E C O N O M I C S   B U L L E T I N

A notion of sufficient input

Bertrand Crettez
Université de Franche−Comté

Philippe Michel
G.R.E.Q.A.M, Université d'Aix−Marseille II, E.U.R.E.Q.U.A, Université de Paris I

Abstract

In this paper, we study a notion of sufficient input, i.e. input that allows to produce at least one unit of output when the other inputs are fixed at any positive level. We show that such an input allows to produce any positive amount of production. The main property of sufficient inputs is as follows. A input is sufficient if and only if the unit cost goes to zero when its price goes to zero.

Citation: Crettez, Bertrand and Philippe Michel, (2003) "A notion of sufficient input." *Economics Bulletin,* Vol. 4, No. 37 pp. 1−4
Submitted: November 17, 2003. **Accepted:** December 17, 2003.
URL: http://www.economicsbulletin.com/2003/volume4/EB−03D20003A.pdf

1. Introduction

In this paper, we study a notion of sufficient input, *i.e.* input that allows to produce at least one unit of output when the other inputs are fixed at any positive level.

In the next section, we show that a sufficient input allows to produce any positive amount of production. As a consequence, reducing the other inputs to an arbitrary positive level can be neutralized by an increase in the quantity of the sufficient input.

We also prove the main property of sufficient inputs. A sufficient input is characterized by the fact that the unit cost goes to zero when its prices goes to zero.

2. Definition and Properties

Let $F : \mathbb{R}_+^n \to \mathbb{R}_+$ be a production function for a monoproduct firm. For a vector $X = (x_1, ..., x_n)$ in $\mathbb{R}_+^n$, we shall use the following notation $X = (x_i, X_{-i})$, where X_{-i} is the vector of coordinates x_j, $j \neq i$ and is denoted $X_{-i} = (x_j)_{j \neq i}$. We shall make the following assumption.

Assumption. *F is homogenous of degree 1, continuous and increasing.*

We recall that the unit cost function $\lambda(.)$:

$$\lambda(W) = \inf \left\{ \sum_{i=1}^{n} w_i z_i \ ; \ Z \in \mathbb{R}_+^n, \ F(Z) \geq 1 \right\}$$

is defined for any vector of input prices $W = (w_1, ..., w_n)$ in $\mathbb{R}_+^n$.

The following properties of λ are well-known: $\lambda(.)$ is non-decreasing with respect of each of its argument (see, for instance, Mas-Colell *et al.* (1995), chapter 5, or Nadiri (1982), and the minimum is realized when W is in $\mathbb{R}_{++}^n$, the set of vectors in $\mathbb{R}_+^n$ whose coordinates are all positive (see, for instance McFadden (1980), section 4, page 10).

Definition. *An input i is sufficient if for all positive values of the other inputs, there is a quantity z_i which allows to produce at least one unit of output. Formally*

$$\forall Z_{-i} = (z_j)_{j \neq i} \in \mathbb{R}_{++}^{n-1}, \exists z_i > 0, \ F(z_i, Z_{-i}) \geq 1$$

Proposition. *Let the input i be sufficient. Let $(Z_{-i}, y) \in \mathbb{R}^n_{++}$ be given. Then there is a quantity z_i which allows to produce at least y: $\exists z_i > 0$, $F(z_i, Z_{-i}) \geq y$. Equivalently: for all Z_{-i} in $\mathbb{R}^{n-1}_{++}$, $\sup_{z_i>0} F(z_i, Z_{-i}) = +\infty$.*

Proof. Let (Z_{-i}, y) in $\mathbb{R}^n_{++}$ be given. By definition of a sufficient input, if $y \leq 1$, the result holds trivially. Now suppose that $y > 1$. For all $\alpha > 0$, αZ_{-i} in $\mathbb{R}^{n-1}_{++}$. By definition, there exists $z_i(\alpha)$ such that $F(z_i(\alpha), \alpha Z_{-i}) \geq 1$. It follows that:

$$F(\frac{z_i(\alpha)}{\alpha}, Z_{-i}) = \frac{1}{\alpha} F(z_i(\alpha), \alpha Z_{-i}) \geq \frac{1}{\alpha}$$

Choosing α such that $1/\alpha > y$ yields the result. Finally, note that $\sup_{z_i>0} F(z_i, Z_{-i}) \geq \frac{1}{\alpha}$ for all $\alpha > 0$. This implies that $\sup_{z_i>0} F(z_i, Z_{-i}) = +\infty$. Q.E.D.

