E C O N O M I C S   B U L L E T I N

Shapley−like values for interval bankruptcy games

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Abstract

In this paper interval bankruptcy games arising from bankruptcy situations with interval claims are introduced. For this class of cooperative games two (marginal−based) Shapley−like values are considered and the relation between them is studied.

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1 Introduction

In a classical bankruptcy situation one has to divide a certain amount of money (estate) among some people (claimants) who have individual claims on the estate, and the total claim is weakly larger than the estate (cf. Aumann and Maschler (1985), O'Neill (1982)).

This paper deals with a generalized version of classical bankruptcy situations where claimants are facing uncertainty regarding their effective rights and, as a result, individual claims can be rather expressed in the form of closed intervals, the estate being still weakly smaller than the sum of the lower claims. Some appealing procedures leading to efficient and reasonable solutions for division problems of this type are described in Branzei et al. (2002). Another extension of classical bankruptcy situations, namely bankruptcy situations with references, is treated in Pulido et al. (2002a, b) by introducing two classical coalitional games for which compromise values are considered.

In this paper we tackle bankruptcy situations with interval claims by introducing a new type of cooperative games, that we call interval bankruptcy games, for which two values based on the Shapley value (Shapley (1953)) are considered. To be more precise, our Shapley-like values use the formula of the Shapley value which is based on marginal vectors. However, other well-known formulae (and axiomatizations) for the Shapley value exist in the literature. We mention here Harsanyi's formula based on dividends (Harsanyi (1959)) and Owen's formula based on the multilinear extension of a TU-game (Owen (1972)). To have an idea about how much attention has been captured by this interesting solution concept, the reader is referred to Roth (1988).

The rest of the paper is organized as follows. In Section 2 bankruptcy problems with interval claims are introduced and the corresponding interval games are constructed. Section 3 describes two Shapley-like values for interval bankruptcy games and shows that they are related to each other via the set inclusion. We conclude in Section 4 with some remarks on further research.

2 Interval bankruptcy games

Let $N = \{1, \ldots, n\}$ be the set of claimants among which an estate E has to be divided. We denote by $I_i = [a_i, b_i]$ the claim interval of claimant $i \in N$, where a_i is the lower bound of the claim interval, while b_i is the upper bound.

Let $\Im$ be the family of closed intervals in $\Re_+$ and $\Im^N$ be the set of all vectors of the form $I = (I_1, \ldots, I_n)$. A *bankruptcy problem with interval claims* is a pair (E, I), where $0 < E \leq \sum_{i \in N} a_i$. We denote by $\Im\mathcal{D}^N$ the set of all bankruptcy problems of the form (E, I).

Note that if all claim intervals I_i, $i \in N$ are degenerated intervals, i.e. $I_i = [a_i, a_i]$, the problem (E, I) coincides with the classical bankruptcy problem (E, a) with $a = (a_1, \ldots, a_n)$ and $0 < E \leq \sum_{i \in N} a_i$. Moreover, all division problems on N with sharp claims w.r.t. the available amount E, of the form (E, d) with $d = (d_1, \ldots, d_n)$ and $0 < E \leq \sum_{i \in N} a_i \leq \sum_{i \in N} d_i$ appear as particular cases of a problem $(E, I) \in \Im\mathcal{D}^N$. In the following we use the notation $\mathcal{D}^N$ to refer to the family of classical division problems related to a division problem with interval claims.

To each problem in $\Im\mathcal{D}^N$ one can associate a set of problems in $\mathcal{D}^N$ based on the idea to compromise the interval claims of each player $i \in N$ by weighting the upper bound of his claim interval with $t_i \in [0, 1]$ and the lower bound with $(1 - t_i)$.

