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Improving the accuracy of the analytical indirect inference estimator for MA models.

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Abstract

We propose to use the analytical generalised least squares (GLS) transformation matrix of Galbraith and Zinde-Walsh (1992) to correct finite sample estimation error of MA(q) processes parameters estimates. Our method may be considered as an iteration of the analytical indirect inference estimator (AIIE) of Galbraith and Zinde-Walsh (1994). Its potential is explored through a series of Monte Carlo experiments.

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1. Introduction

It is well known that estimators for MA(q) models parameters are biased in finite samples. For example, Cordeiro and Klein (1994) derive analytical expressions for the bias of the maximum likelihood estimator (MLE). Bias corrected estimators can be derived from these equations, but they become complicated as the order of the MA process increases.

The bias of an estimator typically is inversely related to sample size. Thus, biased estimators often have quite large finite sample estimation errors, that is, the value of the parameter estimated in a given small sample may be very far from the true value. In this paper, we introduce a simple way to estimate this error and obtain a corrected estimate.

Our method is especially well suited for the analytical indirect inference estimator (AIIE) of Galbraith and Zinde-Walsh (1994). Although not as popular as MLE, this estimator, which exploits the analytical binding functions that exist between the parameters of MA and AR models, is nevertheless often used in applied work, see for example, Tkacz (2007). Among its advantages is the fact that it is more robust to misspecification than MLE. It may also be very useful when one uses a simulation-based estimatior which requires estimation of an MA process at each step, see for example, Hryshko (2006) for such an application.

The theoretical foundation of the AIIE is that any invertible MA(q) process has an AR(∞) form which can be approximated arbitrarily well in finite samples by an autoregression of order p. Among other things, this means that if the ultimate goal is to forecast future values of the process, then one can simply use an AR(p) model. However, the AR(p) approximation, and consequently the forecasts, may be quite imprecise if p is small, so that it may still be preferable to estimate the MA(q) process in small samples. If, on the other hand, the MA(q) process arises from the formulation of a theoretical model, as it does in expectation models such as the one used by Tkacz (2007), then estimation of its parameters is necessary to compute the GLS estimator. Tests of the rational expectation hypothesis in currency exchange markets such as proposed by Hansen and Hodrick (1980) also give rise to regression models with MA error terms.

2. GLS correction for MA(q) models

Consider n observations of an invertible MA(q) process:

$$u_t = \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + ... + \theta_q \varepsilon_{t-q} + \varepsilon_t$$

where the ε_ts are iid innovations with mean 0 and finite variance σ_ε^2. Let $\hat{\theta}$ be a consistent estimator of the q-vector of true parameters θ^0. Let θ_j^0 denote the true value of the j^{th} parameter and assume that $\hat{\theta}_j = \theta_j^0 + b_j$ where b_j is the estimation error of the j^{th} element of $\hat{\theta}$. Also, let $\Psi(\theta^0)$ be the GLS transformation matrix of Galbraith and Zinde-Walsh (1992) evaluated at θ^0 such that $\Psi(\theta^0)\Psi^\top(\theta^0) = \Sigma^{-1}$, where $\Sigma = E(uu^\top)$ and u is a n-vector with typical element u_t. Let $\nu = \Psi^\top(\hat{\theta})u$ denote the $n \times 1$ vector of residuals obtained when Ψ is evaluated at $\hat{\theta}$.

Observation 1.

Assuming that $\varepsilon_0 = \varepsilon_{-1} = ... = \varepsilon_{-q+1} = 0$, which is harmless asymptotically,

$$\nu_t = \sum_{i=1}^{\infty} \gamma_i \nu_{t-i} + \varepsilon_t \tag{1}$$

where

$$\gamma_i = \sum_{k=1}^{\min\{i,q\}} -\gamma_{i-k}\theta_k^0 - b_i, \quad \gamma_0 = 0. \tag{2}$$

Proof

The element in position i, j of the matrix $\Psi^\top(\hat{\theta})$ is determined by:

$$h_{i,j} = \begin{cases} 0 & \text{if } i < j, \\ 1 & \text{if } i = j, \\ -\sum_{k=1}^{\min\{i-1,q\}} (\theta_k^0 + b_k) h_{i-k,j} & \text{otherwise.} \end{cases}$$

