E C O N O M I C S   B U L L E T I N

# Reexamination of the A−J effect

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## *Abstract*

We establish four necessary and sufficient conditions for the existence of the Averch−Johnson effect in a generalized version of their famous model of the rate−of−return regulated firm. The four necessary and sufficient conditions are then compared to the two stronger sufficient conditions for the Averch−Johnson effect found in the literature. Our analysis also permits us to put to rest a somewhat protracted debate about the very existence of the Averch−Johnson effect.

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## 1. Introduction

Since the publication of the seminal paper of Averch and Johnson (1962) modeling the behavior of a firm operating under rate-of-return regulation, numerous researchers have sought to refine and extend the properties of the model, among them Takayama (1969, 1993), Baumol and Klevorick (1970), Zajac (1970, 1972), and El-Hodiri and Takayama (1973). At one juncture, even a somewhat protracted debate took place between Takayama (1969) and El-Hodiri and Takayama (1973) on the one hand, and Pressman and Carol (1971, 1973) on the other, concerning the very existence of the so-called Averch-Johnson effect (hereafter A-J effect), defined as a negative sign for the partial derivative of the optimal value of the regulated input with respect to the rate-of-return parameter. This note puts an end to that debate once and for all. We accomplish this by proving, subject to three basic assumptions on the model, that (i) the optimal value of the Lagrange multiplier corresponding to the rate-of-return constraint does not necessarily lie in the positive unit interval, (ii) the A-J effect cannot be unequivocally derived from the assertion of maximizing behavior alone, and (iii) the A-J effect is equivalent to four economically intuitive conditions emanating from the model. Moreover, we derive these results using a more general model than has typically been contemplated in the literature. Finally, we relate the four necessary and sufficient conditions for the existence of the A-J effect to the two stronger sufficient conditions that are prevalent in the literature.

## 2. The Value of the Lagrange Multiplier

The Averch and Johnson (1962) model of the rate-of-return regulated firm is so well known that we can be relatively succinct in presenting our slight generalization of it. Consider, therefore, a profit maximizing monopolist producing one homogeneous good, say $y \in \Re_+$, employing $M$ factors of production, say $\mathbf{x} \in \Re_+^M$, which are purchased in competitive factor markets at prices $\mathbf{w} \in \Re_{++}^M$. The production function $f(\cdot):\Re_+^M \to \Re_+$ is assumed to be a $C^{(2)}$ function on $\Re_{++}^M$. The monopolist faces the inverse demand function $P(\cdot):\Re_+ \to \Re_+$, which is also assumed to be a $C^{(2)}$ function on $\Re_{++}$. The revenue function $R(\cdot):\Re_+^M \to \Re_+$ of the monopolist is defined by $R(\mathbf{x}) \stackrel{\text{def}}{=} P(f(\mathbf{x}))f(\mathbf{x})$, which given our prior assumptions concerning differentiability, is similarly a $C^{(2)}$ function on $\Re_{++}^M$. The regulatory constraint is the archetype one in that the rate-of-return on capital is not allowed to exceed some "fair" value determined by the regulatory agency. Defining $x_M$ as the capital of the firm, or equivalently, the regulated input, we may write the generalized rate-of-return regulated profit maximizing model of the firm as

$$\pi^*(\mathbf{w},s) \stackrel{\text{def}}{=} \max_{\mathbf{x}} \left\{ R(\mathbf{x}) - \sum_{m=1}^{M} w_m x_m \ \text{ s.t. } \ R(\mathbf{x}) - \sum_{m=1}^{M} w_m x_m \le (s - w_M)x_M \right\}, \tag{1}$$

where $s \in \Re_{++}$ is the fair rate-of-return on capital. Hereafter, we refer to the constrained optimization problem (1) as the generalized A-J model.

We impose and maintain the following three standard assumptions on problem (1) throughout the note:

(A.1) $s > w_M$.