Let $I = (I_1, \ldots, I_n)$ be the vector of interval claims in the problem $(E, I) \in \Im\mathcal{D}^N$ and $t = (t_1, t_2, \ldots, t_n)$, where $t_i \in [0, 1]$ for each $i \in N$. We introduce the *t-compromise claim* $c^t = \left(c_1^{t_1}, c_2^{t_2}, \ldots, c_n^{t_n}\right)$ by $c_i^{t_i} = t_i b_i + (1 - t_i)\, a_i$ for each $i \in N$. Given the amount E, for each t-compromise claim c^t, we can consider the division problem $(E, c^t) \in \mathcal{D}^N$, which we call the *$t$-compromise division problem*.

Now, to each standard (point claims) division problem $(E, c^t) \in \mathcal{D}^N$, one can associate a corresponding cooperative game as follows.

For $x \in \Re$, denote $x_+ = \max(x, 0)$. The associated division game $\langle N, v_{E,c^t} \rangle$ has the characteristic function $v_{E,c^t} : 2^N \to \Re$ where

$$v_{E,c^t}(S) = \left(E - \sum_{i \in N \setminus S} c_i^{t_i}\right)_+ \quad \text{for each } S \in 2^N, \tag{1}$$

i.e. $v_{E,c^t}(S)$ is the amount of the estate that is left if all players outside coalition S receive their claim (cf. Curiel et al. (1987)). The set of all division games $\langle N, v_{E,c^t} \rangle$ is denoted by $\mathcal{F}^N$.

Let $c^1 = (c_1^1, c_2^1, \ldots, c_n^1)$ and $c^0 = (c_1^0, c_2^0, \ldots, c_n^0)$. Now we construct the *cooperative interval game* $\langle N, v_{E,I} \rangle$ that corresponds to a division problem with interval claims $(E, I) \in \Im\mathcal{D}^N$. This game has the characteristic function

$v_{E,I} : 2^N \to \Im$ where, for each $S \in 2^N$, $v_{E,I}(S) = \left\{ v_{E,c^t}(S) \mid t \in [0,1]^N \right\}$ or, equivalently,

$$v_{E,I}(S) = \left[\left(E - \sum_{i \in N \setminus S} b_i \right)_+ , \left(E - \sum_{i \in N \setminus S} a_i \right)_+ \right] = \left[v_{E,c^1}(S), v_{E,c^0}(S) \right]. \tag{2}$$

So, in an interval bankruptcy game each coalition S can receive an amount which is between the amount of the estate that is left if all players outside coalition S receive their upper claim and the amount of the estate that is left if all players outside coalition S receive their lower claim. The set of all cooperative interval games $\langle N, v_{E,I} \rangle$ is denoted by $\mathcal{G}^N$.

3 Two Shapley-like values

In this section we introduce two Shapley-like values for n-person interval bankruptcy games. The *first* Shapley-like value is an indirect one in the sense that it is built on the standard (marginal-based) Shapley value for each cooperative game corresponding to a t-compromise division problem. The *second* Shapley-like value is constructed directly on an interval bankruptcy game and it associates to each such a game one vector of intervals in $\Im^N$. As proved in Proposition 1 the intervals generated via the indirect value are always included in the corresponding intervals generated via the direct value. Moreover, Example 2 shows that these two values are different.

We start by introducing for games $\langle N, v_{E,c^t} \rangle \in \mathcal{F}^N$ and permutations $\sigma : N \to N$ marginal vectors $m^\sigma(v_{E,c^t})$. Let

$$P_\sigma(i) = \left\{ r \in N \mid \sigma^{-1}(r) < \sigma^{-1}(i) \right\}$$

be the set of predecessors of i in σ. Then

$$m_i^\sigma(v_{E,c^t}) = v_{E,c^t}(P_\sigma(i) \cup \{i\}) - v_{E,c^t}(P_\sigma(i))$$

for each $i \in N$. The Shapley value $\varphi(v_{E,c^t})$ is equal to the average of the marginal vectors (cf. Shapley (1953))

$$\varphi_i(v_{E,c^t}) = \frac{1}{n!} \sum_{\sigma \in \pi(N)} m_i^\sigma(v_{E,c^t}) \text{ for all } i \in N,$$

where $\pi(N)$ is the set of all permutations π on N.

