E C O N O M I C S B U L L E T I N

On stability of Bertrand−Nash equilibrium in a simple model of the labour market

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Abstract

We examine a Bertrand−Edgeworth model of competition in a labour market where the workers simultaneously set wages disregarding any influence their current decision may have on opponents' future decisions. The iterated best response process is shown to converge in finite time to a Bertrand−Nash solution, where wages are set at the market−clearing level. This convergence result is also shown to hold when the assumption of static expectations is replaced by milder restrictions on beliefs about opponents' wages.

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1. **Introduction**

Like Cournot (1838), both Bertrand (1883) and Edgeworth (1925) were concerned with adjustment processes under duopoly. Since Theocharis (1959) and Fisher (1961), stability of Cournot equilibrium under oligopoly has been discussed extensively in terms of iterated best responses or more general dynamics. What has prevented similar analyses from being applied to price competition is perhaps that "best responses do not always exist in the pure Bertrand model" (Qin and Stuart, 1997, p. 503; see also Hehenkamp, Qin, and Stuart, 1999, p. 218). This difficulty, however, arises from viewing the price game as a continuous game, what in reality is not the case as there exists a minimum fraction of the money unit, however small it may be.

The present paper purports to analyse stability of Bertrand equilibrium in a discontinuous game for the labour market where, as in Weibull (1987) and Solow (1990, pp. 53-56), the workers simultaneously decide on their wages, whereupon the firm chooses whom to hire. We take the workers as playing a repeated wage game, each one myopically maximizing current expected utility given his beliefs on his opponents' wages. It is shown that, under static expectations, wages converge over time to the unique Nash equilibrium of the constituent game, in which wages are set at the market clearing level. Convergence is also shown to hold under much less restrictive assumptions on beliefs.

2. **Market clearing at a Bertrand-Nash equilibrium of a wage game**

We analyse a labour market with *n* identical workers and a single firm that produces a single output under decreasing returns to labour. With *Y* and *L* denoting quantities of output and labour, we write $Y = F(L)$, with $F(0) = 0$, and $F'(L) > 0$ and $F''(L) < 0$ for any $L \leq n$. $N = \{1, \dots, i, \dots, n\}$ denotes the set of workers, w_i and c_i worker *i*'s wage rate and income, respectively, and l_i the time spent working, as a fraction of the maximum working time allowed by the law. Worker utility is assumed to be $U_i = U(c_i + m \times (1 - l_i))$, with $U'(\ ) > 0, U''(\ ) \leq 0$ and $m > 0$ representing the constant marginal rate of substitution between income and leisure. For simplicity, the employed are allowed no discretion with respect to working time, so $l_i = 1$ for the employed. Therefore, utility is $U_i = U(w_i)$ for the employed and $U_i = U(b + m)$ for the unemployed, with $b \geq 0$ being any benefit granted to the latter. Indifference between the two alternatives occurs at the reservation wage $w^r = b + m$.

The marginal product of labour is the extra output provided by one additional worker, i.e., $MP(L) = F(L+1) - F(L)$. It can either be $MP(n-1) > w^r$ or $MP(n-1) \leq w^r$; to save space, the former is assumed throughout.[1]

We take a preliminary look at the competitive model, where workers and the firm are wage takers. Total labour supply is $L^s(w) = 0$ at $w < w^r$, $L^s(w) = n$ at $w > w^r$, and $L^s(w) \in \{1, 2, \dots, n\}$ at $w = w^r$. For any $w \leq MP(0)$, the profit-maximizing employment, denoted $L(w)$, is such that $MP(L(w)) \leq w \leq MP(L(w) - 1)$. $L(w)$ - and hence labour demand, $L^d(w)$ - decreases in a stepwise fashion as w increases, as in Figure 1. $L(w)$ is a function except at wages $w = MP(L^\circ)$ for some L°, when $L(w) \in \{L^\circ, L^\circ + 1\}$; at any such wage we take labour demand to be $L^d(w) = L^\circ + 1$.

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[1] For a more complete analysis, also dealing with the case $MP(n-1) \leq w^r$ - the one studied by Weibull (1987) and Solow (1990) – see De Francesco (2000).

We denote ε the smallest feasible fraction of the money unit. With $MP(n-1) > w^r$, at Walrasian (market-clearing) equilibrium employment is $L^w = n$ and the wage is $w^w = MP(n-1)$. (The latter is slightly inaccurate: indeed, $L^s(w) = L^d(w) = n$ at any $w \in \left[\max\{w^r,\ MP(n)+\varepsilon\},\ MP(n\text{-}1)\right]$, while we have identified the Walrasian wage with the highest value, $MP(n-1)$, in that range.)

