E C O N O M I C S   B U L L E T I N

The spatial Solow model

Carmen Camacho
IRES

Benteng Zou
IRES

Abstract

In this paper, we solve a Solow model in continuous time and space. We prove the existence of a solution to the problem and its convergence to a stationary solution. The simulations of various scenario in the last section of the paper illustrates the convergence issue.

We appreciate professor Raouf Boucekkine for helpful comments and discussion on earlier drafts of this paper. Perceptible suggestions of an associate editor have greatly improved this work.
Citation: Camacho, Carmen and Benteng Zou, (2004) "The spatial Solow model." *Economics Bulletin,* Vol. 18, No. 2 pp. 1−11
Submitted: January 16, 2004. **Accepted:** January 30, 2004.
URL: http://www.economicsbulletin.com/2004/volume18/EB−04R10001A.pdf

The Spatial Solow Model*

Carmen Camacho
IRES

Benteng Zou
IRES

Abstract

In this paper, we solve a Solow model in continuous time and space. We prove the existence of a solution to the problem and its convergence to a stationary solution. The simulations of various scenarios in the last section of the paper illustrate the convergence issue.

Keywords: Solow model, Growth, Geographic economics.

JEL classification: R1; O0.

1 Introduction

The inclusion of the geographical space in economic analysis has regained relevance in the recent years. The emergence of a new economic geography discipline is indeed one of the major events in the economic literature of the last decade (see Ten Raa(1986), Krugman (1991) and (1993), Fujita, Krugman and Venables (1999), and Fujita and Thisse (2002)). Departing from the early regional science contributions which are typically based on simple flow equations (eg. Beckman (1952), Puu(1982)), the new economic geography models use a general equilibrium framework with a refined specification of local and global market structures, and some precise assumptions on the mobility of production factors.

Two main characteristics of the new economic geography contributions are: (i) the discrete space structure, and (ii) the absence of capital accumulation. Since capital accumulation is not allowed, the new economic geography models are losing a relevant determinant of migrations, and more importantly, an engine of growth. It seems however clear that

*We acknowledge the financial support of the Belgian French speaking community (Grant ARC 99/04-235 "Growth and incentive design") and of the Belgian Federal Government (Grant PAI P5/10, "Equilibrium theory and optimization for public policy and industry regulation"). Address for correspondance: Benteng Zou, Université catholique de Louvain, place Montesquieu 3, B-1348 Louvain-la-Neuve, Belgium. zou@ires.ucl.ac.be

many economic geography problems (eg. uneven regional development) have a preeminent growth component, and *vice versa*. Thus, there is an urgent need to unify in some way the new economic geography and the growth theory, or at least to develop some junction models.

This paper constitutes a first step following exactly this line of research. We study the Solow model with one dimensional space, in which we relocated the geography location (note as $x \in \mathbb{R}$), such that, the rich regions are not far away from the central of the economy, that is, $x = 0$. In another word to say, a region, which is far away from $x = 0$, is a very poor region, where there is no capital flow. Furthermore, as mentioned by Ten Raa(1986), page 528–530, "to avoid simple but unrealistic boundary conditions", we consider that space is continuous and infinite, and capital accumulation are space dependent. From macro-economic general equilibrium point of view one dimension for space is reasonable. Different from Ten Raa(1986) and Puu(1982), where they consider a fluid dynamics, hence obtained wave equations of income, in our case capital flowing follows heat equations because of decreasing return to capital: the regions with the largest capital endowments are those which display the lowest return, hence allowing capital to flow to less capital endowments regions.

In line with Mossay (2003), we shall allow for both divergence and convergence forces. The convergence force is the well known neoclassical mechanism according to which poor regions attract capital because of decreasing returns to this factor. Divergence mechanisms are linked to space heterogeneity, given by region specific technology and/or saving rate. If a region produces using a more advanced technology, it attracts capital from less advanced regions, despite decreasing returns to scale. The same result holds for a region that saves, and therefore invests, at a larger rate.