We introduce the *indirect Shapley-like value* $\varphi(v_{E,I})$ for an interval bankruptcy game $\langle N, v_{E,I}\rangle \in \mathcal{G}^N$ as having as i-th component the set

$$\begin{aligned}\varphi_i(v_{E,I}) &= \left\{\varphi_i(v_{E,c^t}) \mid t \in [0,1]^N\right\} \\ &= \left[\min\left(\varphi_i(v_{E,c^0}), \varphi_i(v_{E,c^1})\right), \max\left(\varphi_i(v_{E,c^0}), \varphi_i(v_{E,c^1})\right)\right]\end{aligned}$$

for all $i \in N$.

The second Shapley-like value is based on the marginal vectors $m_i^\sigma(v_{E,I})$ constructed directly on games $\langle N, v_{E,I}\rangle \in \mathcal{G}^N$, i.e.

$$m_i^\sigma(v_{E,I}) = v_{E,I}(P_\sigma(i) \cup \{i\}) - v_{E,I}(P_\sigma(i)) \tag{3}$$

for each $i \in N$.

Note that by using (2) formula (3) can be rewritten as

$$\begin{aligned}m_i^\sigma(v_{E,I}) = {} & \left[v_{E,c^1}(P_\sigma(i) \cup \{i\}), v_{E,c^0}(P_\sigma(i) \cup \{i\})\right] - \\ & \left[v_{E,c^1}(P_\sigma(i)), v_{E,c^0}(P_\sigma(i))\right]\end{aligned} \tag{4}$$

for each $i \in N$.

The *direct Shapley-like value* $\Phi(v_{E,I})$ of a game $\langle N, v_{E,I}\rangle \in \mathcal{G}^N$ is the average of the marginal vectors of the game, i.e.

$$\Phi(v_{E,I}) = \frac{1}{n!}\sum_{\sigma\in\pi(N)} m^\sigma(v_{E,I}).$$

By using (4) this formula can be rewritten (cf. Moore (1979)) as

$$\Phi(v_{E,I}) = \left[\underline{\Phi(v_{E,I})}, \overline{\Phi(v_{E,I})}\right],$$

where

$$\begin{aligned}\underline{\Phi(v_{E,I})} &= \frac{1}{n!}\sum_{\sigma\in\pi(N)}\left(v_{E,c^1}(P_\sigma(i) \cup \{i\}) - v_{E,c^0}(P_\sigma(i))\right), \\ \overline{\Phi(v_{E,I})} &= \frac{1}{n!}\sum_{\sigma\in\pi(N)}\left(v_{E,c^0}(P_\sigma(i) \cup \{i\}) - v_{E,c^1}(P_\sigma(i))\right).\end{aligned}$$

Proposition 1 *Let* $\langle N, v_{E,I}\rangle \in \mathcal{G}^N$*. Then*

$$\varphi_i\left(v_{E,I}\right) \subset \Phi_i\left(v_{E,I}\right) \text{ for all } i \in N.$$

Proof. Let $i \in N$ and take a $t \in [0,1]^N$. By using (1), we have

$$\begin{aligned} & v_{E,c^t}\left(P_\sigma(i) \cup \{i\}\right) - v_{E,c^t}\left(P_\sigma(i)\right) \\ \geq\ & v_{E,c^1}\left(P_\sigma(i) \cup \{i\}\right) - v_{E,c^t}\left(P_\sigma(i)\right) \\ \geq\ & v_{E,c^1}\left(P_\sigma(i) \cup \{i\}\right) - v_{E,c^0}\left(P_\sigma(i)\right). \end{aligned}$$

