E C O N O M I C S B U L L E T I N

The Le Chatelier Principle in lattices

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Abstract

The standard Le Chatelier Principle states that the long−run demand for a good (in which by definition there are fewer restraints on the variables) is more elastic than short−run demand. The fundamental insight above goes well beyond demand theory, and proofs of this basic idea have been found in various settings. Nearly all of these have been continuous optimization problems requiring assumptions on the continuity of the objective function and on the convexity of the choice set. However, the statement and intuition for the original principle do not seem to rely on any such `technical' assumptions. Work by Milgrom Shannon on monotone [ordinal] comparative statics provides an obvious framework to pursue a broader result in a discrete environment. The present paper therefore formulates and proves a very general Le Chatelier Principle in the context of lattices. A further generalization is that we allow the choice set to vary (potentially as a function of the underlying parameter).

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1 Introduction

The Le Chatelier Principle[1] is a basic but extremely useful property of equilibria. The connection to optimization was made by Paul Samuelson in [**6**] as part of his pioneering work on comparative statics. The simplest form of the result is that the long-run demand for a good is more elastic (i.e. price-sensitive) than the short-run demand, though the fundamental insight – that there is added interaction between variables when fewer constraints are imposed – holds true in a wide variety of situations. Formally, the standard version of the Le Chatelier Principle might start with the optimization problem

$$\textit{maximize } y = f(x_1, ..., x_n) - \sum_{i=1}^{n} \alpha_i x_i$$

where the Hessian of f is negative definite. Then if we let $x_{i,r}^*$ be the ith coordinate of $\arg\max y$ subject to r independent [nested] constraints on x, we have that

$$\frac{\partial x_{i,0}^*}{\partial \alpha_i} \leq \frac{\partial x_{i,1}^*}{\partial \alpha_i} \leq ... \leq \frac{\partial x_{i,n-1}^*}{\partial \alpha_i} \leq \frac{\partial x_{i,n}^*}{\partial \alpha_i} = 0 \tag{1}$$

for all $i = 1, ...n$. E.g., this says that the optimal value of an input x_1 to a production function f will vary more with its price α_1 when the quantity x_2 of another input is allowed to vary than when x_2 is fixed (at its original maximizing level). It also follows that we can consider it part of the Le Chatelier Principle to conclude that all such changes are negative (i.e. that demand decreases with price), though this has not generally been pointed out.[2]

Since the time of Samuelson's original result, the Le Chatelier Principle has been generalized in several ways, mostly involving calculus and a continuous choice variable; see

[1] Originally due to Henri Louis Le Chatelier in 1888 (see [**3**]). As used in chemical thermodynamics, it says roughly that if stress is applied to a system in equilibrium, then the system readjusts to a new equilibrium in which that stress is relieved, if possible.

[2] Note that if we had taken derivatives with respect to the 'actual' coefficient on x_i in y, which is $-\alpha_i$, the inequalities would reverse in sign; this will be useful for comparison below.

[**7**] and references therein. The present paper, based on [**2**] and building on the monotone methodology developed in [**8**] and later [**5**] for discrete choice sets, proves a general version of the Le Chatelier Principle in lattices.

2 Background

Consider a lattice $(X, \leq)$ with meet $\wedge$ and join $\vee$. We use Veinott's *strong set order* $\leq_s$: for $Y, Z \subseteq X$, $Y \leq_s Z$ if for every $y \in Y$ and $z \in Z$ we have $y \wedge z \in Y$ and $y \vee z \in Z$. Thus $Y \leq_s Y$ iff Y is a sublattice of X. We will call a function $f : T \to \wp(X)$ from a poset T to the power set of a lattice *nondecreasing* if it is nondecreasing with respect to the strong set order. Finally, if X and Y are lattices, then we can form the product lattice $X \times Y$, with $(x, y) \geq (x', y')$ only when $x \geq x'$ and $y \geq y'$ in the respective partial orders. In this case we also have coordinate-wise projection functions $\Pi_X(X \times Y)$ and $\Pi_Y(X \times Y)$ onto X and Y respectively.

In order to prove ordinal comparative statics results in the context of optimization over lattices, [**5**] introduces the following definitions[3]. A real-valued function f on a lattice X is called *quasisupermodular* (or qsm) if both

$$\begin{aligned} f(x) &\geq f(x \wedge x') \implies f(x \vee x') \geq f(x') \text{ and} \\ f(x) &> f(x \wedge x') \implies f(x \vee x') > f(x'). \end{aligned}$$

If X is a lattice and T is a poset, then $f : X \times T \to \mathbb{R}$ has the *single crossing property* (scp) in $(x; t)$ if for every $x' \geq x$ and $t' \geq t$

$$\begin{aligned} f(x', t) &\geq f(x, t) \implies f(x', t') \geq f(x, t') \text{ and} \\ f(x', t) &> f(x, t) \implies f(x', t') > f(x, t'). \end{aligned}$$

[3]These extend the analogous cardinal (as opposed to ordinal) definitions and results in [**8**].