Neoclassical economic theory predicts that regions will converge in the long run under perfect competition. However, this is not so. An argument that has been put forward is that technological transfers between regions is far from perfect. The lack of transferee expertise and poor training in the technology importing region, together with Government barriers may impede an effective technological transfer (see Niosi, Hanel and Fiset (1995)). Boucekkine, Martinez and Saglam (2003) point out the role of capital goods technological embodiment in technology adoption decisions. A developing country may not adopt the most sophisticated technique since it implies replacing existing capital and lose their technology-specific skills. The spatial Solow model allows to study the link between technology transfers and development. Indeed, it is flexible enough to study the existence of technological poles with partial transfer to neighboring regions, as well as more complicated patterns of knowledge diffusion across space and time.

The paper is organized as follows. Section 2 presents the spatial Solow problem. Section 3 is devoted to prove the existence of solutions, providing explicit solutions for the Ak case, and their convergence to a steady state. Section 4 presents different scenarios that bring out the relevance of initial conditions and of space dependent technology and savings. Section 5 concludes.

2 The model

Assume that in one economy market, there is only one final good, which can be signed to consume or invest. In contrast to the standard Solow model, the law of motion of capital does not rely entirely on the saving capacity of the economy under consideration: the net flows of capital to a given location or space interval should also be accounted for. Suppose that households locate along the real line. At time $t > 0$, the technology at work in location $x \in \mathbb{R}$ produces output $y(x,t) = A(x,t)f(k(x,t))$, where $k(x,t)$ is capital storing at location x at time $t > 0$, $A(x,t)(\geq 0)$ stands for total factor productivity at (x,t), and $f(\cdot)$ is the production function, which satisfies the following assumptions:

(A1) $f(\cdot)$ is non-negative, increasing and concave;

(A2) $f(\cdot)$ verifies the Inada conditions, that is,

$$f(0) = 0, \quad \lim_{k \to 0} f'(k) = +\infty, \quad \lim_{k \to +\infty} f'(k) = 0.$$

Moreover we assume that the production function is the same whatever is the location. Hence the budget constraint of household $x \in \mathbb{R}$ is

$$\frac{\partial k(x,t)}{\partial t} = s(x,t)A(x,t)f\left(k(x,t)\right) - \delta k(x,t) + \tau(x,t), \tag{1}$$

where δ is the depreciation rate of capital[1] , $s(x,t)(\geq 0)$ is saving rate of household x at time $t(> 0)$, which could be homogenous or heterogenous with respect to time and space, and $\tau(x,t)$ is the household's net trade balance of household x at time t, and also the capital account balance, by the assumption of homogenous depreciation rate of capital, without adjustment cost and no arbitrage opportunities. Since the economy is closed, we have

$$\int_{\mathbb{R}} \left(\frac{\partial k(x,t)}{\partial t} - s(x,t)A(x,t)f\left(k(x,t)\right) + \delta k(x,t) - \tau(x,t) \right) dx = 0.$$

And as mentioned by Brito P.(2003): Let the capital and goods flow between regions. If regions are considered as closed economies, then for any given closed region $[a,b] \subset \mathbb{R}$:

$$\int_{[a,b]} \left(\frac{\partial k(x,t)}{\partial t} - s(x,t)A(x,t)f\left(k(x,t)\right) + \delta k(x,t) - \tau(x,t) \right) dx = 0. \tag{1'}$$

Recall that capital movements tend to eliminate geographical differences. Since without inter-regional arbitrage opportunities, the capital flows from regions with lower marginal productivity of capital to the higher ones, that is equivalent to saying that capital flows from regions with abundant capital toward the ones with relatively less capital. Suppose

[1] Depreciation rate of capital could be heterogenous in time t and space x, but without special mention, we assume that the depreciation rate of capital is homogenous in time t, space x and capital k.

furthermore that there is no institution barriers to capital flow (or do not consider the adjustment speed)[2]. Applying the fundamental theorem of calculus to region $X = [a, b]$, we have

$$\int_X \tau(x,t)dx = -\int_{\partial X} \frac{\partial k}{\partial x} dx = -\left(\frac{\partial k(b,t)}{\partial x} - \frac{\partial k(a,t)}{\partial x} \right) = -\int_X \frac{\partial^2 k}{\partial x^2} dx.$$

Substitute the above equation into equation (1'), we have $\forall X \subset \mathbb{R}$, $\forall t$

$$\int_X \left(\frac{\partial k(x,t)}{\partial t} - \frac{\partial^2 k}{\partial x^2} - (s(x,t)A(x,t)f(k(x,t)) - \delta k(x,t)) \right) dx = 0.$$