Then

$$\begin{aligned} & \frac{1}{n!} \sum_{\sigma \in \pi(N)} \left(v_{E,c^t}\left(P_\sigma(i) \cup \{i\}\right) - v_{E,c^t}\left(P_\sigma(i)\right)\right) \\ \geq\ & \frac{1}{n!} \sum_{\sigma \in \pi(N)} \left(v_{E,c^1}\left(P_\sigma(i) \cup \{i\}\right) - v_{E,c^0}\left(P_\sigma(i)\right)\right), \end{aligned}$$

which is equivalent to

$$\varphi_i\left(v_{E,c^t}\right) \geq \underline{\Phi_i\left(v_{E,I}\right).}$$

Similarly, one obtains

$$\varphi_i\left(v_{E,c^t}\right) \leq \overline{\Phi_i\left(v_{E,I}\right)}.$$

The last two inequalities imply $\varphi_i\left(v_{E,I}\right) \subset \Phi_i\left(v_{E,I}\right)$. ∎

Example 2 *Let* $N = \{1,2\}$*,* $E = 10$*,* $I = (I_1, I_2) = ([4,6], [8,9])$*. So, the division problem with interval claims is* $(10, [4,6], [8,9])$ *and the characteristic function of the corresponding cooperative interval game (see (2)) is as follows:*

$$\begin{aligned} v_{E,I}(\emptyset) &= [0,0], \\ v_{E,I}(\{1\}) &= [1,2], \\ v_{E,I}(\{2\}) &= [4,6], \\ v_{E,I}(\{1,2\}) &= [10,10]. \end{aligned}$$

For the marginal vectors of the corresponding point-claims games we have

$$\begin{aligned} m^{(12)}\left(v_{E,c^0}\right) &= m^{(12)}\left(v_{10,(4,8)}\right) = (2,8), \\ m^{(21)}\left(v_{E,c^0}\right) &= m^{(21)}\left(v_{10,(4,8)}\right) = (4,6), \\ m^{(12)}\left(v_{E,c^1}\right) &= m^{(12)}\left(v_{10,(6,9)}\right) = (1,9), \\ m^{(21)}\left(v_{E,c^1}\right) &= m^{(21)}\left(v_{10,(6,9)}\right) = (6,4), \end{aligned}$$

and the marginal vectors of the interval game are

$$\begin{aligned} m^{(12)}\left(v_{10,([4,6],[8,9])}\right) &= ([1,2]\,,[10,10]-[1,2]) = ([1,2]\,,[8,9])\,, \\ m^{(21)}\left(v_{10,([4,6],[8,9])}\right) &= ([10,10]-[4,6]\,,[4,6]) = ([4,6]\,,[4,6])\,. \end{aligned}$$

The two Shapley-like values are then

$$\begin{aligned} \varphi\left(v_{10,([4,6],[8,9])}\right) &= \left(\left[\varphi_1\left(v_{10,(4,8)}\right),\varphi_1\left(v_{10,(6,9)}\right)\right],\left[\varphi_2\left(v_{10,(6,9)}\right),\varphi_2\left(v_{10,(4,8)}\right)\right]\right) \\ &= \left(\left[\frac{1}{2}(2+4)\,,\frac{1}{2}(1+6)\right],\left[\frac{1}{2}(9+4)\,,\frac{1}{2}(8+6)\right]\right) \\ &= ([3,3.5]\,,[6.5,7]) \end{aligned}$$

and

$$\begin{aligned} \Phi\left(v_{10,([4,6],[8,9])}\right) &= \left(\frac{1}{2}([1,2]+[4,6])\,,\frac{1}{2}([8,9]+[4,6])\right) \\ &= ([2.5,4]\,,[6,7.5])\,. \end{aligned}$$

It is easy to see that for each $i\in\{1,2\}$, $\varphi_i\left(v_{10,([4,6],[8,9])}\right)$ *is strictly included in* $\Phi_i\left(v_{10,([4,6],[8,9])}\right)$ *and, hence, different from it.*

4 Final remarks

The Shapley value has been inspiring for constructing Shapley-like values on different classes of cooperative games (cf., for example, Timmer et al. (2002)) where different formulae for the Shapley value are considered. It could be a topic of further research to apply also different formulae of the Shapley value on interval bankruptcy games and to study their interrelations. Another possibility is connected with the fact that uncertainty is just as likely to affect the estate as to affect claims. Hence, one could assume that the estate is also expressed in the form of a closed interval, and to analyze the connections among the corresponding point claims and interval games.

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