E C O N O M I C S B U L L E T I N

Games with vector−valued payoffs and their application to competition between organizations

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Abstract

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"Games with vector-valued payoff functions and their application to competition between organizations[1]"

by

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1. Introduction

In 1959, Lloyd Shapley wrote a short paper on games with vector payoffs. He analyzed zero-sum matrix games. Here, we extend Shapley's equilibrium concept to general games with vector payoffs, introduce an organizational interpretation of the concept, elaborate the relationship of the original concept to another equilibrium concept where each player can be viewed as running a bargaining game among internal 'factions,' and finally comment upon its relationship to the concept of party unanimity Nash equilibrium (PUNE).

2. Games with vector payoffs

Consider a 'game' with n players, each of whom has several goals. For the sake of simplicity in exposition only, we suppose that n=2, each player has two goals, and that they share a common strategy space S. Denote by $U^a : S \times S \to \mathbb{R}$ and $U^b : S \times S \to \mathbb{R}$ two payoff functions for Player One, and by $V^a : S \times S \to \mathbb{R}$ and $V^b : S \times S \to \mathbb{R}$ two payoff functions for Player Two. We will speak of the players' a- and b-goals.

Players are unable to assign weights to these two goals – they hence do not possess complete preference orders over $S \times S$. They are limited to making comparisons

[1] I am grateful to Michel LeBreton for alerting me to the existence of Shapley's paper.

by dominance only, in the following sense. A best response by Player One to a strategy s_2 by Player Two is a strategy s_1 such that:

$$(/\,s \quad S)(U^a(s,s_2),U^b(s,s_2)) \quad (U^a(s_1,s_2),U^b(s_1,s_2)) \qquad (1)$$

where the vector inequality means at least one component of the left-hand side is strictly larger than its counterpart, and none is smaller. Thus, define:

$$B^1(s_2)=\{s_1 \quad S \mid \text{ statement (1) holds}\}.$$

Define $B^2(s_1)$ in like manner.

The tuple (S,U^a,U^b,V^a,V^b) will be called a *game with vector payoffs*.

Definition. A strategy pair $(s_1,s_2) \quad S \quad S$ is a *vv-equilibrium* of the game (S,U^a,U^b,V^a,V^b) if $s_1 \quad B^1(s_2)$ and $s_2 \quad B^2(s_1)$.

Let $\ ,\ \quad (0,1)$ and consider the standard game whose payoff functions for the two players are:

$$\begin{aligned} {}^1\ (s_1,s_2) &= \ U^a(s_1,s_2)+(1 \qquad)U^b(s_1,s_2), \\ {}^2\ (s_1,s_2) &= \ V^a(s_1,s_2)+(1 \qquad)V^b(s_1,s_2). \end{aligned}$$

Denote this game by $G^{(\ ,\ )}$. If $U^a,U^b,V^a,$ and V^b are concave and continuous then $G^{(\ ,\ )}$ possesses a (standard) Nash equilibrium, call it $(s_1(\ ,\ ),s_2(\ ,\ ))$. But this is also a vv-equilibrium of G. For suppose there were a strategy s such that

$$(U^a(s,s_2(\ ,\ )),U^b(s,s_2(\ ,\ ))) \quad (U^a(s_1(\ ,\ ),s_2(\ ,\ )),U^b(s_1(\ ,\ ),s_2(\ ,\ ))).$$

Then, since $\quad 0 \quad 1 \quad$, we have ${}^1\ (s,s_2(\ ,\ )) > {}^1\ (s_1(\ ,\ ),s_2(\ ,\ ))$, an impossibility.

Thus we have immediately a two dimensional manifold of vv-equilibria. (One should perhaps worry about whether these are distinct.)

Are there other vv-equilibria? Possibly – but not very many. Define the utility possibilities sets:

$$^{1}(s_2) = \{(u^a, u^b) \quad \mathbb{R}^2 \mid (\quad s \quad S)((u^a, u^b) = (U^a(s, s_2), U^b(s, s_2))\},$$

and $\;^{2}(s_1)$ in like manner. Assume free disposability – that is, $\;^{2}(s_1)$ and $\;^{1}(s_2)$ are comprehensive sets in $\mathbb{R}^2$ for all s_1, s_2. Then these sets are convex. For let $s_1, \hat{s}_1, s_2$ be arbitrary.Then

$$(U^a(\quad s_1 + (1 \qquad)\hat{s}_1, s_2), U^b(\quad s_1 + (1 \qquad)\hat{s}_1, s_2)) \qquad (U^a(s_1, s_2), U^b(s_1, s_2)) + (1 \qquad)(U^a(\hat{s}_1, s_2), U^b(\hat{s}_1, s_2))$$

by concavity. Hence, by free disposability, the point on the r.h.s. of the above inequality lies in $\;^{1}(s_2)$, which demonstrates that it is as convex set.

