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On a property of mixed strategy equilibria of the pricing game

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Abstract

Before solving a two−stage capacity and pricing game for oligopoly, Boccard and Wauthy (2000) argue that, as under duopoly, at a mixed strategy equilibrium of the pricing game the largest firm's expected profit is the profit accruing to it as a Stackelberg follower when the rivals supply their entire capacity. We point to a serious mistake in their argument and then we see how this important property can be satisfactorily established.

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1 Introduction

In a recent paper Boccard and Wauthy (2000) (hereafter, BW) analyze a two-stage capacity and pricing game among a fixed number $n \geq 2$ of firms. The setting is basically the one envisaged for a duopoly by Kreps and Scheinkman (1983) (hereafter, KS). Capacities are built in the first stage while prices are set in the second stage in the knowledge of previous capacity decisions. The firms face identical convex costs of capacity building, marginal variable cost is constant up to capacity, and rationing takes place according to the surplus-maximizing rule. Differently from KS, the firms can produce beyond "capacity" at a constant additional unit cost θ. The upshot is a generalization of the Cournot outcome obtained by KS for a duopoly: when θ is sufficiently high, at the subgame perfect equilibrium solution of the capacity and pricing game the n firms invest in Cournot capacity and prices are subsequently set equal to the demand price of total capacity (BW, pp. 283-4).

In order to apply backward induction to the two-stage capacity and pricing game, BW have to characterize the solutions of the pricing subgames in the different regions of the capacity space that may arise from previous capacity decisions. This task is not so easy in the region where no pure strategy equilibrium exists. Since KS it has been known that, under duopoly, at a mixed strategy equilibrium of the pricing game the largest firm's expected revenue is what it obtains as a Stackelberg follower when the rival supplies its entire capacity. An extention of this property to oligopoly was subsequently provided by Brock and Scheinkman (1985) and Vives (1986) for the special case of equally sized firms. To the best of our knowledge, Claim 6 by BW (pp. 282-3) is the only attempt that has been made so far to generalize this important property to asymmetric oligopoly. Unfortunately, while containing one important step towards such a generalization, the proof by BW suffers from several shortcomings and one serious mistake. Besides seeing this, the present comment shows how the property under discussion can be satisfactorily established for asymmetric oligopoly.

2 The pricing game

2.1 Preliminaries

We draw on BW for most of the notation employed. $D(p)$ and $P(q)$ are demand and the inverse demand, respectively, where p is the price and q the total output; $P(q) > 0$ on a bounded interval $[0, \widehat{q})$ on which $P'(\cdot) < 0$ and $P''(\cdot) \leq 0$. We are concerned with the pricing game being played by the firms after the choice of capacities. There are $n \geq 2$ firms; x_i denotes firm i's capacity, $x_{-i} = \sum_{j \neq i} x_j$ the total capacity of i's rivals, and $x = x_i + x_{-i}$ total industry capacity. Without loss of generality, BW assume firm 1 to be (one of) the largest firm(s), i.e., $x_1 \geq x_i$ for any $i \neq 1$. Variable cost is zero up to the firm capacity, hence in the pricing game each $i \leq n$ seeks to maximize expected revenue, denoted Π_i.

For our purposes we can stick to the KS's assumption that the firms cannot produce above capacity. Then, under the surplus-maximizing rationing rule, the residual demand facing firm i is $\max\{0, D(p_i) - x_{-i}\}$ when $p_i > p_j$ for any $j \neq i$.

Finally, it will be helpful to have on hand a few concepts from Cournot competition. Denote $r(x_{-i})$ firm i's Cournot best response to x_{-i} under costless capacity building and $R(x_{-i})$ the corresponding revenue. In other words, $r(x_{-i}) = \arg\max_{x_i} x_i P(x_i + x_{-i})$ and $R(x_{-i}) = r(x_{-i}) P(r(x_{-i}) + x_{-i})$. Note that $x_i P(x_i + x_{-i})$ is concave in x_i and $-1 < r'(x_{-i}) < 0$ for any $x_{-i} < \widehat{q}$, hence $(r(x_{-i}) + x_{-i})$ is increasing in x_{-i}.

