A UNIFIED APPPROACH TO STRATEGY-PROOFNESS FOR SINGLE-PEAKED PREFERENCES

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A Unified Approach to Strategy-Proofness for Single-Peaked Preferences

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Abstract: This article establishes versions of Moulin's [On strategy-proofness and single peakedness, *Public Choice* **35** (1980), 31–38] characterizations of various classes of strategy-proof social choice functions when the domain consists of all profiles of single-peaked preferences on an arbitrary subset of the real line. Two results are established that show that the median of $2n + 1$ numbers can be expressed using a combination of minimization and maximization operations applied to subsets of these numbers when either these subsets or the numbers themselves are restricted in a particular way. These results are used to show how Moulin's characterizations of generalized median social choice functions can be obtained as corollaries of his characterization of min-max social choice functions.

Keywords: generalized median social choice functions, Moulin min-max rules, single-peaked preferences, strategy-proofness

JEL classification numbers: D71, D82.

1 Introduction

Black (1948) has argued that, in practice, there are many social choice problems in which the set of alternatives A is one-dimensional (and so can be thought of as being a subset of the real line $\mathbb{R}$) and individual preferences are single-peaked. For example, this is the case when the alternatives are either quantities of a divisible public good or a finite set of political candidates arrayed on a left-right ideological spectrum. Black was interested in the properties of pairwise majority rule when preferences are single-peaked. For an odd number of individuals, Black demonstrated that pairwise majority rule selects the alternative that is the median of the individual preference peaks and, furthermore, that this rule is strategy-proof.[1]

There is now a substantial literature dealing with strategy-proof social choice with single-peaked preferences and multidimensional generalizations of single-peakedness. For introductions to this literature, see Sprumont (1995) and Barberà (2011).

A social choice function that only depends on each individual's most-preferred alternative is said to satisfy the *tops-only property*. Black's median-voter rule has this property. Moulin (1980) has characterized the set of all strategy-proof social choice functions that satisfy the tops-only property when the domain consists of all profiles of single-peaked preferences on the set of alternatives A. When there are n individuals, each of these functions is characterized by 2^n parameters each of which is either one of the alternatives in A or the infinim or supremum of this set, with one parameter assigned to each subset of the set of individuals.[2] For each profile of preferences, the chosen alternative is determined by (i) first identifying for each subset S of individuals, the maximum of the parameter value assigned to S and the largest preference peak of the individuals in S and (ii) then choosing the smallest of these values over all subsets of individuals. Sprumont (1995) calls such a social choice function a *min-max rule*. Alternative, but equivalent, ways of specifying this class of social choice functions have been developed by Barberà, Gul, and Stacchetti (1993) and Ching (1997).

Moulin (1980) has also characterized the strategy-proof social choice functions satisfying the tops-ony property that are (i) anonymous and (ii) anonymous and Pareto efficient. In case (i), a rule satisfying these properties is characterized by $n+1$ parameters drawn from the same set of admissible parameters as is used for the min-max rules. For each profile of preferences, this rule chooses the median of the individual preference peaks and these $n+1$ parameters. Because the median is being determined from $2n+1$ numbers, this rule is well defined. Black's median-voter rule is the special case in which half of the parameters are set equal to the infinim of A and half are set equal to the supremum of A. In case (ii), the rules are constructed in the same way, but use only $n-1$ parameters. These two classes of rules are known as *generalized median social choice functions*.

Because generalized median social choice functions are strategy-proof and satisfy the tops-ony property, they must be min-max rules. Curiously, Moulin did not establish his theorems about generalized median social choice functions as corollaries of his min-max social choice function theorem; he instead provided independent proofs of his generalized median and min-max results.

[1] When there are an even number of individuals, there may be two median peaks. In this case, Black supposed that one of the individuals is given the power to break ties.

[2] If A is unbounded from below (resp. above), then $-\infty$ (resp. ∞) is used instead of the infinim (resp. supremum).

The main purpose of this article is to show how Moulin's generalized median theorems can be obtained from his min-max theorem. In order to do this, I establish two results that show that the median of $2n + 1$ numbers can be expressed using a combination of minimization and maximization operations applied to subsets of these numbers when either these subsets or the numbers themselves are restricted in a particular way.

A subset S of $A \subseteq \mathbb{R}$ is an *interval of* A if for any two alternatives in S, all alternatives in A that lie between them are also in S. It is now known from the work of Barberà and Jackson (1994), Sprumont (1995), Ching (1997), and Weymark (2008) that if the range of a strategy-proof social choice function f whose domain is the set of all profiles of single-peaked preferences on A is an interval of A, then f satisfies the tops-only property. I show that, in fact, the tops-only property is equivalent to the range being an interval of A.

Moulin (1980) established his results for the case in which the set of alternatives A is all of $\mathbb{R}$, but noted that they also apply to the case in which A is a finite set. Moulin's theorems are often applied to situations in which A is a closed interval of $\mathbb{R}$. I show that they are valid when A is any subset of A containing at least two alternatives. The special structure placed on A in the previous literature is not needed.