By Hahn-Banach Theorem, therefore the budget constraint can be written as:

$$\frac{\partial k(x,t)}{\partial t} - \frac{\partial^2 k(x,t)}{\partial x^2} = s(x,t)A(x,t)f\left(k(x,t)\right) - \delta k(x,t), \ \forall (x,t). \tag{2}$$

The initial distribution of capital, $k(x, 0)$, is assumed to be known and C^0. Moreover, we assume that, if the location is far away from the origin, there is no capital flow, that is

$$\lim_{x \to \pm\infty} \frac{\partial k}{\partial x} = 0.$$

We can write the problem as:

$$P \begin{cases} \frac{\partial k}{\partial t}(x,t) - \frac{\partial^2 k}{\partial x^2}(x,t) = s(x,t)A(x,t)f\left(k(x,t)\right) - \delta k(x,t), \\ k(x,0) = k_0(x) > 0, \quad x \in \mathbb{R}, \\ \lim\limits_{x \to \infty} \frac{\partial k}{\partial x} = \lim\limits_{x \to -\infty} \frac{\partial k}{\partial x} = 0. \end{cases}$$

in $\mathbb{R} \times (0, \infty)$. k_0 is defined in $L^\infty_+(\mathbb{R})$, where $L^\infty_+(\mathbb{R}) = \{y \in L^\infty(\mathbb{R}) | y(x) \geq 0$ for almost every $x \in \mathbb{R}\}$.

3 Mathematical results

3.1 Existence

The literature on Partial Differential Equations provides us with an existence theorem for problem P.

[2]We could assume that there is institution barrier (or adjustment speed) to capital flow basing on location and time(see Ten Raa(1986) and Puu(1982)). But if we assume they are independent of capital k and consumption c, then we can obtain a linear equation with coefficients in front of the Laplacean operator. After some affine transformation, we will get the similar result as below. But if the barriers (or adjustment speed) are functions of k and/or c, we are facing nonlinear results, which we do not consider in this work.

Theorem 1 *If s, A are continuous and if f verifies $(A1)$ and $(A2)$, there exists a unique global continuous nonnegative solution to problem P.*

Proof: If $(x,t) \in \mathbb{R} \times (0,T)$, where $0 < T < \infty$, it is well known there exists a unique bounded solution to problem P (see Ladyzhenskaja, Solonnikov and Ural'ceva (1968)). Following Hofbauer and Simon (2001), we obtain the global existence and uniqueness of the solution to P in $\mathbb{R} \times (0,\infty)$.

Now we use Inada conditions to prove that the solution is nonnegative. Define $k(x,t) = e^{-\delta t} v(x,t)$, then $v(x,t)$ satisfies the following problem:

$$(M) \begin{cases} \dfrac{\partial v}{\partial t}(x,t) - \dfrac{\partial^2 v}{\partial x^2}(x,t) = s(x,t)A(x,t)f\left(e^{-\delta t}v(x,t)\right), \\ v(x,0) = k_0(x) > 0, \quad x \in \mathbb{R}, \end{cases}$$

By the first part of this proof, we know that there exists a unique solution $v(x,t)$ to problem (M). Using a comparison theorem, we assert that the above solution v is nonnegative, provided that the (A1) and (A2) hold. So does $k(x,t)$. □

The following theorem gives an explicit solution for the Ak model.

Theorem 2 *Suppose that the production function, $f(k(x,t)) = k(x,t)$. If A and s are constants, then the solution to problem P is given by*

$$k(x,t) = e^{(sA-\delta)t} \int_{\mathbb{R}} \Gamma_0(x-y,t)k_0(y)dy, \tag{3}$$

where

$$\Gamma_0(x,t) = \begin{cases} \dfrac{1}{(4\pi t)^{\frac{1}{2}}} e^{-\frac{x^2}{4t}}, \ t > 0, \\ 0, \ t < 0. \end{cases}$$

If $A = A(x,t)$, $s = s(x,t)$, the solution to problem (P) is given by

$$k(x,t) = \int_{\mathbb{R}} \Gamma(x-\xi,t)k_0(\xi)d\xi,$$

where Γ is defined as

$$\Gamma(x-\xi,t-\tau) = \Gamma_0(x-\xi,t-\tau) + \int_\tau^t \int_{\mathbb{R}} \Gamma_0(x-\eta,t-\sigma)\Phi(\eta-\xi,\sigma-\tau)d\eta d\sigma,$$

and Φ satisfies

$$\Phi(\eta-\xi,\sigma-\tau) = \sum_{\nu=1}^{\infty} (\mathcal{L}\Gamma_0)_\nu(\eta-\xi,\sigma-\tau).$$