2.2 Mixed strategy equilibria

Before looking at mixed strategy equilibria of the pricing game it will be helpful to explore the region of the capacity space giving rise to pure strategy equilibria. The following proposition is a straightforward generalization of results already achieved under symmetric oligopoly (Vives, 1986).

Proposition 1 *(i) Let $x : D(0) \leq x_{-1}$. Then all firms charging a zero price is the symmetric equilibrium of the pricing game.*

(ii) Let $x : x < D(0); r(x_{-1}) \geq x_1$. Then all firms charging $P(x)$ is the unique equilibrium of the pricing game.

Proof. (i) If follows from $x_{-1} \geq D(0)$ that $x_{-i} \geq D(0)$ for any $i \leq n$. Then, for any i, charging a zero price is a best response to all $j \neq i$ charging a zero price because there would be no residual demand left if charging $p_i > 0$.[1]

(ii) Note that $r(x_{-1}) \geq x_1$ implies $[\partial(p_1(D(p_1) - x_{-1}))/\partial p_1]_{p_1=P(x)} \leq 0$, which in turn implies $[\partial(p_i(D(p_i) - x_{-i}))/\partial p_i]_{p_i=P(x)} \leq 0$ for any $i \leq n$. From concavity of $(p_i(D(p_i) - x_{-i})$ it follows that, for any i, charging $P(x)$ is the best response to all $j \neq i$ charging $P(x)$. Uniqueness of equilibrium can be established along the lines of KS. ■

A pure strategy equilibrium of the pricing game does not exist either at $x : x_{-1} < D(0) \leq x$ or $x : x < D(0); r(x_{-1}) < x_1$. In the region of no existence of a pure strategy equilibrium, existence of a mixed strategy equilibrium is guaranteed by Theorem 5 of Dasgupta and Maskin (1986) (see also BW, p. 281). Let $\Sigma_i \equiv \{p_i \mid f_i(p_i) > 0\}$ denote the support of firm i's equilibrium density $f_i(p_i)$, and $\underline{p}_i$ and $\overline{p}_i$, respectively, the infimum (i.e., the greatest lower bound) and the supremum (the lowest upper bound) of Σ_i. It is important to note that it can be $\overline{p}_i \notin \Sigma_i$. Also, let $\overline{p} \equiv \max_{i \leq n} \overline{p}_i$ and $\overline{H} \equiv \arg\max_{i \leq n} \overline{p}_i$; in words, members of $\overline{H}$ are all $i \leq n$ such that $\overline{p}_i = \overline{p}$. Further, let $F_i(p) \equiv \Pr(p_i < p)$ and denote Π_i^* firm i's equilibrium expected revenue.

The present comment seeks to establish the following property of mixed strategy equilibria, which was already obtained for a duopoly by KS (pp. 331-5).

Proposition 2 *At a mixed strategy equilibrium of the pricing game* $\overline{p} = P(r(x_{-1}) + x_{-1})$ *and (any of) the largest firm(s) has expected profit* $R(x_{-1}) = r(x_{-1}) P(r(x_{-1}) + x_{-1})$, *the profit accruing to it as a Stackelberg follower when its rivals supply their entire capacity.*

Remark. It must be pointed out that our Proposition 2 does not exactly match the characterization of mixed strategy equilibrium to be found in BW. In fact, BW make the following points while proving their Claim 6 (pp. 282-3):[2]

[1] Any price vector $(p_1, ..., p_n)$ such that $\sum_{j \neq i : p_j = 0} x_j \geq D(0)$ for any $i : p_i = 0$ is also an equilibrium.

[2] Claim 6 states that, in the region of the capacity space where no pure strategy equilibrium exists, "$\overline{H} = \{1\}$, the large capacity firm" (p. 282). Besides other problems, this

(a) $\overline{p} = P(r(x_{-1}) + x_{-1})$,

(b) $\overline{p} = P(r(x_{-j}) + x_{-j})$ for any $j \in \overline{H}$, hence "members of $\overline{H}$ must have the same (largest) capacity" (p. 283),

(c) $\Pi_j^* = R(x_{-j})$ for any $j \in \overline{H}$.

