E C O N O M I C S   B U L L E T I N

The Properties of the Watts Poverty Index under Lognormality

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Abstract

Under lognormality assumption, we derive the parametric formula of the Watts measure, one of the main axiomatically sound poverty measures. In these conditions, we derive new properties of the Watts measure, its sensitivity to distribution parameters and its parametric standard error.

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1 Introduction

In order to be meaningful, a poverty measure must satisfy reasonable axioms. Several formulae have been proposed for axiomatically sound poverty measures, sometimes relating them to social welfare functions. The properties of poverty measures are generally considered independently from a priori assumptions on the distribution of living standards. Yet, the fact that the observed distributions show unimodal asymmetric and leptokurtic shapes and incorporate only positive values, suggests that relevant information resides in these shapes.

One of the most popular axiomatically sound poverty measures is the Watts measure (Watts 1968, Zheng 1993). The aim of this article is to show that new properties of the Watts measure arise under lognormality of the living standard distribution, notably parametric formulae for this measure and its standard error. In section 2 we define, the Watts poverty measure. We derive new properties of the Watts measure under lognormality in section 3. Finally, section 4 concludes. The proofs are given in the appendix.

2 The Watts poverty measure

The Watts poverty measure is defined as

$$W = \int_0^z - \ln(y/z)\, d\mu(y) \tag{1}$$

where μ is the cumulative density function of living standards y and z is the poverty line. The Watts measure satisfies the focus[1], monotonicity, transfer and transfer sensitivity axioms. It is also continuous, subgroup consistent, decomposable (Foster, Greer and Thorbecke 1984) and satisfies other attractive properties (Zheng 1993,1997). Because of its axiomatic properties, it is often a better representation of poverty than other frequently used poverty indicators.

3 Properties under lognormality

[1]The focus axiom imposes that the poverty measure does not depend on the living standards of the non-poor.

3.1 A parametric formula

The lognormal distribution is unimodal, asymmetrical, leptokurtic and takes on only positive values. It has been used in applied analysis of living standards (Alaiz and Victoria-Feser 1996, Slesnick 1993), and has sometimes been found statistically consistent with income data (van Praag, Hagenaars and van Eck 1983). The assumption of lognormality of income is also exploited in theoretical economics (Hildenbrand 1998). Other distribution models for living standards or incomes are also used that sometimes yield better goodness-of-fit with empirical income than the lognormal distribution[2]. However, Cramer (1980) has found with U.S. data that when introducing measurement errors, the lognormal distribution is no longer dominated. Moreover, our concern here is not so much the goodness-of-fit of the model as to obtain a parametric expression of the Watts measure, which is not possible with all these distributions. Finally, as showed in Maasoumi (1989), the lognormal distribution can be used to develop series expansion of Theil's inequality indicator with respect to the deviation of the true distribution from lognormality. In that sense, the parametric formula that we derive for the Watts measure under lognormality may also be considered as the first term of a series expansion under the true distribution.

In general, parametric formulae for poverty and inequality measures are hard to find for most families of distributions. However, parametric formulae of the Gini coefficient, the General Entropy inequality measures, and the Head-count index are known for a few distributions, including the lognormal distribution[3]. The parametric formula of the Watts measure under lognormality is not available and we present it now. The other distributions used in the literature do not correspond to an explicit parametric formula of the Watts measure because the global integral cannot be expressed explicitly in these cases.

Proposition 1

[2]Fisk (1961), Salem and Mount (1974), Singh and Maddala (1976), Kloek and van Dijk (1978), McDonald and Ransom (1979), McDonald (1984), Hirschberg and Slottje (1989), Majumder and Chakravarty (1990), McDonald and Mantrala (1995).

[3]Chipman (1974), McDonald and Jensen (1979), McDonald (1981, 1984), Butler and McDonald (1989), Cowell and Victoria-Feser (1996).

If the living standard, y, follows a lognormal distribution law such that ln(y) $\sim$ N(m, σ^2), then the Watts measure is equal to:

$$W = (\ln z - m)\,\Phi\left(\frac{\ln z - m}{\sigma}\right) + \sigma\,\phi\left(\frac{\ln z - m}{\sigma}\right) \tag{2}$$

where ϕ and Φ are respectively the p.d.f. and c.d.f. of the standard normal distribution. The knowledge of Z = (ln z - m)/σ (the 'standardised logarithm of the poverty line', or s.l.p.l.) and σ is sufficient for the knowledge of W.

$$W = \sigma.[Z.\Phi(Z) + \phi(Z)] = \sigma.G(Z) \tag{3}$$

where G is a primitive function of the head-count index under lognormality.

σ^2 is equal to the variance of the logarithms, a common inequality measure (Foster and Ok 1999). Eq. (3) shows that the Watts measure can be decomposed in a product of two terms: σ and G(Z). Since under lognormality the incidence of poverty is equal to $\Phi(Z)$, function G(Z) is a primitive function with respect to Z of the poverty incidence (with value $\frac{1}{\sqrt{2\pi}}$ at Z = 0). We denote G(Z) the "cumulating poverty incidence", which can be used for second-order stochastic dominance analysis of poverty curves when σ is fixed. Then, G(Z) can itself be considered as a poverty measure. As a direct consequence, the elasticity of W with respect to the variation of any variable is the sum of the elasticity of σ and of the elasticity of the cumulating poverty incidence.

3.2 Sensitivity analysis

We now turn to the sensitivity analysis of W, which will clarify the role of distribution parameters and of the poverty line.