The operator $\mathcal{L}$ is recursively defined, and it is given by

$$(\mathcal{L}\Gamma_0)_1 = \mathcal{L}\Gamma_0 = (s(x,t)A(x,t)-\delta)\Gamma_0(x,t),$$

$$(\mathcal{L}\Gamma_0)_{\nu+1}(\eta-\xi,\sigma-\tau) = \int_\tau^\sigma \int_{\mathbb{R}} ((\mathcal{L}\Gamma_0)(\eta-y,\sigma-s))(\mathcal{L}\Gamma_0)_\nu(y-\xi,s-\tau)dyds.$$

Proof: See Ladyzenskaja, Solonnikov and Ural'ceva (1968) and Friedman (1983).

Remark As proved by Ladyzenskaja, Solonnikov and Ural'ceva (1968), if $k_0(x)$ does not increase too rapidly for $|x| \to +\infty$ (for example, not faster than e^{x^2}), then the integral in (3) converges. Hence we can get the same order of growth rate as in the standard Solow model.

Theorem 2 allows to clearly study the long run behavior of $k(x,t)$ when $A(x,t) = A$ and $s(x,t) = s$, where s and A are constants. For if $sA \leq \delta$, then from (3) one can check that

$$\lim_{t \to \infty} k(x,t) = 0.$$

If an economy does not save at least to compensate for depreciation, then it will decay until no capital is left.

If, on the contrary, $sA > \delta$, we obtain that:

$$\lim_{t \to \infty} k(x,t) = \infty.$$

This implies that, as in the 1-dimensional case, the spatial Ak model does not have a steady state.

3.2 Steady State and Convergence

We define a steady state solution to (1) by the standard conditions $\frac{\partial k(x,t)}{\partial t} = 0$, $A(x,t) = A(x)$ and $s(x,t) = s(x)$:

$$P_S \begin{cases} \dfrac{\partial^2 k(x)}{\partial x^2} + s(x)A(x)f\left(k(x)\right) - \delta k(x) = 0, \\ \lim\limits_{x \to \infty} \dfrac{\partial k}{\partial x} = \lim\limits_{x \to -\infty} \dfrac{\partial k}{\partial x} = 0, \end{cases}$$

We can reduce the problem into an ordinary differential equation 2-dimensional system:

$$\begin{aligned} \frac{\partial k(x)}{\partial x} &= w(x), \\ \frac{\partial w(x)}{\partial x} &= \frac{\partial^2 k(x)}{\partial x^2} = -s(x)A(x)f(k(x)) + \delta k(x). \end{aligned}$$

then a solution to this system is given by,

$$\begin{aligned} k(x) &= \int_{-\infty}^{x} w(z)dz, \\ w(x) &= \int_{-\infty}^{x} (-s(x)A(x)f(k(z)) + \delta k(z)dz. \end{aligned}$$

Any solution $(k(x), w(x))_{x\in\mathbb{R}}$ must also verify that,

$$\lim_{x\to\infty} \frac{\partial k}{\partial x}(x) = \lim_{x\to-\infty} \frac{\partial k}{\partial x}(x) = 0, \tag{4}$$

this implies that, $\lim_{x\to\pm\infty} w(x) = 0$,. If k verifies that

$$-s(x)A(x)f(k(x)) + \delta k(x) = 0, \quad \forall x \tag{5}$$

then, the boundary conditions are verified and k is a particular solution of P_S. Unfortunately, the stationary solutions are not unique.

Theorem 3 *If the production function f verifies $(A1)$ and $(A2)$, then the nonnegative solution $k(t, x; k_0)$ to problem P converges to a stationary solution as $t \to \infty$.*

Proof: The proof requires some minor changes to the proof provided in Bandle, Pozio and Tesei (1987) for a similar problem.

4 Dynamic simulations

We illustrate in this section the behavior of solutions to P under different scenarios. In particular, we simulate the spatial Solow model with a Cobb-Douglas production function. We consider various cases depending on initial conditions and on whether A and s are constants or space-dependent.