(A.2) For each $\mathbf{a} \stackrel{\text{def}}{=} (\mathbf{w},s) \in A \subset \Re_{++}^{M+1}$ there exists an interior $C^{(1)}$ solution to problem (1), which we denote by $\mathbf{x} = \mathbf{x}^*(\mathbf{a})$, with $\lambda = \lambda^*(\mathbf{a})$ being the optimal value of the Lagrange multiplier corresponding to the rate-of-return constraint.

(A.3) The rate-of-return constraint is binding, and *not* just binding, at $\mathbf{x}=\mathbf{x}^*(\mathbf{a})$.

Assumption (A.1) is basic in the literature and fundamental to problem (1). It amounts to asserting that the fair rate-of-return on capital is larger than the rental price of one unit of capital, thereby permitting the monopolist to earn a positive profit. It is important to recognize that since the rate-of-return constraint is assumed to be binding (and not just binding) in assumption (A.3), the solution to problem (1) *does not* coincide with the unconstrained solution of problem (1), which would be found by ignoring the rate-of-return constraint. It is also worthwhile to note that in contrast to the existing literature, we have not made any assumptions about the slope of the inverse demand function, the slope of the revenue function with respect to the factors of production or output, the marginal products of the inputs, or the curvature of the production function, nor are we limiting ourselves to two factors of production.

The Lagrangian function for problem (1) is defined as

$$L(\mathbf{x},\lambda;\mathbf{w},s) \stackrel{\text{def}}{=} R(\mathbf{x})-\sum_{m=1}^{M} w_m x_m+\lambda\left[\sum_{j=1}^{M-1} w_j x_j+s x_M-R(\mathbf{x})\right]. \tag{2}$$

To verify the classical rank condition on the constraint function that is prerequisite for the Lagrangian approach, we define

$$g(\mathbf{x};\mathbf{a}) \stackrel{\text{def}}{=} \sum_{j=1}^{M-1} w_j x_j+s x_M-R(\mathbf{x}). \tag{3}$$

The gradient of $g(\cdot)$ with respect to $\mathbf{x}$ is given by

$$\nabla_{\mathbf{x}}\, g(\mathbf{x};\mathbf{a})=\left(w_1-\frac{\partial R}{\partial x_1}(\mathbf{x}), w_2-\frac{\partial R}{\partial x_2}(\mathbf{x}),\ldots, w_{M-1}-\frac{\partial R}{\partial x_{M-1}}(\mathbf{x}), s-\frac{\partial R}{\partial x_M}(\mathbf{x})\right).$$

As long as $\nabla_{\mathbf{x}}\, g(\mathbf{x}^*(\mathbf{a});\mathbf{a})\neq \mathbf{0}_M$, where $\mathbf{0}_M$ is the null vector in $\Re^M$, then Theorem 2.3 of Takayama (1993) implies that the first-order necessary conditions of problem (1) are given by

$$\frac{\partial L}{\partial x_i}=\left[\frac{\partial R}{\partial x_i}(\mathbf{x})-w_i\right](1-\lambda)=0,\quad i=1,2,\ldots,M-1, \tag{4}$$

$$\frac{\partial L}{\partial x_M}=\frac{\partial R}{\partial x_M}(\mathbf{x})-w_M-\lambda\left[\frac{\partial R}{\partial x_M}(\mathbf{x})-s\right]=0, \tag{5}$$

$$\frac{\partial L}{\partial \lambda}=\sum_{j=1}^{M-1} w_j x_j+s x_M-R(\mathbf{x})\geq 0,\ \lambda\geq 0,\ \frac{\partial L}{\partial \lambda}\cdot\lambda=0. \tag{6}$$

Recall that with Eqs. (4)–(6) as necessary conditions, we have $\mathbf{x}=\mathbf{x}^*(\mathbf{a})$ and $\lambda=\lambda^*(\mathbf{a})$ as their simultaneous solutions for each $\mathbf{a}\in A$.