Moulin's proof of the min-max characterization theorem omits many of the details of the argument. In view of the importance of Moulin's min-max theorem for the subsequent literture, I also provide a more complete proof of this result.

In Section 2, I introduce the model and present some background results. In Section 3, I consider the tops-only property. Moulin's min-max social choice functions are introduced in Section 4. In Section 5, I prove Moulin's min-max theorem when the set of alternatives is an arbitrary subset of $\mathbb{R}$ and characterize the set of all min-max social choice functions that are also Pareto efficient. In Section 6, I first establish two propositions that identify situations in which a median can be expresed in terms of minimization and maximization operations and then use these results to show how Moulin's generalized median theorems can be obtained as corollaries of his min-max theorem. Finally, I provide some concluding remarks in Section 7.

2 The Model and Background Results

The set of alternatives A is assumed to be a nonempty subset of $\mathbb{R}$ containing at least two alternatives. As the examples in Section 1 illustrate, it is often natural to suppose that A is either connected or discrete, but this is not assumed in the subsequent analysis. Let $a_- = \inf A$ if this infinum exists and let $a_- = -\infty$ otherwise. Similarly, let $a_+ = \sup A$ if this supremum exists and let $a_+ = \infty$ otherwise. Let $A^* = A \cup \{a_-, a_+\}$ and $\mathbb{R}^* = \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ (the extended real line). For an odd number of alternatives $x_1, \ldots, x_m$ in A, let $\operatorname{med}\{x_1, \ldots, x_m\}$ denote the *median*.

For any $x, y \in \mathbb{R}^*$, $[x, y]$ denotes the closed interval of $\mathbb{R}^*$ that has x and y as its endpoints. A subset S of A is an *interval of* A if $[x, y] \cap A \subseteq S$ for all $x, y \in S$. Even though an interval S of A need not be connected relative to $\mathbb{R}$, relative to A it includes all points in A that lie between any two distinct points in S. The closure of any subset of A is understood to be relative to A. Thus, for all $x, y \in A^*$, $[x, y] \cap A$ is a closed interval of A. It is denoted by $\overline{xy}$. Henceforth, it is understood that $x \leq y$ when I write $\overline{xy}$.

Let $\mathcal{R}$ denote the set of all orderings (that is, reflexive, complete, and transitive binary relations) on A. An ordering $R \in \mathcal{R}$ is interpreted as being a preference. For any $R \in \mathcal{R}$ and any nonempty set $S \subseteq A$, the *top set* of R in S is

$$\tau(R, S) = \{x \in S \mid xRy \text{ for all } y \in S\}.$$

In other words, $\tau(R, S)$ is the set of best alternatives in S according to the preference R. A preference $R \in \mathcal{R}$ is *single-peaked* if there exists an alternative $\pi(R) = \tau(R, A) \in A$, the *peak* of R, such that $\pi(R)PxPy$ whenever $x, y \in A$ and $y < x < \pi(R)$ or $\pi(R) < x < y$. Let $\mathcal{S}$ denote the set of single-peaked preferences on A.

The set of *individuals* is $N = \{1, \ldots, n\}$, where n is finite. While there is no social choice problem unless $n \geq 2$, the results in this article also hold for $n = 1$. This special case is used as part of an induction proof and provides some insight into the structure of the social choice rules considered here. A *profile* is an n-tuple of individual preference orderings $\mathbf{R} = (R_1, \ldots, R_n)$.

A *social choice function* is a function $f \colon \mathcal{D}^n \to A$, where $\mathcal{D} \subseteq \mathcal{R}$ is the common set of admissible preferences for each individual. Thus, the *domain* of f is $\mathcal{D}^n$. In this article, $\mathcal{D} = \mathcal{S}$, the set of all single-peaked preferences. The *range* of f is

$$A^f = \{x \in A \mid f(\mathbf{R}) = x \text{ for some } \mathbf{R} \in \mathcal{D}^n\}.$$

Let $\mathbf{R}^S = (R_i)_{i \in S}$ denote the *subprofile* of preferences of the individuals in S, where $\varnothing \subset S \subset N$. A profile is sometimes written as $\mathbf{R} = (\mathbf{R}^S; \mathbf{R}^{-S})$, where $-S$ is the complement of S. For the social choice function f, the *option set* generated by $\mathbf{R}^S$ is

$$O^f_{-S}(\mathbf{R}^S) = \{x \in A \mid x = f(\mathbf{R}^S; \mathbf{R}^{-S}) \text{ for some } \mathbf{R}^{-S} \in \mathcal{D}^{n-|S|}\}.$$

The option set $O^f_{-S}(\mathbf{R}^S)$ is the set of alternatives that are attainable given that the individuals in S have the subprofile $\mathbf{R}^S$. The widespread use of option sets to characterize properties of strategy-proof social choice functions is due to the influence of the seminal article of Barberà and Peleg (1990). The option-set methodology was first introduced by Laffond (1980), Satterthwaite and Sonnenschein (1981), and Barberà (1983).