Of course, our Proposition 2 would be implied by points (a), (b), and (c), should they hold true. Unfortunately, points (b) and (c) may not hold true. To check this, consider the example of asymmetric duopoly in the appendix of Levitan and Shubik (1972), where it is assumed $D(p) = a - p$ and $x_1 = a > x_2$. In that example, $\overline{p}_1 = \overline{p}_2$: thus, contrary to point (b), $\overline{H}$ also includes the smallest firm.[3] Further, $\Pi_2^* \neq R(x_1)$, which contradicts point (c).

Proof of Proposition 2. As acknowledged below, one major step of our proof is taken over from the argument by which BW seek to establish their Claim 6; on the other hand, their argument is incomplete and, as shown below, is vitiated by a nontrivial mistake.

To begin with, a few features of mixed strategy equilibria are readily established. First, any $j \in \overline{H}$ sells less than x_j when charging $\overline{p}$: otherwise it would be $D(\overline{p}) \geq x$, implying that any $i : F_i(\overline{p}) > 0$ has not made a best response since it can sell x_i by charging $\overline{p}$. Second, $\overline{p}$ is charged with positive probability by one firm at most: indeed, if $\Pr(p_j = \overline{p}) > 0$, charging slightly less than $\overline{p}$ is better than $\overline{p}$ to any $i \neq j$ due to the jump in i's residual demand in the event of j charging $\overline{p}$. Thus, at a mixed strategy equilibrium there exists at least one firm $j \in \overline{H}$ such that no $i \neq j$ charges $\overline{p}$ with positive probability.[4] Arguing along the lines of KS, $\overline{p} = P(r(x_{-j}) + x_{-j})$ and $\Pi_j^* = R(x_{-j}) = \overline{p}r(x_{-j})$ for any such j, with $r(x_{-j}) < x_j$; in words, since firm j is certainly undercut when charging $\overline{p}$ and given that $\overline{p}$ must be an optimal response to j's rival equilibrium strategies, it must be $\overline{p} = \arg\max_{p_j}(D(p_j) - x_{-j})p_j$.

In the remainder of the proof the main task will be to show that a contradiction would arise from assuming $x_j < x_1$ when $j \in \overline{H}$ and no $i \neq j$ charges $\overline{p}$ with positive probability. Recall that $(r(x_{-i}) + x_{-i})$ is increasing in x_{-i}. Therefore, with $x_j < x_1$ it would be $\overline{p}_1 \leq \overline{p} < P(r(x_{-1}) + x_{-1})$. On the other

concise statement is a bit inaccurate. For example, when there are several firms with the largest capacity, it seems to be in contrast with point (b) in the text.

[3]While $\overline{H} = \{1, 2\}$, $\overline{p}$ is charged with positive probability by firm 1 and with zero probability by firm 2 (more than that, $\overline{p} \notin \Sigma_2$).

[4]This means that, for any $i \neq j$, either $\overline{p} \notin \Sigma_i$ or $\overline{p} \in \Sigma_i$ but $\Pr(p_i = \overline{p}) = 0$.

hand, the fact that firm 1 will never charge a price of $P(r(x_{-1})+x_{-1})$ at the equilibrium reveals that $R(x_{-1}) = r(x_{-1})P(r(x_{-1}) + x_{-1}) \leq \Pi_1^*$. Further - and this is the major step we take over from BW - assuming $x_j < x_1$ would also lead to $x_1 R(x_{-j}) < x_j R(x_{-1})$ (see the proof in the Appendix). In view of all this, we can write, along with BW (p. 283),

$$\Pi_j^* = R(x_{-j}) < \frac{x_j}{x_1} R(x_{-1}) \leq \frac{x_j}{x_1} \Pi_1^* < \Pi_1^*. \tag{1}$$