The first thing to note about the solution to problem (1) is that $\lambda^*(\mathbf{a})>0$. To see this, assume to the contrary that $\lambda^*(\mathbf{a})=0$. Then the first-order necessary conditions (4) and (5) would be identical to the first-order necessary conditions for the unconstrained version of problem (1), thereby implying that the solutions to the constrained and unconstrained versions of problem (1) are identical. But this contradicts the implication of assumption (A.3) that the rate-of-return constraint is binding (and not just binding), namely, that the solutions to the constrained and unconstrained versions of problem (1) differ. Therefore, $\lambda^*(\mathbf{a})>0$. We can similarly show that $\lambda^*(\mathbf{a})\neq 1$. To this end, assume to the contrary that $\lambda^*(\mathbf{a})=1$, so that Eq. (5) reduces to $s=w_M$. But this contradicts assumption (A.1), scilicet that $s>w_M$, thereby implying that $\lambda^*(\mathbf{a})\neq 1$. Furthermore, note that because $s>w_M$ by assumption (A.1), it follows from Eq. (5) that if

$\partial R(\mathbf{x}^*(\mathbf{a}))/\partial x_M > s$, then $\lambda^*(\mathbf{a}) > 1$, whereas if $\partial R(\mathbf{x}^*(\mathbf{a}))/\partial x_M < w_M$, then $\lambda^*(\mathbf{a}) \in (0,1)$. Note also that the case $w_M < \partial R(\mathbf{x}^*(\mathbf{a}))/\partial x_M < s$ is ruled out by the condition $\lambda^*(\mathbf{a}) > 0$. In sum, we have shown that imposing only assumptions (A.1)–(A.3) on problem (1), we can only conclude that $0 < \lambda^*(\mathbf{a}) \neq 1$. We state this important conclusion in the ensuing lemma.

**Lemma 1.** *Under assumptions (A.1)–(A.3), the optimal value of the Lagrange multiplier corresponding to the rate-of-return constraint in the generalized A-J model defined by Eq.* (1) *et. seq. obeys* $0 < \lambda^*(\mathbf{a}) \neq 1$.

Since $\lambda^*(\mathbf{a}) \neq 1$, the $M-1$ first-order necessary conditions in Eq. (4) reduce to

$$\frac{\partial R}{\partial x_i}(\mathbf{x}) - w_i = 0, \quad i = 1, 2, \ldots, M-1, \tag{7}$$

exactly the same as the first $M-1$ first-order necessary conditions in the unconstrained version of problem (1). The first-order necessary condition (5) can be rewritten as

$$\frac{\partial R}{\partial x_M}(\mathbf{x}) - w_M = \lambda\left[\frac{\partial R}{\partial x_M}(\mathbf{x}) - s\right]. \tag{8}$$

Because $\mathbf{x} = \mathbf{x}^*(\mathbf{a})$ is not a solution to the unconstrained version of problem (1) and must satisfy the $M-1$ first-order necessary conditions (7), it follows that $\partial R(\mathbf{x}^*(\mathbf{a}))/\partial x_M - w_M \neq 0$. To see this, simply note that if $\partial R(\mathbf{x}^*(\mathbf{a}))/\partial x_M - w_M = 0$, then the solutions to the constrained and unconstrained versions of problem (1) would be identical, thereby violating assumption (A.3). Since $\lambda^*(\mathbf{a}) \neq 0$ from Lemma 1, it follows from Eq. (8) that $\partial R(\mathbf{x}^*(\mathbf{a}))/\partial x_M - w_M \neq 0$ if and only if $\partial R(\mathbf{x}^*(\mathbf{a}))/\partial x_M - s \neq 0$. Thus our assumption above that $\nabla_{\mathbf{x}}\, g(\mathbf{x}^*(\mathbf{a});\mathbf{a}) \neq \mathbf{0}_M$ is in fact valid, since the first $M-1$ components of $\nabla_{\mathbf{x}}\, g(\mathbf{x}^*(\mathbf{a});\mathbf{a})$ are equal to zero by Eq. (7), while its *M*th component is nonzero as we have just demonstrated. This justifies our use of the Lagrangian approach in characterizing the generalized A-J model defined in Eq. (1).