Now let $(\bar{s}_1, \bar{s}_2)$ be a vv-equilibrium of *G*. Consider the convex set

$$P^1(\bar{s}_1, \bar{s}_2) = (U^a(\bar{s}_1, \bar{s}_2), U^b(\bar{s}_1, \bar{s}_2)) \qquad ^{1}(\bar{s}_2).$$

Let $\quad = \{(x, y) \quad \mathbb{R}^2 \mid x \quad 0, y \quad 0\}$ be the non-negative quadrant of $\mathbb{R}^2$. Then

$$P^1(\bar{s}_1, \bar{s}_2) \qquad = \{(0,0)\};$$

if any other point were in the intersection of these two sets, then $\bar{s}_1$ would not be a vv-best response to $\bar{s}_2$. Therefore, by the separating hyperplane theorem, there is a non-zero vector $p \quad \mathbb{R}^2$ such that

$$p \qquad 0, \quad p \;\; P^1(\bar{s}_1, \bar{s}_2) \quad 0.$$

By the first inequality, we must have $p \quad 0$. Therefore, ignoring scale, there is a number $\quad [0,1]$ such that $p = (\;\; , 1 \qquad)$.

In like manner, there is a number $\qquad [0,1]$ such that $(\;\; , 1 \qquad) \;\; P^2(\bar{s}_1, \bar{s}_2) \quad 0$, where

$P^2(\bar{s}_1,\bar{s}_2) = (V^a(\bar{s}_1,\bar{s}_2), V^b(\bar{s}_1,\bar{s}_2)) \quad {}^2(\bar{s}_1).$

Because $(\;,1 \quad) \; P^1(\bar{s}_1,\bar{s}_2) \quad 0$, it immediately follows that $\bar{s}_1$ maximizes ${}^1 \; (s,\bar{s}_2)$; in like manner, $\bar{s}_2$ maximizes ${}^2 \; (\bar{s}_1,s)$. Thus, $(\bar{s}_1,\bar{s}_2)$ is a Nash equilibrium of the standard game $G^{(\; , \;)}$. Summarizing:

<u>Theorem</u> <u>1</u>. Let U^a,U^b,V^a,V^b be continuous, concave payoff functions, let free disposability hold in the vv game $G = (S,U^a,U^b,V^a,V^b)$, and let *S* be convex. Then:

(1) For every $(\;, \;) \quad (0,1)^2$ the Nash equilibria of the game $G^{(\; , \;)}$ are vv-equilibria of *G;*

(2) If $(\bar{s}_1,\bar{s}_2)$ is a vv-equilibrium of *G,* then there exists $(\;, \;) \quad [0,1]^2$ such that $(\bar{s}_1,\bar{s}_2)$ is a Nash equilibrium of $G^{(\; , \;)}$.

Clearly, the theorem generalizes to any number of players each with any number of goals. If there are *n* players and player *i* has n_i goals, then the dimension of the equilibrium manifold is $\quad (g_i \quad 1) = \quad g_i \quad n.$

3. <u>A</u> <u>bargaining</u> <u>interpretation</u>

We motivated the vv game by describing individuals with multiple goals. In applications, it will often by the case that each player is an organization, whose members have different goals. Call the set of members in an organization of share the same goal a *faction*. The organizations might be political parties, firms, or trade unions. In this context, one might propose that factions would bargain with each other, in the face of a strategy put forth by the opposition organization.

This motivates the following set-up. Let here be two organizations, each with an *a* and *b* faction. Each faction possesses von Neumann-Morgenstern preferences over lotteries on $S \quad S$. Let $U^a : S \quad S \quad\ \mathbb{R}$, $U^b : S \quad S \quad\ \mathbb{R}$ be vNM utility functions representing these preferences for the *a* and *b* factions in Organization 1, and let $V^a : S \quad S \quad\ \mathbb{R}, V^b : S \quad S \quad\ \mathbb{R}$ represent the vNM preferences of the factions in Organization 2. Suppose there are 'status quo' or 'impasse' strategies s_1^0 and s_2^0 for the two organizations; if the factions in an organization cannot reach agreement in their bargaining, then the organization plays the impasse strategy. We also take as data a number $\quad [0,1]$, called the bargaining power of faction *a* in Organization 1, and a number $\quad [0,1]$, called the bargaining power of faction *a* in Organization 2. We define an *organizational game* as a tuple $OG = (S, U^a, U^b, V^a, V^b, s_1^0, s_2^0, \ , \ )$. We define the tuple $(S, U^a, U^b, V^a, V^b, s_1^0, s_2^0)$ to be an *organizational game form*.