Proposition 2 *Gradient of W with respect to distribution parameters:*

$\frac{\partial W}{\partial m} = -\Phi(Z) < 0$ and $\frac{\partial W}{\partial \sigma} = \phi(Z) > 0$.

Poverty measured by the Watts measure decreases in the mean of the logarithm of living standards, m. The corresponding gradient component, equal to minus the incidence of poverty, is bounded in [-1,0] . By contrast, it does not necessarily decrease in the mean of living standards, $e^{m+\sigma^2/2}$.

W increases with σ, although it does not necessarily increase with the variance of living standards, $e^{2m+\sigma^2}(e^{\sigma^2}-1)$. The marginal augmentation of W with σ is bounded upwards by $1/\sqrt{2\pi}$ and downwards by 0. Because the level of the relevant poverty line z is generally unknown, it is important to investigate the sensitivity of the poverty measure to an a priori choice of z.

Proposition 3

The marginal variation of W with respect to the poverty line is

$$\frac{\partial W}{\partial z} = \frac{1}{z}\Phi(Z) > 0 \tag{4}$$

Of course, an increase in the poverty line increases poverty. Moreover, the elasticity of W with respect to the poverty line is equal to a ratio of two poverty measures: the incidence of poverty over the Watts measure. Then, W is more sensitive to the choice of z when there are many extremely poor individuals in the population, or when the inequality among the poor is high.

3.3 Parametric standard error

When maximum likelihood estimators (MLE) are used, standard errors of the Watts measure can be derived. The MLE $\hat{m}$ is the empirical mean of the logarithms of y. The MLE $\hat{\sigma}$ or σ is the square root of the empirical variance of the logarithms.

Proposition 4 *Under lognormality, the standard error of the Watts measure estimated with its maximum likelihood estimator (i.e. by replacing the distribution parameters in its formula by the MLE) is:*

$\sigma(W) = \frac{\sigma}{\sqrt{n}}\sqrt{\left(\Phi\left(\frac{\ln z - m}{\sigma}\right)\right)^2 + \frac{1}{2}\left(\phi\left(\frac{\ln z - m}{\sigma}\right)\right)^2}$, *where n is the sample size.*

Under lognormality, $\sigma(W)$ can be consistently estimated by replacing the parameters m and σ by their respective MLE.

4 Concluding Remarks

We have derived a parametric formula under lognormality for the Watts poverty measure and new associated properties. This formula can be useful in several domains. First, it adds a building block to the architecture of parametric forms of poverty and inequality measures, which can be employed for theoretical purposes. Second, it provides a fast and inexpensive way of estimating poverty in an axiomatically valid fashion provided that the values of the distribution parameters are available and when one accepts the lognormal approximation. This is not an infrequent situation since the lognormal distribution has been often used as a model for income distributions. Thus, one can obtain poverty estimates even when access to survey data is not possible, too costly or too time-consuming. A particularly interesting feature of the parametric formula is that the level of the poverty line can be freely and easily adjusted. Then, poverty estimates can be easily produced for a large number of poverty lines. Third, it enables one to conduct explicit sensitivity analysis of poverty change with respect to variations of the distribution parameters or of the poverty line. Four, under lognormality, we provide a parametric formula of the standard error of the Watts measure when it is estimated by plugging MLEs of distribution parameters. Finally, the properties shown for the lognormal case can be considered as 'stylized facts' whose robustness can be investigated. Among the properties which might be extended to other distribution forms and other poverty measures is the existence of a multiplicative decomposition of the poverty measure into fundamental poverty and inequality measures.

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Appendix

Proof of Proposition 3.1:

ln(y) $\sim$ N(m , σ), whose c.d.f. is denoted H. The Watts measure can be decomposed as follows

$W(z) = \int_o^z - \ln(y) \, + \, \ln(z) \, d\mu(y)$

which yields using the transfer theorem with u = ln(y)),

$W(z) = \ln(z).H(\ln(z)) - \int_{-\infty}^{\ln z} u \, dH(u)$

and again with normalisation of u with t $= \frac{u-m}{\sigma}$,

$W(z) = \ln(z).\Phi\left[\frac{\ln(z) - m}{\sigma}\right] - \int_{-\infty}^{\frac{\ln z - m}{\sigma}} \sigma\, t + m\; d\Phi(t)$

where Φ is the c.d.f. of the standard normal law. Then,

$W(z) = (\ln(z) - m).\Phi\left[\frac{\ln(z) - m}{\sigma}\right] - \sigma\,.J(z)$ where $J(z) = \int_{-\infty}^{\frac{\ln z - m}{\sigma}} t\, d\Phi(t)$

.

Integration of the latter equation yields

$J(z) = -\frac{1}{\sqrt{2\,\pi}}\, e^{-\left(\frac{\ln(z)-m}{\sigma}\right)^2/2}$.

Finally, $W = (\ln(z) - m).\,\Phi\left(\frac{\ln z - m}{\sigma}\right) + \frac{\sigma}{\sqrt{2\,\pi}}\, e^{-(\frac{\ln(z)-m}{\sigma})^2/2}$.

Proof of Proposition 3.4:

Proof: The standard errors are derived from the standard errors of the MLE, related to the information matrix. The MLE $(\hat{m}, \hat{\sigma})$ follows asymptotically $N\left[\begin{pmatrix} m \\ \sigma \end{pmatrix}, \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2/2 \end{pmatrix}\right]$. The gradient of W is composed of $\frac{\partial W}{\partial m} = -\Phi\left(\frac{\ln z - m}{\sigma}\right)$ and $\frac{\partial W}{\partial \sigma} = \phi\left(\frac{\ln z - m}{\sigma}\right)$. The application of the delta rule provides the result. QED.