Thus, we have the following equations, where the 0 superscript is omitted for ease of notation:

$$\nu_1 = \varepsilon_1 \tag{3}$$

$$\nu_2 = -b_1\varepsilon_1 + \varepsilon_2 \tag{4}$$

$$\nu_3 = (b_1^2 + \theta_1 b_1 - b_2)\varepsilon_1 - b_1\varepsilon_2 + \varepsilon_3 \tag{5}$$

$$\nu_4 = (\theta_1 b_2 + \theta_2 b_1 + 2b_1 b_2 - b_1^3 - 2\theta_1 b_1^2 - \theta_1^2 b_1 - b_3)\varepsilon_1 + (b_1^2 + \theta_1 b_1 - b_2)\varepsilon_2 - b_1\varepsilon_3 + \varepsilon_4 \quad (6)$$

$$\nu_5 = (\theta_1^3 b_1 + 3\theta_1 b_1^3 + 3\theta_1^2 b_1^2 + b_1^4 - \theta_1^2 b_2 - 2\theta_1\theta_2 b_1 - 4\theta_1 b_1 b_2 - 2\theta_2 b_1^2 - 3b_1^2 b_2 + \theta_1 b_3 + \theta_3 b_1 \quad (7)$$
$$+2b_1 b_3 + \theta_2 b_2 + b_2^2 - b_4)\varepsilon_1 + \left(-\theta_1^2 b_1 - 2\theta_1 b_1^2 - b_1^3 + \theta_1 b_2 + \theta_2 b_1 + 2b_1 b_2 - b_3\right)\varepsilon_2$$
$$+ \left(b_1^2 + \theta_1 b_1 - b_2\right)\varepsilon_3 - b_1\varepsilon_4 + \varepsilon_5.$$

Solving (3), (4), (5) and (6) for ε_1, ε_2, ε_3 and ε_4 and substituting in equation (7),

$$\nu_5 = -b_1\nu_4 + (\theta_1 b_1 - b_2)\nu_3 + \left(-\theta_1^2 b_1 + \theta_1 b_2 + \theta_2 b_1 - b_3\right)\nu_2 \quad (8)$$
$$+ \left(\theta_1^3 b_1 - \theta_1^2 b_2 - 2\theta_1\theta_2 b_1 + \theta_1 b_3 + \theta_3 b_1 + \theta_2 b_2\right)\nu_1 + \varepsilon_5$$

which has the expected form. Further substitutions yield the stated result. •

The equations (2) are generalisations of the equations used by Galbraith and Zinde-Walsh (1994) to develop their AIIE of MA parameters. This can be seen by placing an original estimator such that $\hat{\theta}_i = 0$ for all i in the GLS transformation matrix so that $b_i = -\theta_i^0$ for all i. If $\hat{\theta}$ is the AIIE, then estimating b_i through equations (2) could be considered as an iteration of this estimator.

Based on observation 1, we propose to estimate b_i for each parameter one at a time and to define the corrected estimator in the following recursive way:

1. Use the initial estimate to obtain a vector of filtered data: $\hat{\nu} = \Psi^\top(\hat{\theta})u$.

2. Fit a long autoregression to $\hat{\nu}_t$. Then, estimate the error of $\hat{\theta}_1$ as $\hat{b}_1 = -\hat{\gamma}_1$ and compute the corrected estimator, $\tilde{\theta}_1 \equiv \hat{\theta}_1 - \hat{b}_1$.

3. Use steps similar to 2 to get corrected estimates of any other parameter. That is, compute the corrected estimators $\tilde{\theta}_j \equiv \hat{\theta}_j - \hat{b}_j$ where $\hat{b}_j = -\hat{\gamma}_j - \sum_{j=1}^{\min\{i,q\}} \hat{\gamma}_{i-j}\tilde{\theta}_j$ for $j = 2, ..., q$.