Example 1 (Homogeneous in initial capital and technology) . We shall consider in this first example that, initially, all households are equally endowed with one unit of physical capital. There are no geographical differences, so that they save at a rate $s(x, t) = 0.2$ and the technological coefficient $A(x, t) = 10$. The capital share in the production production, α, equals 1/3 and physical capital depreciation $\delta = 0.05$.

Simulated capital reproduces a neoclassical growth path (see figure 1). Marginal productivity of capital is the same along the real line, so that investors are indifferent among all locations. Since there is no source of heterogeneity, all points produce and grow at the same rates.

Example 2 (Homogeneous in technology, but heterogeneous in initial capital) . We introduce heterogeneity at the initial endowment of capital to study whether differences across regions may persist in the long run. We assume that $k(x, 0) = e^{-x^2}$. The rest of parameters take the same values as in example 1. Figure 2 shows that after some iterations, initial differences are smoothed out and that, in the long run, all points in space will be equally rich.

Example 3 (Heterogeneous in technology). In this example, $A(x, t) = e^{-x^2}$, that is, the central region uses a more advanced technology. Since there is no technology transfer, as

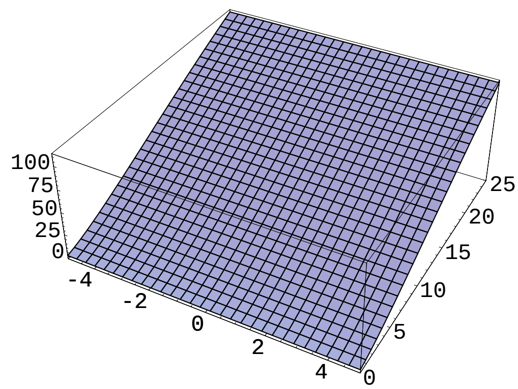

Figure 1: Simulation results when space is homogenous.

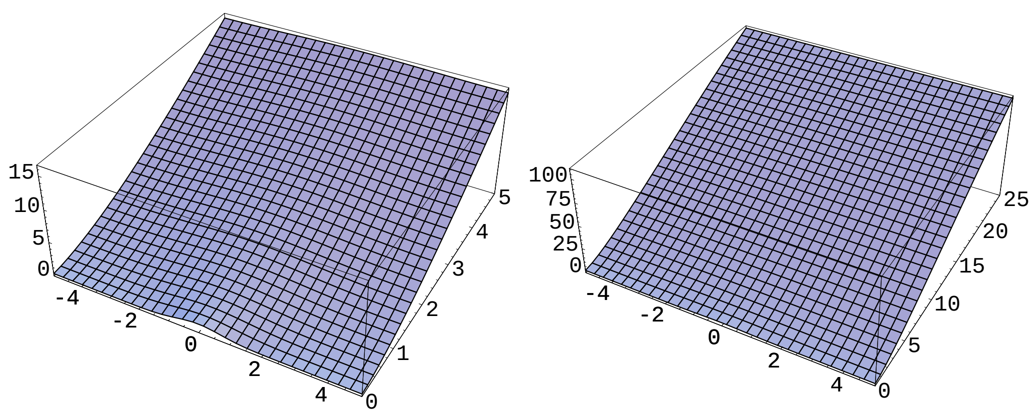

Figure 2: Simulation results for different time horizons. On the left $t \in [0, 5]$ and on the right $t \in [0, 25]$.

a result, they remain leaders forever. The first graph in figure 3 shows the growth path when the initial condition is spatially homogenous, $k(x, 0) = 1$. In the second graph, $k(x, 0) = e^{-x^2}$, which adds a further source of heterogeneity. Results show that whichever the initial condition, any difference in technology which is not subject to modification through time (i.e. if there are no technological spill-overs from the center to the periphery), leads to a non homogenous steady state.