We define:

<u>Definition</u>. A *Nash bargaining solution equilibrium* of the organization game *OG* is a pair of strategies $(\bar{s}_1, \bar{s}_2)$ such that

$$\bar{s}_1 \quad \arg\max_{S} \quad \log(U^a(s,\bar{s}_2) \quad U^a(s_1^0,\bar{s}_2)) + (1 \quad\ )\log(U^b(s,\bar{s}_2) \quad U^b(s_1^0,\bar{s}_2))$$

and

$$\bar{s}_2 \quad \arg\max_{S} \quad \log(V^a(\bar{s}_1,s) \quad V^a(\bar{s}_1,s_2^0)) + (1 \quad\ )\log(V^b(\bar{s}_1,s) \quad V^b(\bar{s}_1,s_2^0)).$$

We say a Nash bargaining solution equilibrium is *non-trivial* if the utility of all four factions at the equilibrium strategy pair is strictly greater than it would be if each organization played its impasse strategy, facing the other organization's equilibrium strategy. (I.e., each faction strictly gains from cooperating with its partner faction.)

The definition says that, given the strategy of Organization 2, strategy $\bar{s}_1$ solves a Nash bargaining game between the factions of Organization 1 (i.e., it maximizes the appropriate weighted product of the utility gains from the impasse point of the two factions), and given the strategy of Organization 1, strategy $\bar{s}_2$ solves the Nash bargaining game between the factions of Organization 2.

We have:

<u>Theorem</u> <u>2</u>. Let $(S,U^a,U^b,V^a,V^b,s_1^0,s_2^0)$ be an organizational game form such that, for every choice of s_2 and s_1, the four functions

$$\begin{array}{l} \log(U^a(\ ,s_2)\quad U^a(s_1^0,s_2)), \log(U^b(\ ,s_2)\quad U^b(s_1^0,s_2)), \\ \log(V^a(s_1,\ )\quad V^a(s_1,s_2^0)), \text{ and } \log(V^b(s_1,\ )\quad V^b(s_1,s_2^0)) \end{array}$$ are concave on the strategy

domains where they are defined (i.e., where the arguments of the log functions are positive). Then for every $(\ ,\ )\quad [0,1]^2$, there exists a Nash bargaining solution equilibrium for the organization game $(S,U^a,U^b,V^a,V^b,s_1^0,s_2^0,\ ,\ )$.

<u>Proof</u>:

Given $(\ ,\ )\quad [0,1]^2$. Define the payoff functions:

$$^1(s_1,s_2) = \begin{array}{c} (U^a(s_1,s_2)\quad U^a(s_1^0,s_2))\ (U^b(s_1,s_2)\quad U^b(s_1^0,s_2))^{1}\quad , \\ \text{if } U^a(s_1,s_2)\quad U^a(s_1^0,s_2) \text{ and } U^b(s_1,s_2)\quad U^b(s_1^0,s_2) \\ 0, \text{ otherwise} \end{array}$$

$$^2(s_1,s_2) = \begin{array}{c} (V^a(s_1,s_2)\quad V^a(s_1,s_2^0))\ (V^b(s_1,s_2)\quad V^b(s_1,s_2^0))^{1}\quad , \\ \text{if } V^a(s_1,s_2)\quad V^a(s_1,s_2^0) \text{ and } V^b(s_1,s_2)\quad V^b(s_1,s_2^0) \\ 0, \text{ otherwise} \end{array}$$

By the log concavity premise, the conditional payoff functions $\ ^1(\ ,s_2)$ and $\ ^2(s_1,\ )$ are quasi-concave for any choice of s_1 and s_2. It therefore follows, by the standard Nash equilibrium existence theorem, that a Nash equilibrium exists for the game $(S,\ ^1,\ ^2)$. But this is Nash bargaining solution equilibrium for the organizational game $(S,U^a,U^b,V^a,V^b,s_1^0,s_2^0,\ ,\ )$. ■

We next exhibit the relationship between Nash bargaining solution equilibrium and vv equilibrium.

