E C O N O M I C S   B U L L E T I N

Whitney topology and spaces of preference relations

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Abstract

The strong Whitney topology on the sets of maps of smooth manifolds induces a topology on the set of preferences in euclidean space. We prove that the obtained space is not connected which implies that there is no continuous social choice function defined on a finite power of this space. We also show that the obtained space is not normal.

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1. Introduction

The set of all preference relations defined on the space of commodity bundles is one of the central elements that determine economy. In order to investigate the varying families of economies and behavior of their economic characteristics such as the sets of equilibria, one has to topologize the set of all preference relations on the spaces of commodity bundles.

Topologizations of the sets of preference relations play also an important role in the theory of topological social choice. One of known results of the theory — Chichilnisky's impossibility theorem — states that there does not exist a continuous social choice function which is anonymous and respects unanimity when the space of all preference relations is not contractible.

There are different approaches to topologization of the sets of preference relations (see, e.g., Debreu 1972, Chichilniski 1980, 1993, Mas-Collel 1985, Schofield 1999). In this note we consider the so called strong Whitney topology on the set of smooth preference relations in euclidean space. The strong Whitney topology allows us to control the behavior of the objects "at infinity" and has numerous applications in differential topology (see, e.g., Hirsch 1976).

The main objective is to prove that the obtained space is neither connected nor normal. In the proof of non-normality we follow the approach of Guran and Zarichnyi 1984 (see also Neves 1991a,b, Serrano 1993), where it was shown that the properties of functional spaces in the strong Whitney topology are close to those of the box products of topological spaces.

A consequence of the main result on non-normality is that there exist two closed classes of economies that cannot be separated by a continuous parameter defined on the space of all economies.

2. Preliminaries

We briefly recall some notions concerning the preference relations; see, e.g. Mas-Colell 1985 for details.

A *preference relation* on a set X is a complete reflective and transitive relation. If, moreover, X is a topological space, then a preference relation is continuous if its graph is a closed subset in $X \times X$. In the sequel, all preference relations are assumed to be continuous.

In the case of an euclidean space $\mathbb{R}^\ell$, $\ell \geq 2$, one of the most important from the point of view of applications to economics, one can introduce special classes of preference relations.

A preference relation $\preceq$ on $\mathbb{R}^\ell$, $\ell \geq 2$, is called a *C^r-preference relation,* $r \geq 2$, if the following hold:

1) the *indifference set* $I = \{(x, y) \in \mathbb{R}^\ell \times \mathbb{R}^\ell | x \sim y\}$ is a C^r-submanifold in $\mathbb{R}^\ell \times \mathbb{R}^\ell$ (hereafter, $x \sim y$ means that both $x \preceq y$ and $y \preceq x$ hold);

2) there exists a C^r-function $u\colon \mathbb{R}^\ell \to \mathbb{R}$ such that $x \preceq y$ if and only if $u(x) \leq u(y)$; we require that the gradient ∂u does not vanish in $\mathbb{R}^\ell$ (such a function u is called a *utility function* of the relation $\preceq$). In the sequel, $\preceq (u)$ denotes the preference relation with utility function u.

The set of all C^r-preference relations on $\mathbb{R}^\ell$ is denoted by $\Gamma^r(\mathbb{R}^\ell)$.

For $x = (x_1, \dots, x_\ell), y = (y_1, \dots, y_\ell) \in \mathbb{R}^\ell$ we write $x \leq y$ if and only if $x_i \leq y_i$ for every $i = 1, \dots, \ell$. A relation $\preceq \in \Gamma^r(\mathbb{R}^\ell)$ is said to be *monotone* if $x \leq y \Rightarrow x \preceq y$. By $\Gamma^r_m(\mathbb{R}^\ell)$ we denote the set of monotone C^r-preference relations on $\mathbb{R}^\ell$.

