E C O N O M I C S   B U L L E T I N

A salary system for the assignment problem

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Abstract

We propose a modification to the concept of the potential of agame à la Hart−Mas Colell to determine a salary system for theassignment problem. We obtain explicit formulas for the potentialof the assignment problem and for it's corresponding salarysystem. Also, we establish some properties of this salary systemand we give an interpretation in terms of the Shapley value.

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1. Introduction

Let us consider a finite set of workers, a set of jobs with equal cardinality and values a_{ij} which measure the efficiency of the assigning worker j to the job i. In the assignment problem each worker is to be assigned to a different job in such a way that the sum of corresponding efficiencies is maximum. In other words,

$$\begin{array}{lll} \max & \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}\ x_{ij} & \\ \text{s.t.} & & \\ & \sum_{i=1}^{n} x_{ij} = 1 & j = 1, \dots, n \\ & \sum_{j=1}^{n} x_{ij} = 1 & i = 1, \dots, n \\ & x_{ij} = 0 \text{ or } 1 & \end{array}$$

where $x_{ij} = 1$ if the worker j is assigned to job i and $x_{ij} = 0$ otherwise.

In this work, we propose a fair salary system for the previous problem. We have three main assumptions: First, we suppose that the maximum efficiency of the system (the optimal value of the objective function) is equal to the amount to be shared in salaries among the workers. Alternatively, the total amount to be shared in salaries could be given exogenously and most of the results remain valid. Second, we only consider problems where the number of workers is equal to the number of jobs. Lastly, we suppose that each worker has the same right to be assigned to any of the jobs, so every worker must agree with the final assignment.

Notice that as a general rule it is not fair to pay according to the efficiency that each worker generates in the optimal solution since, in that case, some workers are usually sacrificed for the sake of global efficiency. For example in the optimal solution of the problem with the following matrix,

$$\begin{array}{cc} & \text{Workers} \\ \text{Jobs} & \begin{bmatrix} 8 & 0 \\ 15 & 10 \end{bmatrix} \end{array} \tag{1.1}$$

the first worker is assigned to job 1 and the second to job 2, however it is not fair to pay 8 units to the first worker and 10 units to the second since the first worker is more productive than the second worker, job by job. So, in order to define a salary for the workers it will be necessary to consider their relative abilities to do the different jobs.

Shapley and Shubik[4] suppose an optimal assignment of sellers to buyers in two-sided markets and proposed a solution based on linear programming to share the maximum total monetary payoff among sellers and buyers. In this work, we suppose an optimal assignment of a set of workers to a set of jobs and propose a solution to share the efficiency of the system only among the set of workers.

We use the concept of potential of a cooperative game to establish this salary system. Hart and Mas-Colell ([2] and [3]) established a new form to generate the value of a game. They start with the definition of a set of games in which a real function is defined, called the potential of the game. This function must be defined in such a way that the following two conditions are satisfied: i) the gain or participation of each player is equal to the difference in potential between the game that is considered and the one that results when he abandons the game, and, ii) the sum of gains or participations add to what all the players get in that game.

Surprisingly, these two conditions determine the gain or participation of the players and also these participations coincide with it's corresponding Shapley value.

For a brief revision of the concepts of cooperative games that are mentioned here, such as the Shapley value, see Appendix A (Finite games and their values) in Aumman and Shapley [1].

Now, we present the previous ideas formally. Let (N, v) be a cooperative game and let us denote by $G = \{(S, v_S) : S \subseteq N \text{ and } v_S = v|_S\}$ the family of subgames of (N, v). The potential of a game is defined as a function

$$P : G \to \mathbb{R}$$

such that

$$\sum_{j \in S} \left[P(S, v_S) - P(S \backslash \{j\}, v_{S \backslash \{j\}}) \right] = v(S) \tag{1.2}$$

with the condition that $P(\emptyset, v_\emptyset) = 0$. Furthermore, it is assumed that the loss of potential

$$P(S, v_S) - P(S \backslash \{j\}, v_{S \backslash \{j\}}) \tag{1.3}$$

when the player j abandons the game, is equal to the amount that corresponds to this player in the game (S, v_S).