Finally, define $M(t, S) = \arg\max_{x \in S} f(x, t)$. They are then able to prove

Theorem 1 *(Milgrom and Shannon) If X is a lattice with $S \subseteq X$, T is a poset, f is a real-valued function on $X \times T$ that is qsm in x and scp in $(x; t)$, then $M(t, S)$ is nondecreasing in t and in S.*

In fact, they also prove the converse of this result, so the conditions are tight. The forward direction, however, is the more relevant here. One immediate corollary is that if S is a sublattice of X and f satisfies qsm and scp, then the maximizing set $M(t, S)$ is a sublattice of S for any t. Technically, only if S is a sublattice can the choice set be held fixed to find that $M(t, S)$ is nondecreasing in t alone.

3 The Le Chatelier Principle

We want to formulate and prove a Le Chatelier Principle for lattices. Our choice variables x_i will be drawn from lattices and our parameters t (analogous to the prices α_i before) from a poset, as above. One extension of the standard Le Chatelier Principle, other than simply the setting, is that we will allow the choice sets to vary in addition to the parameters. So let f be a real-valued function on $X_1 \times ... \times X_n \times T$, where each X_i is a lattice and T is a poset (possibly 'multidimensional' itself in the Euclidean sense, of course). Let $S = \times_{i=1}^n S_i$ and $S' = \times_{i=1}^n S_i'$ be sublattices of $\times_{i=1}^n X_i$. Define $M(t, S) = \arg\max_{\mathbf{x} \in S} f(\mathbf{x}, t)$ to be the set of maximizers as before, for $t \in T$, and $M_i(t, S) = \Pi_{X_i}(M(t, S))$. Finally, define $M^r(t, t'; S, S') = \arg\max_{\mathbf{x} \in S^r} f(\mathbf{x}, t')$, where $S^r = \Pi_{\times_{i=1}^r X_i}(M(t, S)) \times \Pi_{\times_{i=r+1}^n X_i}(S')$. Conceptually, M^r is an in-between maximizing set where we fix the first r components of $\mathbf{x}$ to remain in the argmax set for the original conditions of t and S, whereas the final $n - r$ components are allowed to vary optimally under the new conditions of t' and S'. As always, $M_i^r(t,t';S,S') = \Pi_{X_i}(M^r(t,t';S,S'))$. From now on, for ease of notation, we drop the arguments to $M^r(t,t';S,S')$ and the subscripts to Π (which are always the appropriate section of $\times_{i=1}^n X_i$).

Theorem 2 *In the setting above, if f is qsm in $\mathbf{x}$ and scp in $(\mathbf{x};t)$, then for any $t' \geq t$ in T and $S' \geq_s S$, we have*

$$M_i(t',S') = M_i^0 \geq_s M_i^1 \geq_s ... \geq_s M_i^{n-1} \geq_s M_i^n \geq_s M_i(t,S) \tag{2}$$

for every $i = 1,...n$.

Note the similarities to equation 1 in the introduction; the final inequality would be an equality (this is essentially the comparison to 0 in the standard version) if we leave t fixed and change only S, or if we consider only the least upper bounds of the maximizing sets. It may also be worth pointing out that the statement of the theorem allows S' itself to be a [nondecreasing] function of the parameter t. For instance, this allows a setting in which not only are the prices of inputs changing, but so is the set of available inputs.

Proof. The equality at the beginning of equation 2 follows directly from the definition. Now pick $x_i \in M_i^{r-1}$ and $x_i' \in M_i^r$, with $1 \leq r \leq n$. We need to prove that $x_i \vee x_i' \in M_i^{r-1}$ and that $x_i \wedge x_i' \in M_i^r$. From the definitions, there must exist x_{-i} and x_{-i}' such that $x = (x_i, x_{-i}) \in M^{r-1}$ and $x' = (x_i', x_{-i}') \in M^r$. We begin by showing that $x \vee x' \in S^{r-1}$ and is thus 'eligible' for M^{r-1}. Clearly, both $(x_1,...,x_{r-1})$ and $(x_1',...,x_{r-1}')$ are in $\Pi(M(t,S))$. But $M(t,S)$ is a sublattice as in Theorem 1 (and therefore so is its projection), and $(x \vee x')_1 = x_1 \vee x_1'$ by construction of $\times_{i=1}^n X_i$, so we get that $((x \vee x')_1,...,(x \vee x')_{r-1}) \in \Pi(M(t,S))$ also. Similarly, $(x_{r+1},...,x_n)$ and $(x_{r+1}',...,x_n')$ are in $\Pi(S')$, which is a sublattice, and so

$((x \vee x')_{r+1}, ..., (x \vee x')_n) \in \Pi(S')$. Finally, $x_r \in S'_r$ and $x'_r \in M_r(t,S) \subseteq S_r$, but $S'_r \geq_s S_r$ so we know $x_r \vee x'_r \in S'_r$. Hence, $x \vee x' \in S^{r-1}$ as desired.