A preference relation $\preceq \in \Gamma^r(\mathbb{R}^\ell)$ is called *convex* if for every $x, y \in \mathbb{R}^\ell$ with $x \neq y$ and $x \preceq y$ we have $x \preceq tx + (1-t)y$ for every $t \in [0, 1]$. A preference relation $\preceq \in \Gamma^r(\mathbb{R}^\ell)$ is called *strictly convex* if for every $x, y \in \mathbb{R}^\ell$ with $x \preceq y$ we have $x \prec tx + (1-t)y$ for every $t \in (0,1)$ (as usual, $x \prec y$ means $x \preceq y$ but not $y \preceq x$).

By $\Gamma^R_c(\mathbb{R}^\ell)$ (resp. $\Gamma^R_{sc}(\mathbb{R}^\ell)$) we denote the set of all convex (resp. strictly convex) C^r-preference relations on $\mathbb{R}^\ell$.

We also put

$$\Gamma^r_{msc}(\mathbb{R}^\ell) = \Gamma^r_m(\mathbb{R}^\ell) \cap \Gamma^r_{sc}(\mathbb{R}^\ell),\ \Gamma^r_{mc}(\mathbb{R}^\ell) = \Gamma^r_m(\mathbb{R}^\ell) \cap \Gamma^r_c(\mathbb{R}^\ell).$$

There are different approaches to topologization of the set $\Gamma^r(\mathbb{R}^\ell)$. Some of them are based on the notion of the Gaussian map. The *Gaussian map* for a preference relation $\preceq \in \Gamma^r(\mathbb{R}^\ell)$ is the map $\dfrac{\partial u}{\|\partial u\|}\colon \mathbb{R}^\ell \to S^{\ell-1}$, where $S^{\ell-1}$ denotes the unit $(\ell-1)$-dimensional sphere in $\mathbb{R}^\ell$. In other words, the Gaussian map is a unit vector field in $\mathbb{R}^\ell$. Note that the Gaussian map depends only on the preference relation but not on a particular choice of its utility function.

Therefore, the set $\Gamma^r(\mathbb{R}^\ell)$ can be embedded as a subset in the set $C^{r-1}(\mathbb{R}^\ell, S^{\ell-1})$ of all C^{r-1}-maps from $\mathbb{R}^\ell$ in $S^{\ell-1}$.

In the set $C^{r-1}(\mathbb{R}^\ell, S^{\ell-1})$ we consider the strong Whitney topology (see Hirsch 1976). A base of this topology is formed by the sets

$$(1)\qquad O(f; \{K_i\}_{i=1}^\infty, \{\varepsilon_i\}_{i=1}^\infty) = \Big\{ g \in C^{r-1}(\mathbb{R}^\ell, S^{\ell-1}) \mid$$
$$\left\| \frac{\partial^{|j|} f}{x^j} - \frac{\partial^{|j|} g}{x^j} \right\|_{K_i} < \varepsilon_i \text{ for every multi-index } j,\ |j| \le r-1,\ i = 1, 2, \dots \Big\},$$

where $\{K_i\}_{i=1}^\infty$ is a locally finite family of compact subsets in $\mathbb{R}^\ell$ and $\{\varepsilon_i\}_{i=1}^\infty$ is a sequence of positive numbers (recall that a family of subsets $\{Y_\alpha \mid \alpha \in A\}$ of a topological space X is *locally finite* if for every $x \in X$ there exists a neighborhood U of x such that $|\{\alpha \in A \mid U \cap Y_\alpha \ne \emptyset\}| < \infty$). This definition requires some explanation. We fix a finite atlas in $S^{\ell-1}$ and implicitly assume that the images $f(K_i)$ and $g(K_i)$ belong to the same chart, for every i. Then, in (1), the partial derivatives concern some fixed local coordinate system.

Recall that the *box product* of a family of topological spaces $(X_\alpha)_{\alpha \in A}$ is the Cartesian product $\prod_{\alpha\in A} X_\alpha$ endowed with the so called *box topology.* A base of this topology consists of the sets of the form $\prod_{\alpha \in A} U_\alpha$, where U_α is open in X_α for every $\alpha \in A$. The box product is denoted by $\square_{\alpha \in A} X_\alpha$.