Notice that (1.2) could be rewritten as

$$s \cdot P(S, v_S) = v(S) + \sum_{j \in S} P(S \backslash \{j\}, v_{S \backslash \{j\}}) \tag{1.4}$$

where $s = |S|$. This expression determines P and allows it's calculation in a recursive form. Moreover, it turns out that the difference (1.3) is equal to the Shapley value of player j in the game (S, v_S).

Furthermore, Hart and Mas-Colell give an explicit formula for the potential of a game:

$$P(N, v) = \sum_{S \subseteq N} \frac{(s-1)!\,(n-s)!}{n!} v(S). \tag{1.5}$$

In section 2 we show that these ideas could be applied, with appropriate modification, to determine a salary system for the assignment problem. In section 3 we present some properties of the salary system proposed in this work. First, we see this salary system as a particular case of a more general framework and then we consider the case where the workers do not have any bargaining power.

2. Potential for the assignment problem

We assume that each worker is assigned to one and only one job. Let us also suppose that we have defined a potential on the set of this problems, i.e., a real function on this set.

We need to define the loss of potential when one worker abandons a problem, so we need also to eliminate a job. A priori it could be any job, so we propose the average loss of potential instead. Let us formalize this idea.

Denote by $\mathcal{M}$ the set of jobs, by $\mathcal{N}$ the set of workers and by $\mathcal{A}$ the matrix which entry a_{ij} represents the efficiency of the worker j doing the job i. Notice that rows of $\mathcal{A}$ are indexed by the elements of $\mathcal{M}$ and its columns by the elements in $\mathcal{N}$. We suppose that $\mathcal{M}$ and $\mathcal{N}$ have the same cardinality and we will denote by $\mathcal{A}|(M,N)$ the matrix $\mathcal{A}$ restricted to the rows in M and columns in N. The problem,

$$\begin{array}{lll} \max & \sum_{i\in M}\sum_{j\in N} a_{ij}\ x_{ij} & \\ \text{s.t.} & & \\ & \sum_{i\in M} x_{ij}=1 & j\in N \\ & \sum_{j\in N} x_{ij}=1 & i\in M \\ & x_{ij}=0 \text{ or } 1 & \end{array}$$

defined by the parameters $M\subseteq\mathcal{M}$, $N\subseteq\mathcal{N}$ and $A=\mathcal{A}|(M,N)$ will be denoted by (M,N,A). So that this does not produce confusion, we will use only (M,N) instead of (M,N,A) and sometimes we will represent it just by it's matrix A.

The set of problems that we will consider is as follows,

$$\mathcal{P}=\{(M,N,A): M\subseteq\mathcal{M}, N\subseteq\mathcal{N} \text{ such that } |M|=|N| \text{ and } A=\mathcal{A}|(M,N)\},$$

It is well-known that these kinds of problems always have at least one optimal solution, therefore the following function is well defined,

$$L_P:\mathcal{P}\to\mathbb{R}$$

where $L_P(M,N)$ is the optimal value of the objective function for the problem (M,N). Let us denote

$$\overline{Pot(N\backslash\{j\})}=\sum_{i\in M}\frac{Pot(M\backslash\{i\},N\backslash\{j\})}{n}.$$

Definition 1. *We define the potential of the assignment problem as a function,*

$$Pot:\mathcal{P}\to\mathbb{R}$$

such that $Pot(\emptyset,\emptyset)=0$ *and*

$$\sum_{j\in N}[Pot(M,N)-\overline{Pot(N\backslash\{j\})}]=L_P(M,N). \tag{2.1}$$

As in Hart and Mas-Colell [2] and [3], we will suppose that

$$Pot(M,N)-\overline{Pot(N\backslash\{j\})}$$

is the salary which corresponds to the worker j in the problem (M,N). This number is the average of the loss of potential when the worker j abandons the system. It is easy to see that (2.1) is equivalent to the following recursive expression,

$$Pot(M,N)=\frac{n\ L_P(M,N)+\sum_{j\in N}\sum_{i\in M}Pot(M\backslash\{i\},N\backslash\{j\})}{n^2} \tag{2.2}$$

which determines the potential of the assignment problem. We will denote indistinctly for $\varphi(M,N)$ or for $\varphi(A)$ the vector of salaries that corresponds to the problem (M,N,A), i.e., the vector with entry j is given by

$$\varphi_j(M,N)=Pot(M,N)-\overline{Pot(N\backslash\{j\})} \tag{2.3}$$

and we will call it a salary system for the assignment problem (M,N).