Example. Consider a C.E.S. function with an elasticity $\sigma > 0$, $\sigma \neq 1$:

$$F(Z) = A \left(\sum_{i=1}^{n} \alpha_i z_i^{1-\frac{1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}, \; \alpha_i > 0 \text{ for all } i, \; \sum_{i=1}^{n} \alpha_i = 1$$

If $\sigma > 1$, every good i is sufficient since $F(Z) \geq A\alpha_i^{\frac{\sigma}{\sigma-1}} z_i$.

If $\sigma < 1$, there is no sufficient input. Indeed, $\lim_{z_i \to +\infty} F(Z) = A \left(\sum_{j\neq i} \alpha_j z_j^{1-\frac{1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}}$ is finite for all Z_{-i} in $\mathbb{R}^{n-1}_{++}$.

If $\sigma = 1$, $F(Z) = A \prod_j z_j^{\alpha_j}$ and all inputs are sufficient.

The following result characterizes sufficient input.

Theorem. *Let $\overline{W}_{-i}$ be a vector of input prices in $\mathbb{R}^{n-1}_{++}$. Then: $\lim_{w_i \to 0} \lambda(w_i, \overline{W}_{-i}) = 0$ if and only if input i is sufficient.*

Proof. Let i be a sufficient input. Let $\varepsilon > 0$, be given. We can always choose Z^{ε}_{-i} in $\mathbb{R}^{n-1}_{++}$ such that $\sum_{j\neq i} \overline{w}_j z^{\varepsilon}_j < \frac{\varepsilon}{2}$. By definition, there exists z^{ε}_i such that $F(z^{\varepsilon}_i, Z^{\varepsilon}_{-i}) \geq 1$. This implies

$$\lambda(w_i, \overline{W}_{-i}) \leq w_i z^{\varepsilon}_i + \sum_{j\neq i} \overline{w}_j z^{\varepsilon}_j < w_i z^{\varepsilon}_i + \frac{\varepsilon}{2}$$

Hence, for all $w_i < \frac{\varepsilon}{2 z^{\varepsilon}_i}$, $\lambda(w_i, \overline{W}_{-i}) < \varepsilon$. Thus, if input i is sufficient, then $\lim_{w_i \to 0} \lambda(w_i, \overline{W}_{-i}) = 0$.

Conversely, suppose that i is not sufficient. By definition, there exists Z^0_{-i} in $\mathbb{R}^{n-1}_{++}$, such that for all $z_i > 0$, $F(z_i, Z^0_{-i}) < 1$.

Since the infimum

$$\lambda(w_i, \overline{W}_{-i}) = \inf \left\{ w_i z_i + \sum_{j \neq i} \overline{w}_j z_j \ ; \ Z \geq 0, \ F(Z) \geq 1 \right\}$$

is realized for all $w_i > 0$ (see McFadden (1980)), there exists a vector $\widetilde{Z}$ (depending upon w_i) such that:

$$\lambda(w_i, \overline{W}_{-i}) = w_i \widetilde{z}_i + \sum_{j \neq i} \overline{w}_j \widetilde{z}_j$$

with $F(\widetilde{Z}) \geq 1$. Since, for all positive z_i, $F(z_i, Z^0_{-i}) < 1$, necessarily, for at least one index $j_0 \neq i$, $\widetilde{z}_{j_0} > z^0_{j_0}$ (indeed, if for all $j \neq i$, $\widetilde{z}_j \leq z^0_j$, then $F(\widetilde{Z}) < 1$, which is impossible). Hence, $\sum_{j \neq i} \overline{w}_j \widetilde{z}_j \geq \min_{j \neq i} \overline{w}_j z^0_j = b > 0$. Then, for all $w_i > 0$, $\lambda(w_i, \overline{W}_{-i}) \geq b$. So, $\inf_{w_i} \lambda(w_i, \overline{W}_{-i}) \geq b > 0$. Q.E.D.

Remark. If it is possible to produce only with the input i, in the sense that $F(1, 0) > 0$, then this input is sufficient. Indeed, $\sup_{z_i} F(z_i, Z_{-i}) \geq \sup_{z_i} F(z_i, 0) = \sup_{z_i} z_i F(1, 0) = +\infty$. However, as shown by the Cobb-Douglas production function, this condition is not necessary.

3. Conclusion

This note has presented a notion of sufficient input. Such an input is characterized by the property that the unit cost function goes to zero whenever its price goes to zero. The notion can be easily extended to a subset of coordinates of an input vector.

References

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