The final stage of the argument by BW relies on writing (p. 283)

$$\Pi_1^* = \underline{p}_1 \left[F_{-1}(\underline{p}_1)(D(\underline{p}_1) - x_{-1}) + (1 - F_{-1}(\underline{p}_1))D(\underline{p}_1) \right], \tag{2}$$

where the expression in square brackets is firm 1's expected output when charging $\underline{p}_1$ in the face of rivals' equilibrium strategies. The term $D(\underline{p}_1)$ multiplying $(1 - F_{-1}(\underline{p}_1))$ is mistaken. Notice that $F_{-1}(\underline{p}_1)$ is the probability of the event that $p_i < \underline{p}_1$ for every $i \neq 1$, i.e., $F_{-1}(\underline{p}_1) = \times_{i \neq 1} F_i(\underline{p}_1)$. The complementary event is thus the event that $p_i \geq \underline{p}_1$ *for at least some* $i \neq 1$. Therefore, $(1 - F_{-1}(\underline{p}_1))$ should have been multiplied by $q_1(p_1 = \underline{p}_1 \mid \textit{some}\ p_i \geq \underline{p}_1)$, i.e., 1's expected output when charging $\underline{p}_1$, conditional on $p_i \geq \underline{p}_1$ for some $i \neq 1$. To understand that $q_1(p_1 = \underline{p}_1 \mid \textit{some}\ p_i \geq \underline{p}_1) < D(\underline{p}_1)$ it suffices to show that $x_1 < D(\underline{p}_1)$. Notice that $x_{-j} \geq x_1$ (with $x_{-j} = x_1$ if and only if $n = 2$) and recall that $\overline{p} = P(r(x_{-j}) + x_{-j}) < P(x_{-j})$; further, $\underline{p}_1 < \overline{p}$ since it is assumed that $\overline{p}$ is never charged by firm 1. Consequently, $D(\underline{p}_1) > r(x_{-j}) + x_{-j} > x_1$.[5]

Then, how can we achieve the desired contradiction? Write firm 1's equilibrium expected revenue as $\Pi_1^* = \underline{p}_1 q_1(p_1 = \underline{p}_1)$, where $q_1(p_1 = \underline{p}_1)$ is 1's expected output when charging $\underline{p}_1$. Let $(x_j/x_1)\Pi_1^* \equiv x_j \underline{p}_1 k$, where $k \equiv q_1(p_1 = \underline{p}_1)/x_1 \leq 1$. It follows from (1) that $\Pi_j^* < x_j \underline{p}_1 k$. Notice that firm j sells x_j by charging $\underline{p}_1^-$, i.e., a price slightly less than $\underline{p}_1$: indeed, this results in a residual demand of at least $D(\underline{p}_1) - x_{-j} + x_1 > x_1 > x_j$ because $D(\underline{p}_1) > x_{-j}$ as $\underline{p}_1 < \overline{p} < P(x_{-j})$. Thus $\Pi_j(p_j = \underline{p}_1^-) = \underline{p}_1 x_j \geq x_j \underline{p}_1 k > \Pi_j^*$, a contradiction.

To avoid the contradiction it must be $\overline{p} = P(r(x_{-1}) + x_{-1})$ and $\Pi_i^* = R(x_{-i})$ for any $i : x_i = x_1$. This is immediate when $x_1 > x_i$ for every $i \neq 1$,

[5] Further, if $F_i(\underline{p}_1) > 0$ for some $i \neq 1$, the event that $p_i \geq \underline{p}_1$ for some $i \neq 1$ would also include price vectors by 1's rivals with some $p_i < \underline{p}_1$, resulting in a residual demand for firm 1 less than $D(\underline{p}_1)$ when charging $\underline{p}_1$.

since then it must be that $1 \in \overline{H}$ and no $i \neq 1$ charges $\overline{p}$ with positive probability. The case $x_i = x_1$ for some $i \neq 1$ needs further reflection. The above contradiction is avoided by having $x_j = x_1$ for any $j \in \overline{H}$ for which no $i \neq j$ charges $\overline{p}$ with positive probability; further, $\Pi_i^* = R(x_{-i})$ for any $i : x_i = x_1$. Indeed, suppose the latter does not hold, i.e., $1 \in \overline{H}$ but $\overline{p}_i < \overline{p} = P(r(x_{-1}) + x_{-1})$ and $\Pi_i^* > \Pi_1^* = R(x_{-1})$ for some $i : x_i = x_1$.[6] Similarly as above, firm 1 would sell x_1 by charging $\underline{p}_i^-$,[7] hence $\Pi_1(p_1 = \underline{p}_i^-) = \underline{p}_i x_1 \geq \Pi_i^* > \Pi_1^*$, a contradiction. ■