Example. In this example, we calculate the salary system for the problem with the following matrix,

$$\begin{bmatrix} 9 & 10 & 2 \\ 7 & 8 & 6 \\ 10 & 1 & 3 \end{bmatrix} \tag{2.4}$$

For simplicity of the exposition we will represent the problem (M, N, A) in the following form:

$$A^{L_P(M,N)}_{Pot(M,N)}$$

In order to calculate $Pot(M, N)$ using (2.2) we need to generate all sub problems as the result of removing one worker and one job. We repeat this process in each case until there are no workers. Then we calculate L_P for each of the obtained problems and their potential beginning with the lower cardinality. Notice that for matrices with one element, say $A = [a]$, $L_P(M, N) = Pot(M, N) = a$. While if the matrix is 2×2, then $Pot(M, N)$ is equal to the sum of the entries of the matrix, plus $2L_P(M, N)$, all divided by 4. Thus, the problems that are derived from (2.4) when one worker is missing are:

When worker 1 is missing:

$$\begin{bmatrix} 10 & 2 \\ 8 & 6 \end{bmatrix}^{16}_{14.5} \quad \begin{bmatrix} 10 & 2 \\ 1 & 3 \end{bmatrix}^{13}_{10.5} \quad \begin{bmatrix} 8 & 6 \\ 1 & 3 \end{bmatrix}^{11}_{10} \tag{2.5}$$

When worker 2 is missing:

$$\begin{bmatrix} 9 & 2 \\ 7 & 6 \end{bmatrix}^{15}_{13.5} \quad \begin{bmatrix} 9 & 2 \\ 10 & 3 \end{bmatrix}^{12}_{12} \quad \begin{bmatrix} 7 & 6 \\ 10 & 3 \end{bmatrix}^{16}_{14.5} \tag{2.6}$$

When worker 3 is missing:

$$\begin{bmatrix} 9 & 10 \\ 7 & 8 \end{bmatrix}^{17}_{17} \quad \begin{bmatrix} 9 & 10 \\ 10 & 1 \end{bmatrix}^{20}_{17.5} \quad \begin{bmatrix} 7 & 8 \\ 10 & 1 \end{bmatrix}^{18}_{15.5} \tag{2.7}$$

The potential of the problem with matrix (2.4) is the result of adding 3 times the optimal value of the objective function $(3 \cdot 26)$ with the potential of each of the nine previous problems and divide the result by 9. Thus,

$$\begin{bmatrix} 9 & 10 & 2 \\ 7 & 8 & 6 \\ 10 & 1 & 3 \end{bmatrix}^{26}_{22.555}$$

Now, we can calculate the salaries for the problem with matrix (2.4) using (2.3).

$$\begin{aligned} \overline{Pot(S\backslash\{1\})} &= (14.5 + 10.5 + 10)/3 = 11.667 \\ \overline{Pot(S\backslash\{2\})} &= (13.5 + 12 + 14.5)/3 = 13.333 \\ \overline{Pot(S\backslash\{3\})} &= (17 + 17.5 + 15.5)/3 = 16.667 \end{aligned}$$

giving us

$$\varphi(A) = \begin{bmatrix} 22.555 - 11.667 \\ 22.555 - 13.333 \\ 22.555 - 16.667 \end{bmatrix} = \begin{bmatrix} 10.888 \\ 9.222 \\ 5.888 \end{bmatrix}.$$

As another example, the wages for the problem given in the introduction (see 1.1) are 12.25 for worker 1 and 5.75 for worker 2.

3. Properties of the potential of the assignment problem

The next theorem establishes an explicit expression for the potential of the assignment problem. We will denote by

$$\widehat{L_M(S)} = \sum_{\{R\subseteq M:|R|=|S|\}} L_P(R,S). \tag{3.1}$$

Theorem 1. *For each $(M,N)\in\mathcal{P}$ we have that*

$$Pot(M,N) = \sum_{S\subseteq N} \frac{1}{s}\left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(S)}. \tag{3.2}$$

Theorem 2. *The salary for worker j in the assignment problem (M,N) is given by*

$$\varphi_j(M,N) = \sum_{\{S\subseteq N:j\in S\}} \frac{1}{s}\left[\frac{s!\,(n-s)!}{n!}\right]^2 \left[\widehat{L_M(S)} - \widehat{L_M(N\backslash S)}\right].$$

