{2} > 0, \end{aligned}$$

where (ii) and (iii) follow from (8), the premises (I), (II), and (III') of Theorem 1 in McDonald (1979) are satisfied. Note that $|(n-1) - E_n| \ge (2n+1)(1/2) - (n-1) = 3/2 > 1$, and

$$\lim_{n\to\infty} \frac{|(n-1) - E_n|}{\sqrt{(2n+1)/4}} = \lim_{n\to\infty} \frac{|(n+1) - E_n|}{\sqrt{(2n+1)/4}} = c^*.$$

Therefore, the local limit theorem of McDonald (1979) implies that

$$\lim_{n\to\infty} \frac{\sqrt{V_n} \Pr(S_n = n-1)}{\phi(c_n)} = 1, \tag{9}$$

[7] Note that $\frac{4^n}{\sqrt{\pi n}} = \frac{\sqrt{2\pi(2n)}(2n)^{2n}e^{-2n}}{(\sqrt{2\pi n}n^n e^{-n})^2}$.

and that if $c^* > 0$, then,

$$\lim_{n\to\infty} \frac{\sqrt{V_n}\,\Pr\left(S_n = n+1\right)}{\phi(c_n)} = 1. \tag{10}$$

As we will briefly see in the final paragraph,

$$\begin{aligned} \Pr(S_n^i = n) &\geq \Pr(S_n = n-1), \text{ and} && (11)\\ |\Pr(S_n^i = n) - \Pr(S_n = n+1)| &\leq |\Pr(S_n = n+1) - \Pr(S_n = n-1)|. && (12) \end{aligned}$$

Thus, by (9) and (11), if $c^* = 0$, then, along such subsequences the LHS of (5) diverges to infinity, which equals $4\phi(0)/0$. If $c^* > 0$, then, by (9), (10), and (12),

$$\lim_{n\to\infty} \max_i \left| \frac{\sqrt{V_n}\,\Pr\left(S_n^i = n\right)}{\phi(c_n)} - 1 \right| = 0.$$

Therefore, since $\lim_{n\to\infty} V_n/(2n+1) = 1/4$, as seen in (ii) above, along such subsequences the LHS of (5) converges to $4\phi(c^*)/c^*$.

These imply that $c_n \to c^*$ solving (4) as n grows. (3) is then a simple consequence of the central limit theorem.[8]

Now, we briefly see (11) and (12). Since $q_{i,n} \geq 1/2$ for all i,

$$\begin{aligned} \Pr(S_n^i = n-2) &\leq \Pr(S_n^i = n-1) \leq \Pr(S_n^i = n), \text{ and} && (13)\\ \Pr(S_n^i = n-1) &\leq \Pr(S_n^i = n+1). && (14) \end{aligned}$$

Note that for any m,

$$\Pr(S_n = m) = q_{i,n} \Pr(S_n^i = m-1) + (1 - q_{i,n}) \Pr(S_n^i = m). \tag{15}$$

(13) and (15) imply (11). If $\Pr(S_n^i = n+1) \geq \Pr(S_n^i = n)$, then,

$$\begin{aligned} \Pr(S_n = n+1) &\geq \Pr(S_n^i = n)\\ &\geq q_{i,n} \Pr(S_n^i = n-2) + (1 - q_{i,n}) \Pr(S_n^i = n-1) = \Pr(S_n = n-1), \end{aligned}$$

which implies (12). If $\Pr(S_n^i = n+1) < \Pr(S_n^i = n)$, then, by (13), (14), and (15),

$$\begin{aligned} 0 &< \Pr(S_n^i = n) - \Pr(S_n = n+1)\\ &\leq \Pr(S_n = n+1) - \Pr(S_n^i = n+1)\\ &\leq \Pr(S_n = n+1) - (q_{i,n} \Pr(S_n^i = n-2) + (1 - q_{i,n}) \Pr(S_n^i = n-1))\\ &= \Pr(S_n = n+1) - \Pr(S_n = n-1), \end{aligned}$$

[8]See, e.g., Feller (1971).

where the second inequality follows from $q_{i,n} \geq 1/2$. Thus, (12) holds. $\square$

Our theorem can derive the corresponding result of Martinelli (2004).