3. Results

Proposition 3.1. *The space $\Gamma^r(\mathbb{R}^\ell)$ is not connected.*

Proof. Consider the relation $\preceq_0 \in \Gamma^r(\mathbb{R}^\ell)$ with the utility function

$$u(x_1, \dots, x_\ell) = x_1 + \dots + x_\ell,\ (x_1, \dots, x_\ell) \in \mathbb{R}^\ell.$$

Let $K_i = \{x \in \mathbb{R}^\ell \mid i \le \|x\| \le i+1\}$, $\varepsilon_i = 1/i$. Put

$$U = \{\preceq (v) \mid \lim_{i\to\infty} \|(u - v)\|_{K_i} = 0\}.$$

It is easy to show that the sets U and $\Gamma^r(\mathbb{R}^\ell) \setminus U$ are open in the strong Whitney topology. The set U is a neighborhood of the preference relation $\preceq_0$. In addition, the set $\Gamma^r(\mathbb{R}^\ell) \setminus U$ is nonempty; it contains, for example, the preference relation with utility function $v(x_1, \dots, x_\ell) = 2x_1 + x_2 \dots + x_\ell$.

$\square$

Corollary 3.2. *The space $\Gamma^r(\mathbb{R}^\ell)$ is not contractible.*

One can similarly prove counterparts of the above Proposition and Corollary for the space $\Gamma^r_m(\mathbb{R}^\ell)$.

Proposition 3.3. *The component of $\preceq\in\Gamma^r(\mathbb{R}^\ell)$ is the set*

$$\begin{aligned} C(\preceq) = \Big\{ \preceq'\in\Gamma^r(\mathbb{R}^\ell) \mid \ & \textit{the Gaussian maps of } \preceq \textit{ and } \preceq' \\ & \textit{coincide outside a compact subset of } \mathbb{R}^\ell \Big\}. \end{aligned}$$

Proof. Suppose the contrary and let $\preceq'$ be an element of the component of $\preceq$ in $\Gamma^r(\mathbb{R}^\ell)$ that does not belong to $C(\preceq)$. Then there exist a multi-index j, $|j|\le r-1$, a locally finite infinite family K_i of nonempty compact subsets of $\mathbb{R}^\ell$ such that the following condition holds (here f, f' are the Gaussian maps of $\preceq$, $\preceq'$ respectively):

$$\left\| \frac{\partial^{|j|} f}{x^j} - \frac{\partial^{|j|} f'}{x^j} \right\|_{K_i} = \eta_j > 0.$$

Let U be the set of all $\preceq''\in\Gamma^r(\mathbb{R}^\ell)$ that satisfy the condition (f'' is the Gaussian maps of $\preceq''$):

$$\lim_{i\to\infty} \frac{1}{\eta_i} \left\| \frac{\partial^{|j|} f}{x^j} - \frac{\partial^{|j|} f''}{x^j} \right\|_{K_i} = 0.$$

One can easily verify that both U and $\Gamma^r(\mathbb{R}^\ell)\setminus U$ are open and nonempty (because $\preceq\in U$, $\preceq'\notin U$), which gives a contradiction. $\square$

Recall that a topological space is *normal* if any two disjoint closed subsets in it can be separated by disjoint neighborhoods.

Theorem 3.4. *The space $\Gamma^r(\mathbb{R}^\ell)$ is not normal.*

Proof. Let $U_n = B_1(3n,\dots,3n)$, $n\in\mathbb{N}$, be the open balls of radius 1 centered at $(3n,\dots,3n)$. Consider the set

$$X = \left\{ \preceq\in\Gamma^r(\mathbb{R}^\ell) \mid \frac{\partial u(\preceq)}{\|\partial u(\preceq)\|} = (1/\sqrt{\ell},\dots,1/\sqrt{\ell}) \text{ for every } x\in\mathbb{R}^\ell\setminus\cup_{n\in\mathbb{N}}U_n \right\}.$$