The previous theorem provides us an alternative way to calculate the salary system for the assignment problem. Using (3.1) and (2.5) - (2.7) we have that

$$\begin{aligned}
&\widehat{L_M(\{1,2,3\})} = 26\\
&\widehat{L_M(\{1,2\})} = 17+20+18 = 55\\
&\widehat{L_M(\{1,3\})} = 15+12+16 = 43\\
&\widehat{L_M(\{2,3\})} = 16+13+11 = 40\\
&\widehat{L_M(\{1\})} = 9+7+10 = 26\\
&\widehat{L_M(\{2\})} = 10+8+1 = 19\\
&\widehat{L_M(\{3\})} = 2+6+3 = 11
\end{aligned}$$

and therefore,

$$\varphi_1(A) = \frac{1}{9}(26-40) + \frac{1}{18}(55-11) + \frac{1}{18}(43-19) + \frac{1}{3}26 = 10.888.$$

The next result establishes a relation between the salary system given by (2.3) and the Shapley value. Let us denote by

$$\Theta = \{\theta : N \to M | \theta \text{ bijective}\}$$

the set of the $n!$ possible ways to assign the jobs to the workers. Let us select $\theta\in\Theta$ randomly (each case with probability $\frac{1}{n!}$), in such a way that worker j gets the job $\theta(j)$ and suppose the workers have the possibility of exchanging their assigned jobs. The process of bargaining generates a cooperative game in a natural way: when the coalition of workers S is formed they can generate an efficiency $v^\theta(S) = L(\theta(S),S)$. Thus, when they distribute the efficiency according to the value ϕ, they get an average $\frac{1}{n!}\sum_{\theta\in\Theta}\phi(v^\theta)$. The next theorem establishes that the salary system coincides with the previous process when they use the Shapley value. We will denote by $Sh(N,v)$ the Shapley value of the game (N,v).

Theorem 3. *For $A \in \mathcal{P}$ we have that*

$$\varphi(A) = \frac{1}{n!} \sum_{\theta \in \Theta} Sh(N, v^{\theta}) = Sh(N, \frac{1}{n!} \sum_{\theta \in \Theta} v^{\theta}).$$

The following theorem establishes some properties of the salary system. The first property is that the solution does not change when re-naming the jobs. The second one establishes the symmetry with regard to the workers, in particular for two workers with corresponding equal columns, the same salary is obtained. Also we have a change in the scale of efficiency that gives us an equivalent change in the salary. Lastly, we establish a weak monotony property with respect to the efficiency of the workers and a logical solution for permutation matrices.

Let P be a permutation matrix, i.e., a $n \times n$ matrix where $n-1$ of the entries of each column and of each row are zero and the other one is equal to one. It is well known that for any matrix A, PA is a permutation of rows of A and AP is a permutation of the columns of A. Let $\mathbf{1}$ be the column vector in $\mathbb{R}^n$ of all 1's and A_j the column j of the matrix A.

Theorem 4. *For $A \in \mathcal{P}$ we have that*

a) *$\varphi(PA) = \varphi(A)$ for every permutation matrix P.*
b) *$\varphi(AP) = P\varphi(A)$ for every permutation matrix P.*
c) *$\varphi(\lambda A) = \lambda\varphi(A)$ for every real number λ.*
d) *If $A_k \leq A_j$ then $\varphi_k(A) \leq \varphi_j(A)$.*
e) *$\varphi(P) = \mathbf{1}$ for every permutation matrix P.*

In general, we have that $\varphi(diag(d)) \neq d$, where d is a vector and $diag(d)$ it's corresponding diagonal matrix.

It may also happen for the bargaining power of the workers that a positive salary is assigned to a worker with zero efficiency in all jobs. For example in the problem

$$\left[\begin{array}{cc} 2 & 0 \\ 1 & 0 \end{array}\right]^{2}_{\frac{7}{4}}$$

we get the salary system $(\frac{7}{4}, \frac{7}{4} - \frac{3}{2})^T = (\frac{7}{4}, \frac{1}{4})^T$. This could be explained with Theorem 3, with one of the $\theta \in \Theta$ there is no bargaining but with the other, worker 2 has some bargaining power.