## 3. The A-J Effect

With the exception of McNicol (1973), the principal focus of the existing literature, at least as far as comparative statics are concerned, has been on determining the sign of $\partial x_M^*(\mathbf{a})/\partial s$. Takayama (1969, pp. 259–260) was the first to rigorously demonstrate that $\partial x_M^*(\mathbf{a})/\partial s < 0$ for the two factor case, albeit under stronger assumptions than used here, a point we shall return to in section 4. The economic interpretation of this comparative statics result is that a decrease in the allowable rate-of-return on capital induces the firm to raise its capital stock, a conclusion which is now referred to as the A-J effect.

The principal reason why additional ad hoc assumptions beyond the maximization assertion and assumptions (A.1)–(A.3) are required in order to sign $\partial x_M^*(\mathbf{a})/\partial s$ is due to the assertion by Silberberg (1990, p. 202) that " ... , no refutable hypotheses are implied by the maximization hypothesis alone, for parameters entering the constraint." In other words, Silberberg (1990) has shown that when parameters enter the constraints of an optimization problem one will not, in general, be able to sign *individual* partial derivatives of the decision variables with respect to such parameters under the maximization assertion alone, such as the A-J effect derivative $\partial x_M^*(\mathbf{a})/\partial s$. The implication of Silberberg's (1990, p. 202) analysis, therefore, is that additional assumptions beyond (A.1)–(A.3) must be imposed on problem (1) that are not implied by the maximization assertion in order to unequivocally demonstrate that $\partial x_M^*(\mathbf{a})/\partial s < 0$, i.e., in order to establish the A-J effect. To see that the A-J effect may fail to hold under assumptions (A.1)–(A.3), i.e., that $\partial x_M^*(\mathbf{a})/\partial s > 0$ can occur, let $R(x_1, x_2) \overset{\text{def}}{=} \frac{1}{2}x_1^2 + \alpha x_2$, where $\alpha > s$. One can then show that $x_1^*(\mathbf{a}) = w_1$ and $x_2^*(\mathbf{a}) = \frac{1}{2}w_1^2[\alpha - s]^{-1}$ are the unique, globally optimal interior solutions to the generalized A-J problem (1), where $\lambda^*(\mathbf{a}) = [\alpha - w_2][\alpha - s]^{-1} > 1$. It is then straightforward to see that the A-J effect fails in this case since $\partial x_2^*(\mathbf{a})/\partial s = \frac{1}{2}w_1^2[\alpha - s]^{-2} > 0$. In passing, we remark that even though this observation of Silberberg's (1990) is true in general, it does not rule out interesting and important comparative statics results in optimization problems which have parameters in the constraints, as is manifest in the prototype utility maximization problem.

## 4. Four Necessary and Sufficient Conditions for the A-J Effect

Since one cannot unambiguously establish that $\partial x_M^*(\mathbf{a})/\partial s < 0$ based solely on the maximization assertion and assumptions (A.1)–(A.3), additional assumptions must be imposed on problem (1) in order to do so. The auxiliary assumptions contemplated in the literature include (i) the optimal value of the Lagrange multiplier lies between zero and unity [Takayama (1969, p. 257)], (ii) the revenue function is concave in capital and labor [Baumol and Klevorick (1970, p. 167) and El-Hodiri and Takayama (1973, p. 236)], (iii) the marginal products of capital and labor are positive, the isoquants are strictly convex to the origin, and the marginal product of labor increases with increases in capital along an isoquant [McNicol (1973, pp. 430–431)], (iv) the slope of the revenue function with respect to output is positive [Takayama (1993, p. 215)], and (v) the feasible set of the complementary problem (13) defined below is closed and bounded, i.e., it is compact [Zajac (1972, p. 129)]. None of these assumptions is implied by the constrained maximization hypothesis, but each is either sufficient or necessary and sufficient for proving that $\partial x_M^*(\mathbf{a})/\partial s < 0$, as we now proceed to demonstrate.