Obviously, the set X is closed in $\Gamma^r(\mathbb{R}^\ell)$, so it suffices to prove that X is not normal. Denote, for every $n\in\mathbb{N}$, by Y_n the set

$$\begin{aligned} \Big\{ & \preceq\in\Gamma^r(B_{3/2}(3n,\dots,3n)) \mid \text{ there exists a } C^r\text{-utility function with no critical} \\ & \text{points } u \text{ of } \preceq \text{ such that } \frac{\partial u}{\|\partial u\|} = (1/\sqrt{\ell},\dots,1/\sqrt{\ell}) \\ & \text{for every } x\in B_{3/2}(3n,\dots,3n)\setminus B_1(3n,\dots,3n)\Big\}. \end{aligned}$$

It is proved in Mas-Collel 1985 (see Proposition 2.4.5 therein) that the space of C^r-preference relations in $\mathbb{R}^\ell_{++}$ in the uniform C^r-convergence topology is complete metrizable. We easily derive from this fact that Y_n is metrizable and topologically complete.

The natural map $\Phi\colon \square_{n\in\mathbb{N}}Y_n \to X$ sends every sequence $(\preceq_n)_{n\in\mathbb{N}}\in\square_{n\in\mathbb{N}}Y_n$ into the element $\preceq\in X$ defined as follows. Denote by Z the unit vector field on $\mathbb{R}^\ell$ that restricts on every U_n to the vector field generated by $\preceq_n$ (i.e., to the Gaussian map of $\preceq_n$) and is equal to $(1/\sqrt{\ell},\dots,1/\sqrt{\ell})$ on the complement of the set $\cup_{n\in\mathbb{N}}U_n$. Then, by the definition, Z is the Gaussian map of $\preceq$.

It easily follows from the definition of the strong Whitney topology in X that the map Φ is a homeomorphism. We only show that Φ is continuous. Consider a neighborhood V of $\preceq = \Phi((\preceq_n)_{n\in\mathbb{N}})$, $V = O(f; \{K_i\}_{i=1}^{\infty}, \{\varepsilon_i\}_{i=1}^{\infty})$, where $f\colon \mathbb{R}^\ell \to S^{\ell-1}$ is the Gaussian map of $\preceq$, $\{K_i\}_{i=1}^{\infty}$ is a locally finite family of compact sets in $\mathbb{R}^\ell$ and $\{\varepsilon_i\}_{i=1}^{\infty}$ is a set of positive numbers. For every $n \in \mathbb{N}$, put $\eta_n = \min\{\varepsilon_i \mid K_i \cap U_n \neq \emptyset\}$. It follows from the compactness of U_n and local finiteness of the family $\{K_i\}_{i=1}^{\infty}$ that $\eta_n > 0$. Then $\Phi((\preceq'_n)_{n\in\mathbb{N}}) \in V$ for every $(\preceq'_n)_{n\in\mathbb{N}} \in \square_{n\in\mathbb{N}} Y_n$ such that

$$\left\| \frac{\partial^{|j|} f_n}{x^j} - \frac{\partial^{|j|} f'_n}{x^j} \right\|_{K_i} < \varepsilon_i \text{ for every multi-index } j,\ |j| \leq r-1,$$

where f_n (respectively f'_n) is the Gaussian map of $\preceq$ (respectively $\preceq'$).

Note that, obviously, Y_n is infinite and therefore it contains a closed copy of the space of irrationals P_n as well as a convergent sequence S. Therefore, we conclude that the space X contains a closed copy of the box product $S \square (\square_{n\geq 2} P_n)$. By a result of van Douwen 1985, X is not normal. $\square$

One can similarly prove the following result.

Theorem 3.5. *The space $\Gamma^r_m(\mathbb{R}^\ell)$ is not normal.*

Theorem 1 and 2 remain valid if we replace $\mathbb{R}^\ell$ by either

$$\mathbb{R}^\ell_{++} = \{(x_1, \ldots, x_\ell) \mid x_i > 0,\ i = 1, \ldots, \ell\}$$

or $\mathbb{R}^\ell_+ \setminus \{0\}$, where $\mathbb{R}^\ell_+ = \{x \in \mathbb{R}^\ell \mid x \geq 0\}$.