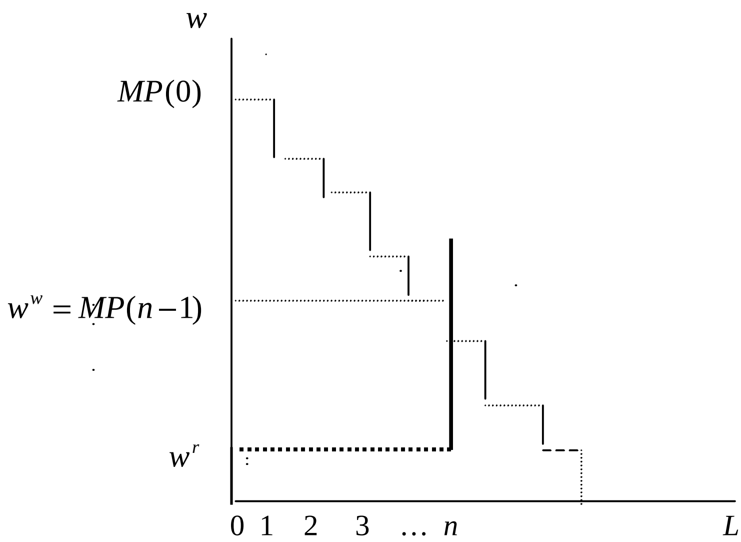

Figure 1

Now we examine the wage game, where the workers simultaneously decide on their wages, whereupon the firm chooses whom to hire. As we look for a subgame-perfect equilibrium of this two-stage game the decision of whom to hire – the "hiring decision" - is made according to a strategy that is optimal for the firm, such that at any wage vector $\boldsymbol{w} = (w_1, \ldots, w_i, \ldots, w_n)$ only hiring decisions that maximize profits at $\boldsymbol{w}$ are chosen. Before specifying the firm's strategy, we must then look closely at the properties of profit-maximizing hiring decisions.

Let $\boldsymbol{w}$ be such that $w_i \le MP(0)$ and $w_j \ge MP(n-1)$ for some $i, j \in N$. (Finding optimal hiring decisions is trivial when $w_i > MP(0)$ or when $w_i < MP(n-1)$ for all $i \in N$.) We identify three critical dimensions of a hiring decision, L_H, w^{h_e}, and w^{l_u}, these being, respectively, employment, the highest wage among the workers employed and the lowest wage among the workers unemployed. In fact, three conditions are necessary for the hiring decision to be optimal for the firm: (i) $w^{h_e} \le MP(L_H - 1)$, otherwise it would pay to reduce employment by laying-off the most expensive worker hired; (ii) $w^{l_u} \ge w^{h_e}$, otherwise it would pay to replace the most expensive worker hired with the least expensive unemployed worker; (iii) $w^{l_u} \ge MP(L_H)$, otherwise it would pay to increase employment by hiring the least expensive unemployed worker.[2]

Denote $L_H(\boldsymbol{w})$, $w^{h_e}(\boldsymbol{w}^\circ)$ and $w^{l_u}(\boldsymbol{w}^\circ)$, respectively, the levels of L_H, w^{h_e}, and w^{l_u} resulting, at $\boldsymbol{w}$, from the optimal strategy selected by the firm. This strategy is assumed to be such that, at any $\boldsymbol{w}$, the firm picks with equal probability any optimal hiring decision, or, whenever applicable, any of the optimal hiring decisions involving the *highest* level of employment. To see how this rule works, consider vectors $\boldsymbol{w}$ involving several optimal

[2] Sufficiency of these conditions can also be easily checked.

hiring decisions. Then, L_H is normally the same for all optimal hiring decisions. For example, let $\boldsymbol{w} = (w,...,w)$, with $w \neq MP(L)\ \forall\ L \leq n$. Each of the $\binom{n}{L_H}$ hiring decisions with $L_H : MP(L_H) < w < MP(L_H - 1)$ is optimal and is chosen with the same probability, so that each worker is hired with probability L_H / n. But look now at $\boldsymbol{w} : \#\{i : w_i < MP(L°)\} \leq L° < \#\{i : w_i \leq MP(L°)\}$ for some $L° < n$. Then, there are two types of profit-maximizing hiring decisions, those with $L_H = L°$, $w^{h_e} \leq MP(L°)$ and $w^{l_u} = MP(L°)$, and those with $L_H = L° + 1$, $w^{h_e} = MP(L°)$ and $w^{l_u} \geq MP(L°)$; according to the strategy specified above, the firm chooses with equal probability any of the latter. For example, let $\boldsymbol{w} = (w,...,w) : w = MP(L°)$ for some $L° < n$. Then, although hiring decisions with $L_H = L°$ are also optimal, the firm chooses any of those with $L_H = L° + 1$, and hence $L_H(\boldsymbol{w}) = L° + 1$.

At any wage vector $\boldsymbol{w} = (w_1,...,w_i,...,w_n)$, worker i's expected utility is $E(U_i | \boldsymbol{w}) = U(w_i) p(i_e | \boldsymbol{w}) + U(w^r)(1 - p(i_e | \boldsymbol{w}))$, where $p(i_e | \boldsymbol{w})$ stands for i's probability of being hired at $\boldsymbol{w}$. The following result is easily established.

Proposition 1. *The Walrasian wage vector* $\boldsymbol{w}^w = (w^w,...,w^w)$ *is an equilibrium of the game.*

Proof. Replying w^w to $\boldsymbol{w}_{-i}{}^w = (w^w,...,w^w)$ yields $E(U_i | \boldsymbol{w}^w) = U(w^w)$, which is higher than both $E(U_i | w_i > w^w, \boldsymbol{w}_{-i}{}^w) = U(w^r)$ or $E(U_i | w_i < w^w, \boldsymbol{w}_{-i}{}^w) = U(w_i)$. *QED*

The next step is to show that any $\boldsymbol{w} \neq \boldsymbol{w}^w$ is not an equilibrium. Some $\boldsymbol{w}$ are immediately disposed of. Note that, irrespective of $\boldsymbol{w}_{-i}$, $E(U_i | w_i = w^w, \boldsymbol{w}_{-i}) = U(w^w)$. In view of this, any wage $w_i < w^w$ is a strictly dominated strategy as $E(U_i | w_i < w^w, \boldsymbol{w}_{-i}) = U(w_i)$; and the same holds for any $w_i > MP(0)$ as $E(U_i | w_i > MP(0), \boldsymbol{w}_{-i}) = U(w^r)$.