<u>Theorem 3</u>. Let U^a,U^b,V^a,V^b be payoff functions on the domain $S\quad S$, and let s_1^0 and s_2^0 be impasse strategies for organizations 1 and 2. Then:

(1) Let $(\ ,\ )\quad [0,1]^2$ and let $(\overline{s}_1,\overline{s}_2)$ be a Nash bargaining solution equilibrium for the organizational game associated with these data. Then $(\overline{s}_1,\overline{s}_2)$ is a vv-equilibrium of the vv game $G=(S,U^a,U^b,V^a,V^b)$.

(2) Let the four functions $\log(U^a(\ ,s_2)\quad U^a(s_1^0,s_2))$, etc., be concave on the domains where they are defined. Let $(\overline{s}_1,\overline{s}_2)$ be a vv-equilibrium of the game $G=(S,U^a,U^b,V^a,V^b)$ such that:

$$(2a)\quad \begin{array}{l} U^a(\overline{s}_1,\overline{s}_2)>U^a(s_1^0,\overline{s}_2) \\ U^b(\overline{s}_1,\overline{s}_2)>U^b(s_1^0,\overline{s}_2) \\ V^a(\overline{s}_1,\overline{s}_2)>V^a(\overline{s}_1,s_2^0) \\ V^b(\overline{s}_1,\overline{s}_2)>V^b(\overline{s}_1,s_2^0) \end{array}$$

Then there is an ordered pair $(\ ,\ )\quad [0,1]^2$ such that $(\overline{s}_1,\overline{s}_2)$ is a Nash bargaining solution equilibrium of the organizational game $(S,U^a,U^b,V^a,V^b,s_1^0,s_2^0,\ ,\ )$.

<u>Proof:</u>

<u>Part (1).</u> Given (,) and $(\bar{s}_1, \bar{s}_2)$ according to the premise. Define $K^1 = U^b(\bar{s}_1, \bar{s}_2)$. We know that the strategy $\bar{s}_1$ solves the program

$$\begin{aligned} &\max_S U^a(s, \bar{s}_2) \\ &s.t.\, U^b(s, \bar{s}_2) \quad K^1 \end{aligned};$$

this uses the fact that $(\bar{s}_1, \bar{s}_2)$ is a non-trivial Nash bargaining solution equilibrium. But this means that $\bar{s}_1 \quad P^1(\bar{s}_2)$. In like manner, $\bar{s}_2 \quad P^2(\bar{s}_1)$. The claim is proved.

<u>Part (2).</u> Let $(\bar{s}_1, \bar{s}_2)$ be a vv-equilibrium which satisfies the premise (2a). Define the sets

$$A = \{(x, y) \quad \mathbb{R}^2 \mid (\; s \quad S)((x, y) = (\log(U^a(s, \bar{s}_2) \quad U^a(s_1^0, \bar{s}_2)), \log(U^b(s, \bar{s}_2) \quad U^b(s_1^0, \bar{s}_2)))\},$$

$$B = \{(x, y) \quad \mathbb{R}^2 \mid (\; s \quad S)((x, y) = (\log(V^a(\bar{s}_1, s) \quad V^a(\bar{s}_1, s_2^0)), \log(V^b(\bar{s}_1, s) \quad V^b(\bar{s}_1, s_2^0)))\}.$$

By inequalities (2a), these sets are non-empty. By the log concavity assumption, they are convex sets. Since $(\bar{s}_1, \bar{s}_2)$ is a vv-equilibrium, the point $(x(\bar{s}_1), y(\bar{s}_1)) \quad A$ generated by the strategy $\bar{s}_1$ lies on *A*'s northeast boundary. Therefore, there exists a supporting line for *A* containing this point whose normal vector is non-negative. It follows that there is a number $0 \qquad 1$ such that

$$\bar{s}_1 \quad \arg\max_S \quad \log(U^a(s, \bar{s}_2) \quad U^a(s_1^0, \bar{s}_2)) + (1 \qquad) \log(U^b(s, \bar{s}_2) \quad U^b(s_1^0, \bar{s}_2)).$$

In like manner, there is a number $0 \qquad 1$ such that

$$\bar{s}_2 \quad \arg\max_S \quad \log(V^a(\bar{s}_1, s) \quad V^a(\bar{s}_1, s_2^0)) + (1 \qquad) \log(V^b(\bar{s}_1, s) \quad V^b(\bar{s}_1, s_2^0)).$$

It follows that $(\bar{s}_1, \bar{s}_2)$ is a Nash bargaining solution equilibrium of the constructed organizational game. ■

Theorem 3 tells us that, under suitable conditions, we can interpret vv equilibria as solutions of ‘hypothetical’ organizational games in which factions representing different ‘interests’ of the individual players are bargaining with each other.