APPENDIX

Here we report (with a few integrations) the argument by which BW establish that $x_1 R(x_{-j}) < x_j R(x_{-1})$, where, by assumption, $x_1 > x_j$, $j \in \overline{H}$ and no $i \neq j$ charges $\overline{p}$ with positive probability at a mixed equilibrium. Let $\overline{m} \equiv \sum_{i \neq j,1} x_i$, so that $x_{-j} = \overline{m} + x_1$ and $x_{-1} = \overline{m} + x_j$. By letting $\Theta(z) \equiv zR(\overline{m}+z) = zr(\overline{m}+z)P(\overline{m}+z+r(\overline{m}+z))$ our task is to prove that $\Theta(x_1) - \Theta(x_j) < 0$. By the envelope theorem, $\dot{\Theta}(z) = (r(\overline{m}+z)-z)P(\overline{m}+z+r(\overline{m}+z))$. In the following, we repeteadly use the fact that $\partial r(\overline{m}+z)/\partial z < 0$ for $\overline{m}+z < \widehat{q}$. If $r(\overline{m}+x_j) < x_j$, then $\dot{\Theta}(z) < 0$ for any $z \in [x_j, x_1]$, implying $\Theta(x_1) - \Theta(x_j) < 0$. The same implication is drawn if $r(\overline{m} + x_j) = x_j$ since then $\dot{\Theta}(z) < 0$ for $z \in (x_j, x_1]$. The argument is somewhat involved when $r(\overline{m} + x_j) > x_j$. Then, $r(\overline{m} + x) = x$ is solved for $x = x^* > x_j$. Similarly, $r(\overline{m} + x) = x_j$ is solved for $x = y^* > x^*$. Finally, in view of the latter and recalling that $r(\overline{m} + x_1) = r(x_{-j}) < x_j$, it is understood that $y^* < x_1$. This leads to[8]

$$\begin{gathered}
\Theta(x_1) - \Theta(x_j) = \int_{x_j}^{x^*} \dot{\Theta}(z)dz + \int_{x^*}^{x_1} \dot{\Theta}(z)dz < \int_{x_j}^{x^*} \dot{\Theta}(z)dz + \int_{x^*}^{y^*} \dot{\Theta}(z)dz \\
= \Theta(y^*) - \Theta(x_j) = y^* R(r^{-1}(x_j)) - x_j R(\overline{m} + x_j) \\
= y^* x_j P(x_j + r^{-1}(x_j)) - x_j R(\overline{m} + x_j) = x_j \left[y^* P(x_j + \overline{m} + y^*) - R(\overline{m} + x_j) \right].
\end{gathered}$$

[6] The case $\Pi_i^* < \Pi_1^* = R(x_{-1})$ is immediately dismissed since firm i earns no less than $R(x_{-1})$ by charging $\overline{p}$. (It earns $R(x_{-1})$ if firm 1 charges $\overline{p}$ with zero probability.)

[7] By charging $\underline{p}_i^-$ firm 1 obtains a residual demand of at least $D(\underline{p}_i) - x_{-1} + x_i > x_i = x_1$ since $\underline{p}_i < \overline{p} < P(x_{-1})$.

[8] Actually, BW (p. 283) write $\int_{x_j}^{x^*} \dot{\Theta}(z)dz + \int_{x^*}^{x_1} \dot{\Theta}(z)dz \leq \int_{x_j}^{x^*} \dot{\Theta}(z)dz + \int_{x^*}^{y^*} \dot{\Theta}(z)dz$. Strict inequality must hold, however, because $\dot{\Theta}(z) < 0$ for $z \geq y^*$ so that $\int_{y^*}^{x_1} \dot{\Theta}(z)dz < 0$.

Finally, $y^* P(x_j + \overline{m} + y^*) \leq R(\overline{m} + x_j)$ as $R(\overline{m} + x_j)$ is by definition the maximum payoff in response to an aggregate output of $\overline{m} + x_j$ by rivals.[9] Thus $\Theta(x_1) - \Theta(x_j) < 0$.

References

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[9] BW conclude that $y^* P(x_j + \overline{m} + y^*) < R(\overline{m} + x_j)$. Equality cannot be excluded, however (it occurs if, by a fluke, $y^* = r(\overline{m} + x_j)$.)