There remain wage vectors $\boldsymbol{w}° = (w_1°,...,w_i°,...,w_n°) : w^w \leq w_i° \leq MP(0)$ for all $i \in N$. We now introduce two particular functions of $\boldsymbol{w}°$, denoted $w'(\boldsymbol{w}°)$ and $w''(\boldsymbol{w}°)$, where $w'(\boldsymbol{w}°) = max\{MP(L_H(\boldsymbol{w}°)), w^{h_e}(\boldsymbol{w}°) - \varepsilon\}$ and $w''(\boldsymbol{w}°) = min\{MP(L_H(\boldsymbol{w}°) - 1), w^{l_u}(\boldsymbol{w}°) - \varepsilon\}$. Note that $w'(\boldsymbol{w}°)$ is the highest wage which any $j : p(j_e | \boldsymbol{w}°) < 1$ might have replied to $\boldsymbol{w}_{-j}°$ to be hired with unit probability; similarly, $w''(\boldsymbol{w}°)$ is the highest wage which would result in any $j : p(j_e | \boldsymbol{w}°) = 1$ being hired with unit probability in the face of $\boldsymbol{w}_{-j}°$.

With $\boldsymbol{w}° > \boldsymbol{w}^w$, there obviously exists some j: $p(j_e | \boldsymbol{w}°) < 1$. The following two results show that any such worker should have quoted a lower wage.

Lemma 1. *At* $\boldsymbol{w}° > \boldsymbol{w}^w$*, any j:* $0 < p(j_e | \boldsymbol{w}°) < 1$ *has failed to make a best reply to* $\boldsymbol{w}_{-j}°$*, this instead being* $w'(\boldsymbol{w}°)$.

Proof. As regards notation, $w_j°$ here denotes the wage quoted at $\boldsymbol{w}°$ by any j: $0 < p(j_e | \boldsymbol{w}°) < 1$. It must be understood that $0 < p(j_e | \boldsymbol{w}°) < 1$ for any

$j : MP\left(\#\{i \in N : w_i{}^\circ \le w_j{}^\circ\} - 1\right) < w_j{}^\circ < MP\left(\#\{i \in N : w_i{}^\circ < w_j{}^\circ\}\right)$.[3] Indeed, when this condition holds, optimal hiring decisions are those with $w^{h_e} = w^{l_u} = w_j{}^\circ$, differing only in terms of who is hired amongst the workers quoting $w_j{}^\circ$; employment is $L_H(\boldsymbol{w}^\circ) : MP(L_H(\boldsymbol{w}^\circ)) < w_j{}^\circ \le MP(L_H(\boldsymbol{w}^\circ) - 1)$ and, given that each optimal hiring decision is chosen with a positive probability, $0 < p(j_e|\boldsymbol{w}^\circ) < 1$ for each of the workers quoting $w_j{}^\circ$.

In the present case, $w'(\boldsymbol{w}^\circ) = w^{h_e}(\boldsymbol{w}^\circ) - \varepsilon = w_j{}^\circ - \varepsilon$, hence $E\left(U_j \middle| w_j = w'(\boldsymbol{w}^\circ), \boldsymbol{w}_{-j}{}^\circ\right) = U(w_j{}^\circ - \varepsilon)$ for any *j*: $p(j_e|\boldsymbol{w}^\circ) < 1$.[4] As to *j*: $0 < p(j_e|\boldsymbol{w}^\circ) < 1$, negligibility of ε implies $E\left(U_j|\boldsymbol{w}^\circ\right) = U(w_j{}^\circ)p(j_e|\boldsymbol{w}^\circ) + U(w^r)(1 - p(j_e|\boldsymbol{w}^\circ)) < U(w_j{}^\circ - \varepsilon)$, so that $w_j{}^\circ - \varepsilon$ is a better reply than $w_j{}^\circ$. Besides, $w_j{}^\circ - \varepsilon$ is actually the best reply, as $E\left(U_j \middle| w_j > w_j{}^\circ, \boldsymbol{w}_{-j}{}^\circ\right) = U(w^r)$ and $E\left(U_j \middle| w_j < w_j{}^\circ - \varepsilon, \boldsymbol{w}_{-j}{}^\circ\right) = U(w_j)$, both of which are lower than $U(w_j{}^\circ - \varepsilon)$. *QED*

Lemma 2. *At* $\boldsymbol{w}^\circ > \boldsymbol{w}^w$*, any j:* $p(j_e|\boldsymbol{w}^\circ) = 0$ *has failed to make a best reply to* $\boldsymbol{w}_{-j}{}^\circ$*, this instead being* $w'(\boldsymbol{w}^\circ)$.