4. Appendix

Proof of Theorem 1. The proof is by induction on the common cardinality of M and N. For $|N| = |M| = 1$, both (2.2) and (3.2) are equal to $L_P(M, N)$. Let us suppose that (2.2) and (3.2) are equal for $(M\backslash\{i\}, N\backslash\{j\})$ for every $i \in M$ and $j \in N$. Then,

$$\frac{1}{n^2} \sum_{i \in M} \sum_{j \in N} Pot(M\backslash\{i\}, N\backslash\{j\}) =$$

$$= \frac{1}{n^2} \sum_{i \in M} \sum_{j \in N} \sum_{S \subseteq N\backslash\{j\}} \sum_{\{R \subseteq M\backslash\{i\} : |R|=s\}} \frac{1}{s} \left[\frac{s!\,(n-s-1)!}{(n-1)!}\right]^2 L_P(R, S)$$

Now, since the double sum $\sum_{i \in M} \sum_{\{R \subseteq M\backslash\{i\}:|R|=s\}}$ is equivalent to $\sum_{\{R \subseteq M:|R|=s\}} \sum_{i \in M\backslash R}$ and the double sum $\sum_{j \in N} \sum_{S \subseteq N\backslash\{j\}}$ is equivalent to $\sum_{\{S \subset N: S \neq N\}} \sum_{j \in N\backslash S}$ we have that

$$= \frac{1}{n^2} \sum_{\{S\subset N: S\neq N\}} \sum_{j\in N\setminus S} \sum_{\{R\subseteq M:|R|=s\}} \sum_{i\in M\setminus R} \frac{1}{s} \left[\frac{s!\,(n-s-1)!}{(n-1)!}\right]^2 L_P(R,S)$$

$$= \sum_{\{S\subset N: S\neq N\}} \sum_{\{R\subseteq M:|R|=s\}} \frac{1}{s} \left[\frac{s!\,(n-s)!}{n!}\right]^2 L_P(R,S) = Pot(M,N) - \frac{1}{s} L_P(M,N)$$

□

Proof of Theorem 2. By the previous theorem we have that

$$\frac{1}{n}\sum_{i\in M} Pot(M\setminus\{i\}, N\setminus\{j\}) = \frac{1}{n}\sum_{i\in M} \sum_{S\subseteq N\setminus\{j\}} \sum_{\{R\subseteq M\setminus\{i\}:|R|=s\}} \frac{1}{s} \left[\frac{s!\,(n-s-1)!}{(n-1)!}\right]^2 L_P(R,S)$$

Since the double sum $\sum_{i\in M}\sum_{\{R\subseteq M\setminus\{i\}:|R|=s\}}$ is equivalent to $\sum_{\{R\subset M:|R|=s\}}\sum_{i\in M\setminus R}$ we have that

$$= \sum_{S\subseteq N\setminus\{j\}} \sum_{\{R\subset M:|R|=s\}} \frac{(n-s)}{n}\frac{1}{s} \left[\frac{s!\,(n-s-1)!}{(n-1)!}\right]^2 L_P(R,S)$$

$$= \sum_{S\subseteq N\setminus\{j\}} \sum_{\{R\subset M:|R|=s\}} \frac{n}{n-s}\frac{1}{s} \left[\frac{s!\,(n-s)!}{n!}\right]^2 L_P(R,S)$$

$$= \sum_{S\subseteq N\setminus\{j\}} \frac{n}{n-s}\frac{1}{s} \left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(S)}$$

therefore

$$\varphi_j(M,N) = Pot(M,N) - \frac{1}{n}\sum_{i\in M} Pot(M\setminus\{i\}, N\setminus\{j\})$$

$$= \sum_{\{S\subseteq N: j\in S\}} \frac{1}{s} \left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(S)} + \sum_{S\subseteq N\setminus\{j\}} \frac{1}{s} \left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(S)}$$

$$- \sum_{S\subseteq N\setminus\{j\}} \frac{n}{n-s}\frac{1}{s} \left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(S)}$$

$$= \sum_{\{S\subseteq N: j\in S\}} \frac{1}{s} \left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(S)} +$$

$$+ \sum_{S\subseteq N\setminus\{j\}} \left[1 - \frac{n}{n-s}\right] \frac{1}{s} \left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(S)}$$

$$= \sum_{\{S\subseteq N: j\in S\}} \frac{1}{s} \left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(S)} - \sum_{S\subseteq N\setminus\{j\}} \frac{1}{n-s} \left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(S)}$$