To begin, substitute $\mathbf{x} = \mathbf{x}^*(\mathbf{a})$ into Eq. (6), use the fact that $\lambda^*(\mathbf{a}) > 0$ by Lemma 1, differentiate the resulting identity with respect to $s$, and then simplify the result using the first-order necessary conditions in Eq. (7) to get

$$\frac{\partial x_M^*(\mathbf{a})}{\partial s} = \frac{x_M^*(\mathbf{a})}{\partial R(\mathbf{x}^*(\mathbf{a}))/\partial x_M - s}.$$

Since $x_M^*(\mathbf{a}) > 0$ by assumption (A.2), it follows from the foregoing comparative statics expression that

$$\frac{\partial x_M^*(\mathbf{a})}{\partial s} < 0 \iff \frac{\partial R(\mathbf{x}^*(\mathbf{a}))}{\partial x_M} - s < 0. \tag{9}$$

This conclusion asserts that the capital stock of the firm increases with a decrease in the fair rate-of-return, if and only if, the marginal revenue product of the capital stock evaluated at the optimal solution is less than the fair rate-of-return.

Next observe that the relation $(1 - \lambda^*(\mathbf{a})) = -(s - w_M)/[\partial R(\mathbf{x}^*(\mathbf{a}))/\partial x_M - s]$ follows from Eq. (8). Furthermore, since $s > w_M$ by assumption (A.1) and $\lambda^*(\mathbf{a}) > 0$ by Lemma 1, the previous equation implies that

$$\lambda^*(\mathbf{a}) \in (0,1) \iff \frac{\partial R(\mathbf{x}^*(\mathbf{a}))}{\partial x_M} - s < 0. \tag{10}$$

An inspection of Eqs. (9) and (10) shows that $\lambda^*(\mathbf{a}) \in (0,1) \Leftrightarrow \partial x_M^*(\mathbf{a})/\partial s < 0$, i.e., the optimal value of the Lagrange multiplier lying inside the positive unit interval is equivalent to the existence of the A-J effect.

We can now appeal to the second-order necessary condition

$$\sum_{k=1}^{M}\sum_{\ell=1}^{M}\frac{\partial^2 L}{\partial x_k \partial x_\ell}\left(\mathbf{x}^*(\mathbf{a}),\lambda^*(\mathbf{a});\mathbf{a}\right)h_k h_\ell \le 0\ \forall\,\mathbf{h}\in\Re^M \ni \sum_{m=1}^{M}\frac{\partial g}{\partial x_m}\left(\mathbf{x}^*(\mathbf{a});\mathbf{a}\right)h_m = 0$$

of problem (1) to arrive at another intuitive condition equivalent to $\lambda^*(\mathbf{a})\in(0,1)$. Using Eq. (3), which defines the constraint function $g(\cdot)$, and the first-order necessary conditions in Eq. (7), we concluded in section 2 that

$$\frac{\partial g}{\partial x_i}\left(\mathbf{x}^*(\mathbf{a});\mathbf{a}\right) = w_i - \frac{\partial R}{\partial x_i}\left(\mathbf{x}^*(\mathbf{a})\right) \equiv 0,\ i = 1,2,\ldots,M-1,$$

and also that

$$\frac{\partial g}{\partial x_M}\left(\mathbf{x}^*(\mathbf{a});\mathbf{a}\right) = s - \frac{\partial R}{\partial x_M}\left(\mathbf{x}^*(\mathbf{a})\right) \ne 0.$$