Proof. Here $w_j{}^\circ$ denotes the wage quoted at $\boldsymbol{w}^\circ$ by any $j : p(j_e|\boldsymbol{w}^\circ) = 0$. With $\boldsymbol{w}^\circ > \boldsymbol{w}^w$, $L_H(\boldsymbol{w}^\circ) < L^w = n$; therefore, $MP(L_H(\boldsymbol{w}^\circ)) \ge w^w$, in turn implying that $w'(\boldsymbol{w}^\circ) \ge w^w$. Consequently, for any *j*: $p(j_e|\boldsymbol{w}^\circ) = 0$, $w'(\boldsymbol{w}^\circ)$ is a better reply than $w_j{}^\circ$, given that $E\left(U_j \middle| w_j = w'(\boldsymbol{w}^\circ), \boldsymbol{w}_{-j}{}^\circ\right) = U(w') \ge U(w^w)$ while $E\left(U_j|\boldsymbol{w}^\circ\right) = U(w^r)$. Moreover, $w'(\boldsymbol{w}^\circ)$ is readily seen to be the unique best reply to $\boldsymbol{w}_{-j}{}^\circ$. While obviously $E\left(U_j \middle| w_j = w'(\boldsymbol{w}^\circ), \boldsymbol{w}_{-j}{}^\circ\right) > E\left(U_j \middle| w_j < w'(\boldsymbol{w}^\circ), \boldsymbol{w}_{-j}{}^\circ\right)$, the argument showing that $E\left(U_j \middle| w_j = w'(\boldsymbol{w}^\circ), \boldsymbol{w}_{-j}{}^\circ\right) > E\left(U_j \middle| w_j > w'(\boldsymbol{w}^\circ), \boldsymbol{w}_{-j}{}^\circ\right)$ is slightly different depending whether $w'(\boldsymbol{w}^\circ) = MP(L_H(\boldsymbol{w}^\circ))$ or $w'(\boldsymbol{w}^\circ) = w^{h_e}(\boldsymbol{w}^\circ) - \varepsilon$. In the former case, $L_H\left(w_j > w'(\boldsymbol{w}^\circ), \boldsymbol{w}_{-j}{}^\circ\right) = L_H(\boldsymbol{w}^\circ)$ and $p\left(j_e \middle| w_j > w_j{}^\circ, \boldsymbol{w}_{-1}{}^\circ\right) = 0$, given that $w_j > MP(L_H(\boldsymbol{w}^\circ)) > w^{h_e}(\boldsymbol{w}^\circ)$; consequently, $E\left(U_j \middle| w_j > w'(\boldsymbol{w}^\circ), \boldsymbol{w}_{-j}{}^\circ\right) = U(w^r)$. Turn now to the latter case, where $w'(\boldsymbol{w}^\circ) + \varepsilon = w^{h_e}(\boldsymbol{w}^\circ)$. First, along similar lines it is argued that $E\left(U_j \middle| w_j > w'(\boldsymbol{w}^\circ) + \varepsilon, \boldsymbol{w}_{-j}{}^\circ\right) = U(w^r)$. Second, $L_H\left(w_j = w'(\boldsymbol{w}^\circ) + \varepsilon, \boldsymbol{w}_{-j}{}^\circ\right) = L_H(\boldsymbol{w}^\circ)$ and $0 < p\left(j_e \middle| w_j = w'(\boldsymbol{w}^\circ) + \varepsilon, \boldsymbol{w}_{-j}{}^\circ\right) < 1$: indeed, if $j : p(j_e|\boldsymbol{w}^\circ) = 0$ changes to quoting $w'(\boldsymbol{w}^\circ) + \varepsilon = w^{h_e}(\boldsymbol{w}^\circ)$ employment remains unchanged as $w^{h_e}(\boldsymbol{w}^\circ) > MP(L_H(\boldsymbol{w}^\circ))$, with *j* now competing with the other worker(s) quoting $w^{h_e}(\boldsymbol{w}^\circ)$. Making use of Lemma 1, we thus have $E\left(U_j \middle| w_j = w'(\boldsymbol{w}^\circ) + \varepsilon, \boldsymbol{w}_{-j}{}^\circ\right) < E\left(U_j \middle| w_j = w'(\boldsymbol{w}^\circ), \boldsymbol{w}_{-j}{}^\circ\right)$. *QED*

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[3] It may be useful to explain this notation in simple words: for example, $MP\left(\#\{i \in N : w_i{}^\circ < w_j{}^\circ\}\right)$ denotes the marginal product of labour when the workers employed are those who quoted wages lower than $w_j{}^\circ$.

[4] Incidentally, $w''(\boldsymbol{w}^\circ) = w'(\boldsymbol{w}^\circ)$ in the present case, as one can easily check; therefore, $w^{h_e}(\boldsymbol{w}^\circ) - \varepsilon$ is also the highest wage that would have led to any $j : p(j_e|\boldsymbol{w}^\circ) = 1$ being hired with unit probability.

The next proposition follows immediately from Proposition 1 and Lemmas 1 and 2.

Proposition 2. *The Walrasian wage vector* $\boldsymbol{w}^w$ *is the unique equilibrium of the game.*

Thus, as is often the case with Bertrand oligopoly models (see, for example, Vives, 1986), at the unique equilibrium of the wage game, wages are set at the market-clearing level. A further result is now provided, which will be used subsequently.

Lemma 3. *At* $\boldsymbol{w}^\circ > \boldsymbol{w}^w$, *for any j:* $p\left(j_e \middle| \boldsymbol{w}^\circ\right) = 1$ *the unique best reply to* $\boldsymbol{w}_{-j}{}^\circ$ *is* $w''(\boldsymbol{w}^\circ)$.