4. Strictly monotone preferences

The possibility of approximation of convex preferences by strictly convex ones is one of the important properties which is used in proofs of various results in the mathematical economics (see, e.g. Mas-Collel 1985). It turns out that such an approximation does not exist in the case of strong Whitney topology.

The following example demonstrates that the set $\Gamma^r_{msc}(\mathbb{R}^\ell)$ is not dense in the set $\Gamma^r_{mc}(\mathbb{R}^\ell)$. Suppose first that $\ell = 2$. Let $\preceq_0 = \preceq (u_0)$, where $u_0(x_1, x_2) = x_2$. Then $f_0(x_1, x_2) = \dfrac{\partial u_0}{\|\partial u_0\|} = (0, 1)$ is the Gaussian map for $\preceq_0$. For all maps $f = (f_1, f_2)\colon \mathbb{R}^2 \to S^1$ sufficiently close to f_0, we see that $f_2(x_1, x_2)$ does not vanish in $\mathbb{R}^2$. Then the vector $(f_2(x_1, x_2), -f_1(x_2, x_1))$ is the tangent vector to the indifference curve of the preference relation $\preceq$ for which f is the Gaussian map. Therefore, every indifference curve of $\preceq$ is the graph of a function $x_2 = g(x_1)$. For the derivative of this function, we obtain (by t we denote the angle coordinate on S^1):

$$\frac{dx_2}{dx_1} = -\frac{dt/dx_1}{dt/dx_2} = -\frac{f_1(x_1, x_2)}{f_2(x_1, x_2)}.$$

Obviously, f can be chosen so close to f_0 that $|dx_2/dx_1| \leq 1/n^2$ as $|x_1| \geq n$. It easily follows from this fact that the graph of the function g has horizontal asymptotes as $|x_1| \to \infty$. Since the preference relation $\preceq$ is assumed to be convex, the epigraph of g is a convex set and therefore g is constant. This means that all the indifference sets of $\preceq$ are horizontal lines, i.e. $\preceq = \preceq_0$. We conclude that $\preceq_0$ is an isolated point in the set $\Gamma^r_{mc}(\mathbb{R}^2)$.

Note that this example can be easily generalized to arbitrary $\mathbb{R}^\ell$, $\ell \geq 2$.

5. Remarks and open questions

Chichilniski and Heal 1983 proved that if the space of preferences, P, is a finite disjoint union of connected parafinite CW-complexes, then a continuous, anonymous, unanimity preserving social choice function $\varphi\colon P^k \to P$ (here k is the number of agents) exists if and only if every component of P is contractible. We do not know whether every component of any $\preceq\in \Gamma^r(\mathbb{R}^\ell)$ is contractible. Moreover, one can easily show that the cardinality of the family of components of the space $\Gamma^r(\mathbb{R}^\ell)$ is infinite. Therefore, the following natural question arises: is there a continuous, anonymous, unanimity preserving social choice function $\varphi\colon \left(\Gamma^r(\mathbb{R}^\ell)\right)^k \to \Gamma^r(\mathbb{R}^\ell)$? A similar question can be formulated for another spaces of preferences ($\Gamma^r_c(\mathbb{R}^\ell)$, $\Gamma^r_m(\mathbb{R}^\ell)$ etc).

The topologization based on the Gaussian map is meaningless for the case $r = 0$. In subsequent papers we are going to consider a different approach and define the strong Whitney topology on the set of all continuous ($= C^0$-) preference relations.

A natural question arises whether the connected component of a fixed preference relation $\preceq\in \Gamma^r(\mathbb{R}^\ell)$ is contractible.

Chichilniski 1980 regarded the sets of preference relations as the sets of transversally oriented C^r-foliations of codimension 1. Therefore, the spaces of preference relations can be topologized as subspaces of spaces of C^r-foliations; see, e.g., Epstein 1977 for topologizations of these spaces. The spaces of foliations equipped with the strong Whitney topology are considered in Zarichnyi, Tkach 1990.

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