5 Conclusion

In this paper, we solve a Solow model in continuous time and space. We prove at the same time, the existence of a solution to the problem and the convergence to a stationary solution. Results coincide with the non-spatial neoclassical intuition. We obtain that

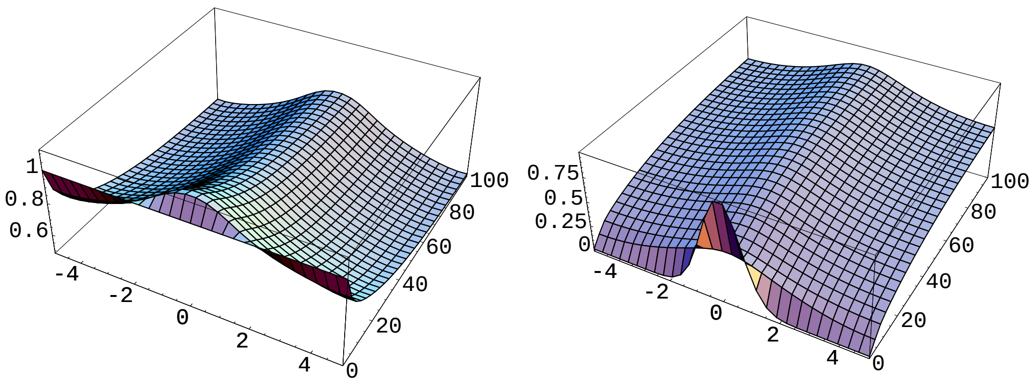

Figure 3: Left: $k(x,0) = 1$, $\forall x$. Right: $k(x,0) = e^{-x^2}$, $\forall x$

in the Ak case, the model does not have a steady state; furthermore, with a standard neoclassical production function, this steady state exists and we prove convergence. If space is homogenous, i.e. if all locations produce using the same technology and they save at the same rate, then at the steady state, all locations have the same level of physical capital. This is true whatever the initial condition. However, if spatial heterogeneity is introduced at the level of the technology or savings rate, regional differences persist.

Further research in this field should lead to the generalization of our results. A natural continuation is the extension to the Ramsey model, in which we are already working.

References

Beckman, M. (1952), *A continuous model of transportation*, Econometrica 20, 643-660.

Bandle C., Pozio M. A. and Tesei A. (1987), *The asymptotic behavior of the solutions of degenerate parabolic equations*, Trans. Amer. Math. Soc., Vol 303, (2), 487–501.

Boucekkine R., Martinez B. and Saglam C. (2003), *The Development Problem under Embodiment*, A discusion, WP-AD 2003-08.

Friedman A. (1975), *Stochastic Differential Equations and Applications* , Volume I, Academic Press, New York, San Francisco, London.

Friedman A. (1983), *Partial Differential equations of parabolic type*, Malabar Fla. Kriger.

Fujita, M. and JF. Thisse (2002), *Economics of agglomeration.* Cambridge University Press.

Fujita, M., P. Krugman and A. Venables (1999), *The spatial economy. Cities, regions and international trade*. MIT Press.

Hofbauer J. and P.L. Simon (2001), *An existence theorem for parabolic equations on $\mathbb{R}^N$ with discontinuous nonlinearity*, E. J. Qualitative Theory of Diff. Equ. No 8, 1-9.

Krugman, P. (1991), *Increasing returns and economic geography*, Journal of Political Economy 99, 483-499.

Krugman, P. (1993), *On the number and location of cities*, European Economic Review 37, 293-298.

Ladyzhenskaja O. A., Solonnikov V. A. and Ural'ceva N. N.(1968), *Linear and quasilinear equations of parabolic type*, Amer. Math. Soc. Providence, RI.

De Mottoni P., Schiaffino A. and Tesei A.(1984), *Attractivity properties of nonnegative solutions for a class of nonlinear degenerate parabolic problems*, Ann. Mat. Pura. Appl., Vol 136, 35–48.

Mossay, P. (2003), *Increasing returns and heterogeneity in a spatial economy*, Regional Science and Urban Economics 33, 419-444.

Niosi J., Hanel P. and Fiset L. (1995), *Technology transfer to developing countries through engineering firms: The Canadian Experience*, *World Development 23*, 1815-1824.

Pozio M. A. and Tesei A. (1987), *Support properties of solutions for a class of degenerate parabolic problems*, Commun. in Part. Diff.equations, 12(1), 47–75.

Puu T. (1982), *Outline of a trade cycle model in continuous space and time*, Geographical Analysis 14, 1–9.

Ten Raa T. (1986), *The initial value problem for the trade cycle in Euclidian Space*, Regional Science and Urban Economics, 16(4), 527–546.