Proof. This result is derived straightforwardly from the preceding ones by means of a simple experiment. Let any $j : p\left(j_e \middle| \boldsymbol{w}^\circ\right) = 1$ change to quoting $w_j > w''(\boldsymbol{w}^\circ)$, so that the wage vector becomes $\left(w_j > w''(\boldsymbol{w}^\circ), \boldsymbol{w}_{-j}{}^\circ\right)$. Then $0 \le p\left(j_e \middle| w_j > w''(\boldsymbol{w}^\circ), \boldsymbol{w}_{-j}{}^\circ\right) < 1$ - recall that $w''(\boldsymbol{w}^\circ)$ represents the highest wage resulting in any $j : p\left(j_e \middle| \boldsymbol{w}^\circ\right) = 1$ certainly being hired given $\boldsymbol{w}_{-j}{}^\circ$. By Lemmas 1 and 2, at wage vectors $\left(w_j > w''(\boldsymbol{w}^\circ), \boldsymbol{w}_{-j}{}^\circ\right)$ worker *j* has not made a best reply, this instead being the highest wage resulting in *j* certainly being hired in the face of $\boldsymbol{w}_{-j}{}^\circ$, i.e., $w'\left(w_j > w''(\boldsymbol{w}^\circ), \boldsymbol{w}_{-j}{}^\circ\right)$. Of course, $w'\left(w_j > w''(\boldsymbol{w}^\circ), \boldsymbol{w}_{-j}{}^\circ\right) = w''(\boldsymbol{w}^\circ)$ as the highest wage leading to a worker certainly being hired only depends on his opponents' strategy profile; hence $w''(\boldsymbol{w}^\circ)$ is the best reply for any $j : p\left(j_e \middle| \boldsymbol{w}^\circ\right) = 1$. *QED*

3. Convergence to Bertrand-Nash equilibrium in a repeated wage game

It is now shown how repetition of the wage game might lead to the emergence of Bertrand equilibrium. As in Bertrand (1883) and Edgeworth (1912) analyses of duopoly, each worker is assumed to be concerned with the current payoff of his wage decision. Therefore, the wage decision depends on the wages that the other workers are expected to quote in the current period. These expectations in turn depend on past wages, about which each worker has perfect information.

Furthermore, in each period the firm follows the optimal strategy specified above. Consequently, there is no path dependency in hiring decisions. For example, with a wage vector $\boldsymbol{w} = (w,...,w) > \boldsymbol{w}^w$ constant over time, each worker is hired with probability $L_H(\boldsymbol{w})/n$ in each period, no matter who was hired in the past.

Variables are now indexed by the time period – $\boldsymbol{w}(t)$ is the wage vector in period *t*, $L_H(t)$ the corresponding level of employment (i. e., $L_H(t) = L_H(\boldsymbol{w}(t))$, $w^{h_e}(t)$ the highest wage among worker hired ($w^{h_e}(t) = w^{h_e}(\boldsymbol{w}(t))$), and so on. Further, $\hat{w}(t)$ denotes the highest wage call in period *t*. In view of the analysis above, $\boldsymbol{w}(t) \ge \boldsymbol{w}^w$. Initially, stability of Bertrand equilibrium is established under static expectations.

Proposition 3. *With static expectations:*
(a) $\hat{w}(t+1) < \hat{w}(t)$ *when* $\boldsymbol{w}(t) > \boldsymbol{w}^w$;
(b) $\boldsymbol{w}(t \ge t^*) = \boldsymbol{w}^w$, *for* t^* *large enough.*

Proof. *(a)* Let $\boldsymbol{w}(t) > \boldsymbol{w}^w$. With static expectations, Lemmas 1 and 2 imply $w_j(t+1) = w'(t) = max\{MP(L_H(t)), w^{h_e}(t) - \varepsilon\}$ for $j : p(j_e|\boldsymbol{w}(t)) < 1$, while Lemma 3 implies $w_j(t+1) = w''(t) = min\{MP(L_H(t)-1), w^{l_u}(t) - \varepsilon\}$ for $j : p(j_e|\boldsymbol{w}(t)) = 1$. Both $w'(t)$ and $w''(t)$ are lower than $\hat{w}(t)$, as can be easily checked; hence $\hat{w}(t+1) < \hat{w}(t)$.

(b) With $\boldsymbol{w}(0) > \boldsymbol{w}^w$, from repeated application of (a) it follows that $\exists t^* > 0 : \boldsymbol{w}(t^*-1) > \boldsymbol{w}^w ; \boldsymbol{w}(t^*) = \boldsymbol{w}^w$. Furthermore, $\boldsymbol{w}(t > t^*) = \boldsymbol{w}^w$: since everyone at t^* has made a best reply, everyone continues to quote w^w at t^*+1, and thereafter. *QED*

From the proof of part (a) of the above proposition it follows immediately that $w_j(t+1) < w_j(t)$ for $j : p(j_e|\boldsymbol{w}(t)) < 1$ while $w_j(t+1) > w_j(t)$ for $j : w_j(t) < w''(t)$. Consequently, under static expectations many wage estimates are biased in disequilibrium. As these biases are likely to be noticed, it is important to see whether the stability result just obtained remains once the assumption of static expectations is removed. The analysis below provides – first under single-valued expectations and then introducing subjective uncertainty - more general conditions that are sufficient for stability.

Denote $\boldsymbol{w}_{-j}{}^j(t+1)$ worker *j*'s single-valued expectations on the wages quoted at $t+1$ by all $i \neq j$; its generic component, $w_i{}^j(t+1)$, is the wage that *j* expects *i* to quote. Furthermore, $w_j'(\boldsymbol{w}_{-j}{}^j(t+1))$ denotes the highest wage leading to *j* being hired with unit probability in the face of $\boldsymbol{w}_{-j}{}^j(t+1)$. We then have the following result.