Taken together, these two results imply that

$$\sum_{m=1}^{M}\frac{\partial g}{\partial x_m}\left(\mathbf{x}^*(\mathbf{a});\mathbf{a}\right)h_m = \left[s - \frac{\partial R}{\partial x_M}\left(\mathbf{x}^*(\mathbf{a})\right)\right]h_M = 0,$$

which in turn implies that $h_M = 0$. Noting that $\partial^2 L/\partial x_k \partial x_\ell = (1-\lambda)[\partial^2 R/\partial x_k \partial x_\ell]$ from the definition of the Lagrangian function in Eq. (2), and using $h_M = 0$, we can reduce the above second-order necessary condition to

$$\left(1-\lambda^*(\mathbf{a})\right)\sum_{i=1}^{M-1}\sum_{j=1}^{M-1}\frac{\partial^2 R}{\partial x_i \partial x_j}\left(\mathbf{x}^*(\mathbf{a})\right)h_i h_j \le 0\ \forall\,(h_1,h_2,\ldots,h_{M-1})\in\Re^{M-1}.$$

Since $\lambda^*(\mathbf{a}) > 0$ by Lemma 1, we can immediately conclude from the preceding quadratic form that

$$\lambda^*(\mathbf{a})\in(0,1)\ \Leftrightarrow\ \sum_{i=1}^{M-1}\sum_{j=1}^{M-1}\frac{\partial^2 R}{\partial x_i \partial x_j}\left(\mathbf{x}^*(\mathbf{a})\right)h_i h_j \le 0\ \forall\,(h_1,h_2,\ldots,h_{M-1})\in\Re^{M-1}. \tag{11}$$

That is, an optimal value of the Lagrange multiplier that lies inside the positive unit interval is equivalent to the local concavity of the revenue function in the first $M-1$ inputs (excluding the capital stock).

Similarly, an inspection of Eqs. (9), (10), and (11) yields

$$\frac{\partial x_M^*(\mathbf{a})}{\partial s} < 0\ \Leftrightarrow\ \sum_{i=1}^{M-1}\sum_{j=1}^{M-1}\frac{\partial^2 R}{\partial x_i \partial x_j}\left(\mathbf{x}^*(\mathbf{a})\right)h_i h_j \le 0\ \forall\,(h_1,h_2,\ldots,h_{M-1})\in\Re^{M-1}. \tag{12}$$

This conclusion asserts that the A-J effect is equivalent to the local concavity of the revenue function in the first $M-1$ inputs (excluding the capital stock). Thus Eqs. (9), (10), (11), and (12) establish an *equivalence* between the A-J effect, the range of the optimal value of the Lagrange multiplier, the marginal revenue product of the capital stock evaluated at the optimal solution relative to the fair rate-of-return, and the local concavity of the revenue function in all the inputs but the capital stock. This is the central result of our note. We therefore summarize it in the following lemma.

**Lemma 2.** *Under assumptions (A.1)–(A.3), the ensuing four conditions are equivalent in the generalized A-J model defined by Eq.* (1) *et. seq.:*

$$\frac{\partial x_M^*(\mathbf{a})}{\partial s} < 0,$$

$$\lambda^*(\mathbf{a}) \in (0,1),$$
$$\frac{\partial R\left(\mathbf{x}^*(\mathbf{a})\right)}{\partial x_M} - s < 0,$$
$$\sum_{i=1}^{M-1}\sum_{j=1}^{M-1}\frac{\partial^2 R}{\partial x_i \partial x_j}\left(\mathbf{x}^*(\mathbf{a})\right)h_i h_j \le 0 \;\forall\,(h_1,h_2,\ldots,h_{M-1}) \in \Re^{M-1}.$$

Lemma 2 shows that the assumed global concavity of the revenue function in all the inputs by Baumol and Klevorick (1970, p. 167) and El-Hodiri and Takayama (1973, p. 236), although sufficient, is certainly not necessary for the A-J effect. Just as important is the fact that Lemma 2 lays to rest the rather protracted controversy concerning the existence of the A-J effect between Takayama (1969) and El-Hodiri and Takayama (1973) on the one hand, and Pressman and Carol (1971, 1973) on the other, and does so with a more general model.