Corollary (Martinelli (2004)). *Suppose that all voters have an identical cost function, C. Then, the aggregate accuracy converges to $\Phi(c^*)$ as n grows, where c^* solves*

$$4\frac{\phi(c^*)}{c^*} = C''\left(1/2\right). \tag{16}$$

To be precise, only symmetric equilibria (i.e., $q_{i,n} = q_n$ for all i) are considered in Martinelli (2004). Note that $q_{i,n} \neq q_{j,n}$ with $C_i = C_j$ can occur under our equilibrium concept. Thus, in allowing the possibility of asymmetric equilibria, we generalize his result even within cases with homogeneous cost functions.

4. Discussions

Our result provides some answers to the topics below. Note that one could not find them if one misinterpreted the result of (16) in the homogeneous cost cases and used the average of $C_i''(1/2)$ as the proxy.

4.1 Severity of Convergence to 100% Accuracy

Martinelli's (2004) result of (16) implies that, if cost functions are homogeneous, then the aggregate accuracy cannot converge to 1 unless $C_i''(1/2) = 0$ for *all* voters.[9] One may conclude, based on this, that the convergence to 100% accuracy is implausible. However, our result suggests that such a conclusion is too premature. Observe that, if $C_k''(1/2) = 0$ for some k, then, however small $\alpha_k^* > 0$ is,

$$\sum_k \alpha_k^* \frac{1}{C_k''\left(1/2\right)} = +\infty,$$

which implies $\Phi(c^*) = 1$. Thus, to attain convergence to 100% accuracy, extreme cost functions satisfying $C_i''(1/2) = 0$ are required only for an *arbitrarily small* fraction of voters.

4.2 Heterogeneous Utilities and the Effect of Pie Allocation

[9] Note that $\Phi(c^*) = 1$ corresponds to the RHS being 0.

Consider that all the members have an identical cost function, $C(\cdot)$, but may differ in their utilities, r_i, from the correct choice by the society, i.e., the payoffs of (1) are replaced by

$$r_i \Pr\left(\sum_{j=1}^{2n+1} x_j(q_{j,n}) \geq n+1\right) - C(q_{i,n}).$$

We can deal with this situation within our framework by letting cost functions as $C_k = C/r_k$.

Then, how does the change in the distribution of the utilities, $\{r_k\}_{k=1,\ldots,K}$, affect the aggregate accuracy? Our result provides a clear-cut answer. In fact,

$$\sum_k \alpha_k^* \frac{1}{C_k''(1/2)} = \frac{\sum_k \alpha_k^* r_k}{C''(1/2)}.$$

Thus, the aggregate accuracy depends only on $\sum_k \alpha_k^* r_k$, the average utility among the members.

Note that, if the average is unchanged, then the sum is also unchanged. Thus, we can interpret changing the distribution of the utilities, but retaining its average, as changing the way of allocating the pie resulting from the society's choice. Then, the above result means that, unless we assume a difference in voting abilities (represented by cost functions) among members, at least asymptotically, the aggregate accuracy is not affected by the manner of allocating the pie; for example, we can allocate more to vulnerable people in the society without causing any deterioration of the aggregate accuracy.

References

Austen-Smith, D. and Banks, J. S. (1996) "Information Aggregation, Rationality, and the Condorcet Jury Theorem" *American Political Science Review* **90**, 34–45.

Berend, D. and Paroush, J. (1998) "When is Condorcet's Jury Theorem Valid?" *Social Choice and Welfare* **15**, 481–88.

Condorcet, M. de (1785) *Essai sur l'Application de l'Analyse à la Probabilité des Décisions Rendue à la Pluralité des Voix*: Paris.

Downs, A. (1957) *An Economic Theory of Democracy*, HarperCollins Publishers: New York.

Feller, W. (1971) *An Introduction to Probability Theory and Its Applications*, John Wiley & Sons: New York.

Martinelli, C. (2004) "Would Rational Voters Acquire Costly Information?" forthcoming in *Journal of Economic Theory.*

McDonald, D. (1979) “A Local Limit Theorem for Large Deviations of Sums of Independent, Nonidentically Distributed Random Variables” *The Annals of Probability* **7**, 526–531.

Mukhopadhaya, K. (2003) “Jury Size and the Free Rider Problem” *The Journal of Law, Economics and Organization* **19**, 24–44.