Proposition 4. *Let expectations be such that, with* $\boldsymbol{w}(t) \geq \boldsymbol{w}^w$,

$$\boldsymbol{w}_{-j}{}^j(t+1) : \#\{i : w_i{}^j(t+1) \leq \hat{w}(t)\} \geq L_H(t) \quad \forall j \in N. \tag{1}$$

This is sufficient for:

(a) $\hat{w}(t+1) < \hat{w}(t)$ *when* $\boldsymbol{w}(t) > \boldsymbol{w}^w$*;*

(b) $\boldsymbol{w}(t \geq t^*) = \boldsymbol{w}^w$*, for* t^* *large enough.*

Proof. *(a)* To start with, note that $\hat{w}(t) > MP(L_H(t))$ when $\boldsymbol{w}(t) > \boldsymbol{w}^w$. Also, Lemmas 1, 2 and 3 straightforwardly imply that $w_j(t+1) = w_j'(\boldsymbol{w}_{-j}{}^j(t+1))\ \forall j \in N$; therefore, to prove that $\hat{w}(t+1) < \hat{w}(t)$ it has to be shown that $w_j'(\boldsymbol{w}_{-j}{}^j(t+1)) < \hat{w}(t)\ \forall j \in N$. Suppose to the contrary that $w_j'(\boldsymbol{w}_{-j}{}^j(t+1)) \geq \hat{w}(t)$ for some *j*. This being so, at wage vector $(w_j = \hat{w}(t), \boldsymbol{w}_{-j}{}^j(t+1))$ at *t*+1, worker *j* is hired with unit probability and the same must then be true for any $i : w_i{}^j(t+1) \leq \hat{w}(t)$. Taking account of (1), this implies that $L_H(w_j = \hat{w}(t), \boldsymbol{w}_{-j}{}^j(t+1)) > L_H(t)$ - employment at the stipulated wage vector is higher than at t – so that $MP(L_H(t)) > MP(L_H(w_j = \hat{w}(t), \boldsymbol{w}_{-j}{}^j(t+1)))$. Recalling that $\hat{w}(t) > MP(L_H(t))$, this in turn implies that $\hat{w}(t) > MP(L_H(w_j = \hat{w}(t), \boldsymbol{w}_{-j}{}^j(t+1)) - 1)$, which violates condition (i) for profit maximization. Therefore, the conjecture that $w_j'(\boldsymbol{w}_{-j}{}^j(t+1)) \geq \hat{w}(t)$ must be rejected to avoid this contradiction.

(b) With $\mathbf{w}(0) > \mathbf{w}^w$, repeated application of (a) leads to the conclusion that $\exists t^* > 0 : \mathbf{w}(t^* - 1) > \mathbf{w}^w ; \mathbf{w}(t^*) = \mathbf{w}^w$. Then, it is easily seen that condition (1) implies static expectations concerning wage quotes made at $t^* + 1$; consequently, everyone will quote w^w at $t^* + 1$, and thereafter. *QED*

We now move a step forward by relaxing the assumption of single-valued expectations. To deal with subjective uncertainty, let $B^j(t+1)$ denote worker *j*'s set of beliefs at date *t*+1 on the wages that the other workers are about to quote in period *t*+1, $\beta_i^j(t+1) \in B^j(t+1)$ worker *j*'s belief about worker *i*'s wage in *t*+1, $S_i^j(t+1)$ the support of $\beta_i^j(t+1)$, i.e., the set of wages that *j* believes will be quoted with positive probability by *i* in *t*+1, $w_i^j(t+1)$ any $w_i(t+1) \in S_i^j(t+1)$, and $\mathbf{w}_{-j}^j(t+1)$ any $\mathbf{w}_{-j}(t+1) \in \prod_{i \neq j} S_i^j(t+1)$. The following result is a generalization of Proposition 4.

Proposition 5. *Let beliefs be such that, with* $\mathbf{w}(t) \ge \mathbf{w}^w$,

$$B^j(t+1) : \#\left\{i : S_i^j(t+1) \subseteq \left\{w^w, w^w + \varepsilon, \ldots, \hat{w}(t)\right\}\right\} \ge L_H(t) \quad \forall j \in N. \qquad (2)$$

This is sufficient for:

(a) $\hat{w}(t+1) < \hat{w}(t)$ *when* $\mathbf{w}(t) > \mathbf{w}^w$;

(b) $\mathbf{w}(t \ge t^*) = \mathbf{w}^w$ *for t* large enough.*

Proof. See Appendix.

It is worth spelling out the restriction on beliefs imposed by condition (2) – which includes (1) as a special case: with $\mathbf{w}(t) \ge \mathbf{w}^w$, each worker expects his opponents' wages will certainly not exceed $\hat{w}(t)$ at *t*+1, for at least as many of them as were employed in *t*. This restriction seems reasonable and is consistent with simple learning procedures. Just to give an example, beliefs might be such that any worker is expected certainly not to switch to a wage that, if quoted in the previous period, would not have raised his expected utility. In such a case, any *j* rules out the event of any $i \neq j$ quoting more than $\hat{w}(t)$ in *t*+1 because *i* would have obtained an expected utility of $U(w^r) \le E(U_i | \mathbf{w}(t))$ by quoting such a wage at *t*.

Further, and most important, restriction (2) on beliefs is never contradicted during the adjustment process, given that in fact it implies $\hat{w}(t+1) < \hat{w}(t)$.

4. Conclusion

We have explored stability of Bertrand-Nash equilibrium in a simple labour market containing *n* wage-setting workers. The iterated best response process has been shown to converge in finite time to the unique equilibrium of the constituent game, in which wages are set at the market-clearing level. Also, convergence has been shown to still hold under milder restrictions on each worker's beliefs about his opponents' wages.