$$= \sum_{\{S\subseteq N: j\in S\}} \frac{1}{s} \left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(S)} - \sum_{\{S\subseteq N: j\in S\}} \frac{1}{s} \left[\frac{s!\,(n-s)!}{n!}\right]^2 \widehat{L_M(N\setminus S)}$$

$$= \sum_{\{S\subseteq N: j\in S\}} \frac{1}{s} \left[\frac{s!\,(n-s)!}{n!}\right]^2 \left[\widehat{L_M(S)} - \widehat{L_M(N\setminus S)}\right]$$

□

Proof of Theorem 3. Let μ be the cooperative game given by

$$\mu(S) = \frac{1}{n!}\sum_{\theta\in\Theta} v^{\theta}(S)$$

then

$$\mu(S) = \frac{1}{n!}\sum_{\theta\in\Theta} L(\theta(S), S)$$

$$= \sum_{\{R\subseteq M:|R|=s\}} \frac{s!(n-s)!}{n!} L(R,S)$$

Now, by (3.2) we have that

$$Pot(M,N) = \sum_{S\subseteq N}\sum_{\{R\subseteq M:|R|=s\}} \frac{1}{s}\left[\frac{s!\,(n-s)!}{n!}\right]^2 L(R,S)$$

$$= \sum_{S\subseteq N} \frac{(s-1)!\,(n-s)!}{n!} \sum_{\{R\subseteq M:|R|=s\}} \frac{s!\,(n-s)!}{n!} L(R,S)$$

$$= \sum_{S\subseteq N} \frac{(s-1)!\,(n-s)!}{n!}\mu(S) = P(N,\mu)$$

which is the potential of the game μ.

In a similar way

$$\overline{Pot(N\backslash\{j\})} = \frac{1}{n}\sum_{i\in M} Pot(M\backslash\{i\}, N\backslash\{j\}) =$$

$$= \frac{1}{n}\sum_{i\in M}\sum_{S\subseteq N\backslash\{j\}}\sum_{\{R\subseteq M\backslash\{i\}:|R|=s\}} \frac{1}{s}\left[\frac{s!\,(n-s-1)!}{(n-1)!}\right]^2 L(R,S)$$

$$= \sum_{S\subseteq N\backslash\{j\}} \frac{1}{n}\frac{1}{s}\left[\frac{s!\,(n-s-1)!}{(n-1)!}\right]^2 \sum_{i\in M}\sum_{\{R\subseteq M\backslash\{i\}:|R|=s\}} L(R,S)$$

$$= \sum_{S\subseteq N\backslash\{j\}} \frac{1}{n}\frac{1}{s}\left[\frac{s!\,(n-s-1)!}{(n-1)!}\right]^2 \sum_{\{R\subseteq M:|R|=s\}} (n-s)L(R,S)$$

$$= \sum_{S\subseteq N\backslash\{j\}} \frac{(s-1)!\,(n-s-1)!}{(n-1)!}\frac{s!\,(n-s)!}{n!} \sum_{\{R\subseteq M:|R|=s\}} L(R,S)$$

$$= \sum_{S\subseteq N\backslash\{j\}} \frac{(s-1)!\,(n-s-1)!}{(n-1)!}\mu(S) = P(N\backslash\{j\}, \mu_{N\backslash\{j\}})$$

□

Proof of Theorem 4.

a) It follows by Theorem 1 and using the fact that

$$\widehat{L_M(S)} = L_{\widehat{\theta(M)}}(S)$$

for every $\theta\in\Theta$.

b) It follows by $\varphi(A) = Sh(N, \frac{1}{n!}\sum_{\theta\in\Theta} v^{\theta})$ and the symmetry of the Shapley value.

c) The proof is straightforward since

$$L_P(R, S, \lambda B) = \lambda L_P(R, S, B).$$

d) Notice that the marginal contributions are monotonic, so, d) follows by Theorem 3.

e) Using b) and a) we have that

$$P\varphi(I) = \varphi(IP) = \varphi(P) = \varphi(PI) = \varphi(I)$$

Thus, $\varphi_i(I) = \varphi_j(I)$ for every i, $j \in N$. By the efficiency property $\varphi_j(I) = 1$ for every $j \in N$. $\square$

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