This correction scheme can be iterated by using $\tilde{\theta}$ in the GLS transformation matrix so as to obtain a new vector of filtered data $\tilde{\nu} = \Psi^\top(\tilde{\theta})u$ and going through steps 2 and 3 again. Similar results exist for AR(p) and ARMA(p, q) models (see Richard, 2007). Note that the proposed method is only valid if the MA process described by $\hat{\theta}$ is invertible since the analytical transformation matrix requires this assumption. Also, notice that if one replaces $\hat{\theta}$ by $E(\hat{\theta})$ in observation 1, then equations (2) become bias equations and $\tilde{\theta}$ becomes a bias corrected estimator. This avenue, which is a bit less intuitive than what we do here (indeed, it is not clear how to link $\Psi(E(\hat{\theta}))$ and $\Psi(\hat{\theta})$), is explored in Richard (2007).

3. Simulations

We now use Monte Carlo simulations to assess the quality of the GLS correction. Throughout this section, error terms are drawn from a N(0,1) distribution and the order of the autoregressions necessary to compute the AIIE and the GLS bias correction are chosen with the Akaike information criterion. Because of its close relation with the GLS correction, we focus on the AIIE of Galbraith and Zinde-Walsh (1994). GLS correction of the MLE is considered in Richard (2007) and is not in general recommendable.

Figures 1 and 2 consider the estimation of a MA(1) model with a sole parameter θ, meaning that the data is generated from the process $y_t = \theta\varepsilon_{t-1} + \varepsilon_t$ with $\varepsilon_t \sim NID(0,1)$, for all $t = 1, 2, ..., N$. It can be seen that the GLS correction greatly improves the small sample properties of the AIIE estimator when θ is close to the non-invertibility region and only slightly increases its mean squared error (MSE) for moderate values. Iterating the GLS correction appears to be useful in small samples when θ is extreme. On the other hand, figure 2 indicates that there is no advantage (nor inconvenient) in using the GLS correction in very large samples.

Figure 3 considers the estimation of an MA(2) process. The data is generated from the process $y_t = \theta_1\varepsilon_{t-1} + \theta_2\varepsilon_{t-2} + \varepsilon_t$ with $\varepsilon_t \sim NID(0,1)$, for all $t = 1, 2, ..., N$. The determinant of the MSE is plotted as a function of the modulus closest to 1 of the roots of the lag polynomial corresponding to the different MA(2) processes used. Evidently, the GLS correction is very useful, although iterating it does not seem to produce any gain.

4. Conclusion

We propose a bias correction technique for the estimation of MA(q) models based on the analytical GLS transformation matrix of Galbraith and Zinde-Walsh (1992). Our simulations indicate that it may provide accuracy gains when used along with the AIIE of Galbraith and Zinde-Walsh (1994). A similar technique can be devised for AR(p) and ARMA(p, q) models but unreported simulations indicate that it does not perform as well as the one proposed here. It is likely that one could use it along with the AIIE for VMA models of Galbraith, Ullah and Zinde-Walsh (2002).

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Figures

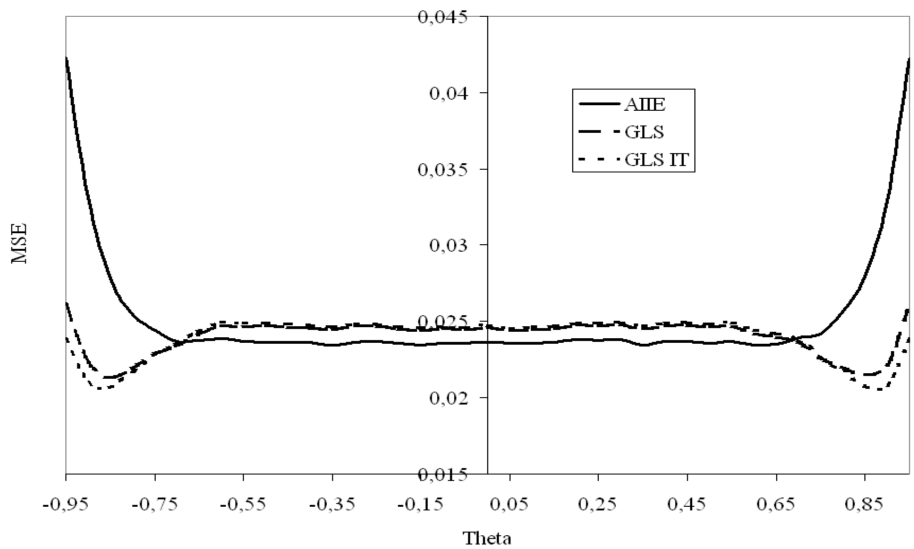

Figure 1. MSE, MA(1) model, N=50, 100 000 Monte Carlo samples.