Two limitations of our analysis are worth bearing in mind. First, the market-clearing wage has been assumed to be higher than the reservation wage. This was just for the sake of brevity; we are confident that similar convergence results would also obtain if the market-clearing wage were equal to the reservation wage. Second, it has been assumed, for simplicity, that there was only one firm operating in the labour market. Whether similar convergence results would hold in the multi-firm case is a task we leave for future research.

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Appendix

Proof of Proposition 5. (a) What needs to be shown is that, with $\boldsymbol{w}(t) > \boldsymbol{w}^w$, condition (2) implies $w_j(t+1) < \hat{w}(t)\ \forall j \in N$. Denote $p\left(j_e \middle| w_j, B^j(t+1)\right)$ and $E\left(U_j \middle| w_j, B^j(t+1)\right)$, respectively, worker j's probability of being hired and expected utility in period t+1, conditional on w_j and j's beliefs. When quoting $\hat{w}(t)$ expected utility is

$$E\left(U_j \middle| \hat{w}(t), B^j(t+1)\right) = U(\hat{w}(t)) \times p\left(j_e \middle| \hat{w}(t), B^j(t+1)\right) + U(w^r) \times \left(1 - p\left(j_e \middle| \hat{w}(t), B^j(t+1)\right)\right). \quad (3)$$

In this equation either $p\left(j_e \middle| \hat{w}(t), B^j(t+1)\right) = 0$ or $0 < p\left(j_e \middle| \hat{w}(t), B^j(t+1)\right) \leq 1$. If the former occurs, j's optimal reply is readily seen to be some wage lower than $\hat{w}(t)$: indeed, $E\left(U_j \middle| w_j \geq \hat{w}(t), B^j(t+1)\right) = U(w^r)$, whereas, for example, $E\left(U_j \middle| w^w, B^j(t+1)\right) = U(w^w)$.

Turn now to the case where $0 < p\left(j_e \middle| \hat{w}(t), B^j(t+1)\right) \leq 1$. Note that this probability can be written as

$$\begin{aligned} &p\left(j_e \middle| \hat{w}(t), B^j(t+1)\right) = Pr\left(\boldsymbol{w}_{-j}{}^j(t+1) : \#\left\{i : w_i{}^j(t+1) < \hat{w}(t)\right\} < L_H(t)\right) \times \\ &p\left(j_e \middle| \hat{w}(t), \boldsymbol{w}_{-j}{}^j(t+1) : \#\left\{i : w_i{}^j(t+1) < \hat{w}(t)\right\} < L_H(t)\right) + \\ &Pr\left(\boldsymbol{w}_{-j}{}^j(t+1) : \#\left\{i : w_i{}^j(t+1) < \hat{w}(t)\right\} \geq L_H(t)\right) \times \\ &p\left(j_e \middle| \hat{w}(t), \boldsymbol{w}_{-j}{}^j(t+1) : \#\left\{i : w_i{}^j(t+1) < \hat{w}(t)\right\} \geq L_H(t)\right), \end{aligned}$$

which reduces to

$$\begin{aligned} &p\left(j_e \middle| \hat{w}(t), B^j(t+1)\right) = Pr\left(\boldsymbol{w}_{-j}{}^j(t+1) : \#\left\{i : w_i{}^j(t+1) < \hat{w}(t)\right\} < L_H(t)\right) \times \\ &p\left(j_e \middle| \hat{w}(t), \boldsymbol{w}_{-j}{}^j(t+1) : \#\left\{i : w_i{}^j(t+1) < \hat{w}(t)\right\} < L_H(t)\right), \end{aligned} \quad (4)$$

as $p\left(j_e \middle| \hat{w}(t), \boldsymbol{w}_{-j}{}^j(t+1) : \#\left\{i : w_i{}^j(t+1) < \hat{w}(t)\right\} \geq L_H(t)\right) = 0$. First of all, it must be noted that $p\left(j_e \middle| \hat{w}(t), \boldsymbol{w}_{-j}{}^j(t+1) : \#\left\{i : w_i{}^j(t+1) < \hat{w}(t)\right\} < L_H(t)\right) < 1$ so that, by equation (4), $p\left(j_e \middle| \hat{w}(t), B^j(t+1)\right) < 1$. Assume, to the contrary, that $p\left(j_e \middle| \hat{w}(t), \boldsymbol{w}_{-j}{}^j(t+1) : \#\left\{i : w_i{}^j(t+1) < \hat{w}(t)\right\} < L_H(t)\right) = 1$. Then, at wage vectors $\left(\hat{w}(t), \boldsymbol{w}_{-j}{}^j(t+1) : \#\left\{i : w_i{}^j(t+1) < \hat{w}(t)\right\} < L_H(t)\right)$, any $i : w_i{}^j(t+1) \leq \hat{w}(t)$ would also be hired with unit probability. This in turn implies that $L_H\left(\hat{w}(t); \boldsymbol{w}_{-j}{}^j(t+1) : \#\left\{i : w_i{}^j(t+1) < \hat{w}(t)\right\} < L_H(t)\right) > L_H(t)$ given that, by condition (2), $\#\left\{i : w_i{}^j(t+1) \leq \hat{w}(t)\right\} \geq L_H(t)$ with unit probability. As one can easily check, the result just achieved leads to the same contradiction as in the proof of Proposition 1. To avoid the contradiction, it must then be that $0 < p\left(j_e \middle| \hat{w}(t), \mathbf{w}_{-j}{}^{j}(t+1) : \#\left\{i : w_i{}^{j}(t+1) < \hat{w}(t)\right\} < L_H(t)\right) < 1$.