We next show that $x \wedge x' \in S^r$. Just as above, it is easy to see that $((x \wedge x')_1, ..., (x \wedge x')_{r-1}) \in \Pi(M(t,S))$ and that $((x \wedge x')_{r+1}, ..., (x \wedge x')_n) \in \Pi(S')$. Now $x'_r \in M_r(t,S)$ so $\exists y$ such that $y = (x'_r, y_{-r}) \in M(t,S) \subseteq S$. Thus $x \wedge y \in S$ (using that S_i is a sublattice for $i < r$ and that $S_i \leq_s S'_i$ for $i \geq r$) and $(x \wedge y)_r = (x \wedge x')_r$. So either $(x \wedge x')_r \in M_r(t,S)$ and we're done, or $(x \wedge y)_r \notin M_r(t,S)$ and $f(y,t) > f(x \wedge y, t)$, since $y \in M(t,S)$. But in that case, quasisupermodularity of f implies that $f(x \vee y, t) > f(x,t)$, and hence $f(x \vee y, t') > f(x,t')$ by single crossing. By the reasoning in the paragraph above, $x \vee y \in S^{r-1}$ (using the final argument not just for the rth coordinate but for all $i \geq r$), which would lead to the contradictory conclusion that $x \notin M^{r-1}$. So it must have been that $(x \wedge x')_r \in M_r(t,S)$ and $x \wedge x' \in S^r$ as we wished.

It remains to prove that $f(x \vee x', t')$ and $f(x \wedge x', t')$ are sufficiently large. Since $x' \in M^r$ and $x \wedge x' \in S^r$, $f(x',t') \geq f(x \wedge x', t')$ and so by qsm $f(x \vee x', t') \geq f(x,t')$. But $x \in M^{r-1}$ and $x \vee x' \in S^{r-1}$, so we must also have $x \vee x' \in M^{r-1}$, and thus $x_i \vee x'_i = (x \vee x')_i \in M_i^{r-1}$. Analogously, $x \in M^{r-1}$ and $x \vee x' \in S^{r-1}$ imply that $f(x,t') \geq f(x \vee x', t')$. Then qsm says $f(x',t') \not> f(x \wedge x', t')$, i.e. $f(x \wedge x', t') \geq f(x',t')$. Using $x' \in M^r$ and $x \wedge x' \in S^r$, we see that $x \wedge x' \in M^r$ and hence $x_i \wedge x'_i \in M_i^r$. [In fact, it is clear at this point that $f(x,t') = f(x \vee x', t')$ and $f(x',t') = f(x \wedge x', t')$.]

From the definition of $\geq_s$, this completes the proof of everything but the last inequality. Since $S^n = M(t,S)$, $M^n \subseteq M(t,S)$. So pick $x \in M^n$ and $x' \in M(t,S)$, a sublattice. Then $x \wedge x' \in M(t,S) \subseteq S$ and $f(x',t) \geq f(x \wedge x', t)$. Now qsm implies that $f(x \vee x', t) \geq f(x,t)$, and so by single crossing $f(x \vee x', t') \geq f(x,t')$. Thus, $x \vee x' \in M^n$ and $M^n \geq_s M(t,S)$. ■

Intuitively, what the theorem says is that the maximizing values of $\mathbf{x}$ are increasing in the parameters and choice set, but that they increase more when there are more degrees of freedom, which is the unifying conceptual notion of a Le Chatelier Principle. Clearly, Theorem 1 (as stated) follows from Theorem 2. In both cases, this is a verification of the idea that these results (i.e. various comparative statics in the case of [**5**]; Le Chatelier in the present case) do not depend on the standard classical framework, but rather that the underlying intuition continues to hold true in a more flexible and widely applicable setting.

One previous paper, following [**2**], discusses the idea of a Le Chatelier-type principle over lattices. They state an elementary version (Theorem 3 in [**4**]) as an immediate corollary of Theorem 1 above. Their statement requires $n = 2$ & $S = S'$, and is point-valued (it compares the largest elements of the maximizing sets M_i). The general version (Theorem 2 above) does not follow from Theorem 1, and is thereby able to dispense with these extra assumptions.[4]

It is difficult to directly compare this result to the classical Le Chatelier Principle. For instance, if the Hessian matrix of the production function f (as in the introduction) is negative definite with non-negative off-diagonal elements, then f is quasisupermodular and we can apply Theorem 2. However, there are specific examples where a Le Chatelier-type conclusion obtains despite f not being quasisupermodular; see [**7**] for further discussion.

The main innovation of the lattice approach is that it applies to non-differentiable objective functions and discrete choice sets. An additional advantage, although the techniques may at first be unfamiliar, is that the lattice-theoretic proofs are generally more elementary and far more intuitive than the standard ones. For instance, Samuelson's original proof invokes

[4] For instance, because they consider only point-valued comparisons, they need an order continuity restriction to ensure that the supremum of the maximizing set is itself a maximizer. And with $n = 2$ only, they have nothing to say regarding the 'in-between' maximizing sets.

a "well-known theorem on determinants (Jacobi)", but it is apparently so well-known that finding a reference to it proved difficult! (see [**1**])

References

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