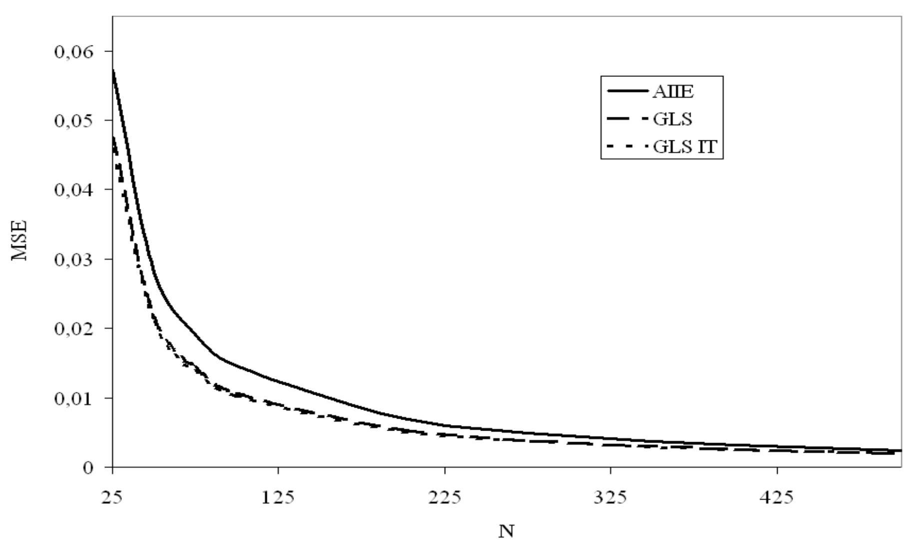

Figure 2. MSE, MA(1) model, $\theta = -0.85$, 100 000 Monte Carlo samples.

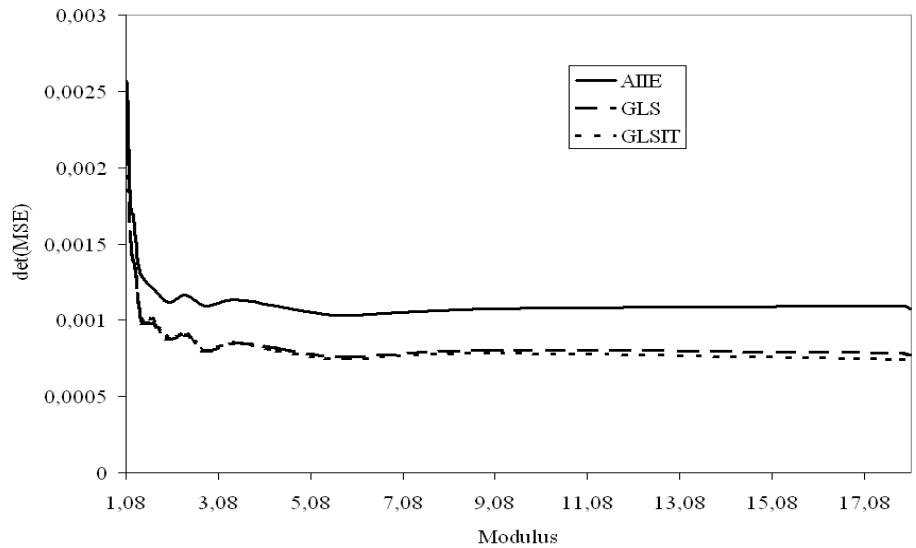

Figure 3. Det(MSE), MA(2) model, N=50, 10 000 Monte Carlo samples.