Now we see that, with $0 < p\left(j_e \middle| \hat{w}(t), B^{j}(t+1)\right) < 1$, worker *j*'s best reply is again some wage strictly less than $\hat{w}(t)$. A wage higher than $\hat{w}(t)$ is a worse reply than $\hat{w}(t)$: indeed, it follows immediately from (2) that $E\left(U_j \middle| w_j > \hat{w}(t), B^{j}(t+1)\right) = U(w^r)$, whereas $E\left(U_j \middle| w_j = \hat{w}(t), B^{j}(t+1)\right) > U(w^r)$ in the present case in which $p\left(j_e \middle| \hat{w}(t), B^{j}(t+1)\right) > 0$. On the other hand, there is some wage lower than $\hat{w}(t)$ yielding more than $\hat{w}(t)$. One such wage is $\hat{w}(t) - \varepsilon$, as we now see. Worker *j*'s probability of being hired at this wage is

$$
\begin{aligned}
&p\left(j_e \middle| \hat{w}(t) - \varepsilon, B^{j}(t+1)\right) = Pr\left(\mathbf{w}_{-j}{}^{j}(t+1) : \#\left\{i : w_i{}^{j}(t+1) < \hat{w}(t)\right\} < L_H(t)\right) \times \\
&p\left(j_e \middle| \hat{w}(t) - \varepsilon, \mathbf{w}_{-j}{}^{j}(t+1) : \#\left\{i : w_i{}^{j}(t+1) < \hat{w}(t)\right\} < L_H(t)\right) + \\
&Pr\left(\mathbf{w}_{-j}{}^{j}(t+1) : \#\left\{i : w_i{}^{j}(t+1) < \hat{w}(t)\right\} \geq L_H(t)\right) \times \\
&p\left(j_e \middle| \hat{w}(t) - \varepsilon, \mathbf{w}_{-j}{}^{j}(t+1) : \#\left\{i : w_i{}^{j}(t+1) < \hat{w}(t)\right\} \geq L_H(t)\right). \qquad (5)
\end{aligned}
$$

On reflection, it follows from $0 < p\left(j_e \middle| \hat{w}(t), \mathbf{w}_{-j}{}^{j}(t+1) : \#\left\{i : w_i{}^{j}(t+1) < \hat{w}(t)\right\} < L_H(t)\right) < 1$ that $p\left(j_e \middle| \hat{w}(t) - \varepsilon,\ \mathbf{w}_{-j}{}^{j}(t+1) : \#\left\{i : w_i{}^{j}(t+1) < \hat{w}(t)\right\} < L_H(t)\right) = 1$ (see the proof of Lemma 1). Equation (5) thus becomes

$$
\begin{aligned}
&p\left(j_e \middle| \hat{w}(t) - \varepsilon, B^{j}(t+1)\right) = Pr\left(\mathbf{w}_{-j}{}^{j}(t+1) : \#\left\{i : w_i{}^{j}(t+1) < \hat{w}(t)\right\} < L_H(t)\right) + \\
&Pr\left(\mathbf{w}_{-j}{}^{j}(t+1) : \#\left\{i : w_i{}^{j}(t+1) < \hat{w}(t)\right\} \geq L_H(t)\right) \times \\
&p\left(j_e \middle| \hat{w}(t) - \varepsilon, \mathbf{w}_{-j}{}^{j}(t+1) : \#\left\{i : w_i{}^{j}(t+1) < \hat{w}(t)\right\} \geq L_H(t)\right) \qquad (6)
\end{aligned}
$$

By comparing (6) with (4) it is seen that $p\left(j_e \middle| \hat{w}(t) - \varepsilon, B^{j}(t+1)\right) > p\left(j_e \middle| \hat{w}(t), B^{j}(t+1)\right)$. Now write expected utility when quoting $\hat{w}(t) - \varepsilon$:

$$
\begin{aligned}
&E\left(U_j \middle| \hat{w}(t) - \varepsilon, B^{j}(t+1)\right) = U\left(\hat{w}(t) - \varepsilon\right) \times p\left(j_e \middle| \hat{w}(t) - \varepsilon, B^{j}(t+1)\right) + \\
&U(w^r) \times \left(1 - p\left(j_e \middle| \hat{w}(t) - \varepsilon, B^{j}(t+1)\right)\right). \qquad (7)
\end{aligned}
$$

Recalling that $p\left(j_e \middle| \hat{w}(t) - \varepsilon, B^{j}(t+1)\right) > p\left(j_e \middle| \hat{w}(t), B^{j}(t+1)\right)$ and the negligibility of ε, by comparing (7) with (3) it is seen that $E\left(U_j \middle| \hat{w}(t) - \varepsilon, B^{j}(t+1)\right) > E\left(U_j \middle| \hat{w}(t), B^{j}(t+1)\right)$.

(b) With $\mathbf{w}(0) > \mathbf{w}^w$, repeated application of (a) leads us to conclude that $\exists t^* > 0 : \mathbf{w}(t^* - 1) > \mathbf{w}^w ; \mathbf{w}(t^*) = \mathbf{w}^w$ for *t** large enough. At *t**+1, condition (2) amounts to saying that $S_i{}^{j}(t^* + 1) = \left\{w^w\right\}\ \forall i, j \in N$, i.e., each worker has static expectations about wages quoted in *t**+1. Therefore, since everyone has made a best reply at *t**, everyone continues quoting w^w at *t**+1, and thereafter. *QED*