The next matter to elucidate pertains to the following statement by Takayama (1993, p. 215): "The statement of $0 < \lambda^*(\mathbf{a}) < 1$ shown above has been debated in the literature. Averch and Johnson's (1962, p. 1056) proof amounts to assuming it. El-Hodiri and Takayama (1973) showed that this assumption can be dispensed with if (the revenue) function $G$ [ $R(\cdot)$ in our notation] is concave. What we have shown above is that such a concavity assumption can be dispensed with." This is a misleading statement, as we now proceed to show. Takayama (1993, p. 215) had to assume that the slope of the revenue function with respect to output is positive at the optimum, i.e., marginal revenue is positive at the optimal solution, in order to prove that $\lambda^*(\mathbf{a}) \in (0,1)$. However, this assumption is equivalent to $\lambda^*(\mathbf{a}) \in (0,1)$, as can be seen by using Takayama's (1993, p. 215) result that $(1-\lambda^*(\mathbf{a}))MR(y^*(\mathbf{a})) > 0$, where $MR(y^*(\mathbf{a}))$ is the value of marginal revenue at the optimal output level $y^*(\mathbf{a})$. Since $\lambda^*(\mathbf{a}) > 0$ by Lemma 1, the strict inequality $(1-\lambda^*(\mathbf{a}))MR(y^*(\mathbf{a})) > 0$ implies that $\lambda^*(\mathbf{a}) \in (0,1) \Leftrightarrow MR(y^*(\mathbf{a})) > 0$. Therefore, assuming that $MR(y^*(\mathbf{a})) > 0$, as Takayama (1993, p. 215) does, is fully equivalent to assuming that $\lambda^*(\mathbf{a}) \in (0,1)$, hence by our Lemma 2 also equivalent to assuming that the revenue function is locally concave in all the inputs but the capital stock, or that the marginal revenue product of the capital stock evaluated at the optimal solution is less than the fair rate-of-return, or that the A-J effect holds. Thus the assumption that $MR(y^*(\mathbf{a})) > 0$ is the fourth necessary and sufficient condition for the A-J effect. In sum, we have shown that while it is true that in order to derive the A-J effect "the concavity assumption can be dispensed with," the price paid in order to do so is an assumption that is implied by the one discarded and fully equivalent to the A-J effect itself. Note, in passing, that Takayama (1993) could have equivalently used his assumption that the marginal product of labor is positive at the optimal solution to reach his misleading conclusion in the above quote.

We close this section by relating the work of Zajac (1972) to the results presented here. In order to do so, we begin by writing down what Zajac (1972, p. 127) defines as the *complement* of problem (1), videlicet

$$\max_{\mathbf{x}}\left\{(s-w_M)x_M \;\text{ s.t. }\; (s-w_M)x_M \le R(\mathbf{x}) - \sum_{m=1}^{M} w_m x_m\right\}. \tag{13}$$

Zajac (1972 p. 129) shows that if one assumes that the feasible set of the complementary problem (13) is closed and bounded, i.e., that it is compact, then (i) the solution of problem (13) occurs where the constraint binds, (ii) the solution of problem (1) is identical to the solution of the complementary problem (13) and occurs where the constraint of problem (1) also binds, and (iii)

$\lambda^*(\mathbf{a}) \in (0,1)$. By Lemma 2, therefore, the assumption that the feasible set of the complementary problem (13) is compact is a sufficient condition for the existence of the A-J effect. It is important to recognize, however, that in general, the feasible set of problem (13) is neither closed nor bounded under assumptions (A.1)–(A.3). For example, when $R(x_1, x_2) \overset{\text{def}}{=} \frac{1}{2}x_1^2 + \alpha x_2$ and $\alpha > s$, the feasible set of the complementary problem (13) is unbounded and hence not compact. In passing, observe that the assumption of compactness of the feasible set of the complementary problem (13) is weaker than concavity of the revenue function, as noted by Zajac (1972, p. 130).