There are, it seems to me, two uses of this theorem. The first is in the case where the players are actually individual persons, who have multiple interests, and who have incomplete preference orders over $S \quad S$, because they are unable to aggregate their multiple interests into a coherent preference order. Here, the interpretation is that any equilibrium can be rationalized as an equilibrium of an associated organizational game, where each interest is represented by a faction, and the factions bargain à la Nash. In other words, Nash bargaining is all that one needs to resolve the incompleteness of preferences generated by multiple interests. The second interpretation is in the case where the players are actually organizations with factions. The interpretation is in this case that, regardless of what the actual bargaining process between the factions is, as long as bargaining is *efficient* (i.e., produces a strategy from which no further mutual gains are possible for the factions), then an equilibrium (which will therefore be a vv-equilibrium) can always be interpreted as a pair of strategies in which each organization’s factions *are* bargaining à la Nash, with specified bargaining powers. Thus, the Nash bargaining model is ‘all we need’ to characterize any kind of efficient bargaining in a situation of competing organizations.

4. PUNE

In Roemer (1998, 1999) I proposed a political equilibrium concept, in which parties with factions compete for voters. Within parties, factions with different interests bargain with each other. Having learned, six years later, about Shapley's paper, it is now clear that PUNE is a special case of a vector-valued game. In Roemer (2001), I described the relationship between vv-equilibrium and Nash bargaining solution equilibrium for the political games.

The reader is referred to the above citations for a precise definition of PUNE. The context is one in which there are two (say) political parties, competing on a multi-dimensional policy space, which is the strategy space. Voters are defined by their preferences over policies. Each party contains an ideological faction, which has a stipulated preference order over policies, and an opportunist faction, which desires to maximize its vote share (or the probability of victory, in another variant). A party-unanimity Nash equilibrium (PUNE) is a pair of policies, one played by each party, such that neither party, facing the other's strategy, can find a policy that makes one of its factions better off, while not reducing the welfare of the other faction. This is exactly a vv-equilibrium. (In a more articulated model, the ideological faction does not have an exogenous preference order over policies; rather, it represents those who vote for the party. PUNE with 'endogenous' ideological factions can still be viewed as a special case of vv equilibrium.)

An important observation about this application is that the payoff functions of the opportunist factions are (virtually) never quasi-concave in the case of multi-dimensional policy spaces. Hence, the existence theorem 1 does not apply. Indeed, I have as yet no interesting, general existence theorem for PUNE. However, I find in many applications that PUNEs exist, and they can be interpreted as Nash bargaining solution equilibria; the set of ordered pairs (,) which support Nash bargaining solution equilibria is never the entire unit square, but is typically a two dimensional manifold in the unit square. Thus, there are only certain pairs of relative bargaining powers which will support vv equilibrium in the set-up of PUNE.

I suggest that vv equilibrium can be fruitfully applied in many areas of social science. One application that immediately comes to mind is oligopolistic competition, where firms have different factions – shareholders, perhaps, who wish to maximize profits, and managers, who wish to maximize firm size (perhaps). If strategies are multi-dimensional (price, quality,…) the experience with PUNE suggests that vv equilibria will exist. Another application could be to a simple (profit maximizing) firm competing with a labor union that has factions with different goals (say, young and old workers, or skilled an unskilled workers). The strategy space for these contests, in reality, is typically multi-dimensional: again, one can expect vv equilibria to exist, where Nash equilibria in pure strategies of organizations that are more simply conceived (ones with single payoff functions) will typically not exist.

In political theory, PUNE was a solution to the non-existence of Nash equilibrium in political games with multi-dimensional policy spaces when political actors (parties or candidates) were conceived of as maximizing single payoff functions. Generally speaking, in games with multi-dimensional strategy spaces, existence of Nash equilibrium in pure strategies is often a problem: the resolution often has been to consider mixed strategies, or to model the problem as a stage game. I am suggesting that a third alternative is to conceive of players as having multiple interests, and using vv equilibrium. And non-quasi-concavity of factional payoff functions, which is a problem when it comes to proving general existence theorems, is in fact a *good thing* – because it reduces the size of the equilibrium manifold. In the PUNE case, the equilibrium manifold is often quite localized in the space $S \quad S$, and so multiplicity of equilibrium does not greatly reduce the predictive power of the equilibrium concept.

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