To put the Zajac (1972) analysis in perspective, recall that Lemma 1 shows that the optimal value of the Lagrange multiplier satisfies $0 < \lambda^*(\mathbf{a}) \neq 1$ under assumptions (A.1)–(A.3), *not* the stronger conclusion $\lambda^*(\mathbf{a}) \in (0,1)$, the latter being necessary and sufficient for the A-J effect by Lemma 2. Moreover, under the maximization assertion and assumptions (A.1)–(A.3), we have shown that one *cannot* unambiguously establish that $\partial x_M^*(\mathbf{a})/\partial s < 0$, thereby demonstrating that the A-J effect does not necessarily follow from the assertion of optimizing behavior and assumptions (A.1)–(A.3).

## 5. Conclusions

The Averch and Johnson (1962) model of the rate-of-return regulated profit maximizing firm has been in existence for over three decades, and a good deal of effort has been expended in order to fully understand its economic content and extend its reach. We have added to this literature by proving that subject to three basic assumptions on the model, that (i) the optimal value of the Lagrange multiplier corresponding to the rate-of-return constraint does not necessarily lie in the positive unit interval, (ii) the A-J effect cannot be unequivocally derived from the assertion of maximizing behavior alone, and (iii) the A-J effect is equivalent to each of the four economically intuitive conditions emanating from the model. Moreover, we derived these results using a more general model than has typically been contemplated in the literature, and have related them to the two stronger sufficient conditions that are prevalent in the literature. Lemma 2 and the paragraph that follows it, therefore, put to rest a protracted debate about the very existence of the A-J effect since they establish four economically intuitive conditions equivalent to the A-J effect.

We close by remarking that even though we have established our results in a more general model than has been heretofore considered, two further generalizations come to mind. One generalization would allow for multiple regulated inputs in the model, while another would relax the assumption that inputs are perfectly divisible. Such generalizations may be worthy of future research.

## 6. References


Averch, H., and L.O. Johnson (1962) "Behavior of the Firm under Regulatory Constraint" *American Economic Review* **52**, 1052–1069.

Baumol, W.J., and A.K. Klevorick. (1970) "Input Choices and Rate-of-Return Regulation: An Overview of the Discussion" *The Bell Journal of Economics and Management Science* **1**, 162–190.

El-Hodiri, M.A., and A. Takayama (1973) "Behavior of the Firm under Regulatory Constraint: Clarifications" *American Economic Review* **63**, 235–237.

McNicol, D.L. (1973) "The Comparative Statics Properties of the Theory of the Regulated Firm" *The Bell Journal of Economics and Management Science* **4**, 428–453.

Pressman I., and A Carol (1971) "Behavior of the Firm under Regulatory Constraint: Note" *American Economic Review* **61**, 210–212.

Pressman I., and A Carol (1973) "Behavior of the Firm under Regulatory Constraint: Reply" *American Economic Review* **63**, 238.

Silberberg, E. (1990) 2nd Edition *The Structure of Economics: A Mathematical Analysis*, McGraw-Hill Publishing Company: New York.

Takayama, A. (1969) "Behavior of the Firm under Regulatory Constraint" *American Economic Review* **59**, 255–260.

Takayama, A. (1993) *Analytical Methods in Economics*, The University of Michigan Press: Ann Arbor.

Zajac, E.E. (1970) "A Geometric Treatment of the Averch-Johnson's Behavior of the Firm Model." *American Economic Review* **60**, 117–125.

Zajac, E.E. (1972) "Lagrange Multiplier Values at Constrained Optima." *Journal of Economic Theory* **4**, 125–131.