A social choice function f is *manipulable* by individual $i \in N$ at the profile $\mathbf{R} \in \mathcal{D}^n$ via $\bar{R}_i \in \mathcal{D}$ if $f(R_1, \ldots, R_{i-1}, \bar{R}_i, R_{i+1}, \ldots, R^n)P_i f(\mathbf{R})$. A social choice function is *strategy-proof* if it is never manipulable.

Definition. A social choice function f is *strategy-proof* if there is no individual $i \in N$, no profile $\mathbf{R} \in \mathcal{D}^n$, and no preference $\bar{R}_i \in \mathcal{D}$ such that f is manipulable by individual i at $\mathbf{R}$ via $\bar{R}_i$.

Any strategy-proof social choice function for which the domain is the Cartesian product of the same set of individual preferences has the property that if everybody agrees that the same alternative is best on the range, then this alternative must be chosen. See Le Breton and Weymark (1999, Proposition 2). For the domain $\mathcal{S}^n$, this property of a social choice function also follows from Zhou (1991, Lemma 2) and Barberà and Jackson (1994, Lemma A-1).

Lemma 1. *If $f\colon \mathcal{S}^n \to A$ is strategy-proof, then for all $\mathbf{R} \in \mathcal{S}^n$, if $\tau(R_i, A^f) = \{x\}$ for all $i \in N$, then $f(\mathbf{R}) = x$.*

Strategy-proofness by itself also places structure on the range of a social choice function. For the domain $\mathcal{S}^n$ considered here, note that for any $x \in A$, there is a preference $R \in \mathcal{S}$ such that $\pi(R) = x$. It therefore follows from Le Breton and Weymark (1999, Proposition 5) that the range of a strategy-proof social choice function must be a closed set when its domain is $\mathcal{S}^n$.[3]

Lemma 2. *If $f\colon \mathcal{S}^n \to A$ is strategy-proof, then A^f is closed.*

The option set $O^f_{-S}(\mathbf{R}^S)$ generated by the subprofile $\mathbf{R}^S$, where $\varnothing \subset S \subset N$, is the range of the $(n - |S|)$-person social choice function $g\colon \mathcal{D}^{n-|S|} \to A$ defined by setting, for all $\mathbf{R}^S \in \mathcal{D}^{n-|S|}$,

$$g(\mathbf{R}^S) = f(\mathbf{R}^S; \mathbf{R}^{-S}).$$

If f is strategy-proof, so is g. Therefore, Lemma 2 implies that is $O^f_{-S}(\mathbf{R}^S)$ is closed when $\mathcal{D} = \mathcal{S}$ and f is strategy-proof.[4]

Lemma 3. *If $f\colon \mathcal{S}^n \to A$ is strategy-proof, then for all nonempty $S \subset N$ and all $\mathbf{R}^S \in \mathcal{D}^{n-|S|}$, $O^f_{-S}(\mathbf{R}^S)$ is closed.*

Anonymity is the requirement that a social choice function treats individuals symmetrically.

Definition. A social choice function f is *anonymous* if for all $\mathbf{R}, \mathbf{R}' \in \mathcal{D}^n$ for which $\mathbf{R}'$ is a permutation of $\mathbf{R}$, $f(\mathbf{R}) = f(\mathbf{R}')$.

An alternative $x \in A$ is *Pareto optimal* if there does not exist an alternative $y \in A$ such that yP_ix for all $i \in N$.[5] *Pareto efficient* social choice functions always choose Pareto optimal alternatives.

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[3] Lemma 2 is also a special case of Lemma 1 in Barberà and Jackson (1994). See also Zhou (1991, p. 113).

[4] Lemma 3 is a special case of Le Breton and Weymark (1999, Proposition 6).

[5] More precisely, this is the definition of a weakly Pareto optimal alternative. For the problem considered here, the sets of weakly and strongly Pareto optimal alternatives coincide.

Definition. A social choice function f is *Pareto efficient* if for all $\mathbf{R} \in \mathcal{D}^n$, $f(\mathbf{R})$ is Pareto optimal.

3 The Tops-Only Property

For a single-peaked preference, the top set on any closed interval of A can be identified either (i) by performing a combination of minimization and maximization operations that only consider the endpoints of the interval and the peak of the preference or (ii) by determing the median of these three alternatives. Furthermore, in the first of these cases, the two operations can be employed in either order.

Lemma 4. *For any closed interval $\overline{xy}$ of A and any $R \in \mathcal{S}$,*

$$\tau(R, \overline{xy}\,) = \min\{y, \max\{\pi(R), x\}\}, \tag{1}$$

$$\tau(R, \overline{xy}\,) = \max\{x, \min\{\pi(R), y\}\}, \tag{2}$$

and

$$\tau(R, \overline{xy}\,) = \mathrm{med}\{x, y, \pi(R)\}. \tag{3}$$

Proof. If $x < \pi(R) < y$, then $\tau(R, \overline{xy}\,) = \pi(R)$. Because $\max\{\pi(R), x\} = \pi(R)$ and $\min\{\pi(R), y\} = \pi(R)$, (1) and (2) are satisfied.

If $\pi(R) \leq x$, then $\tau(R, \overline{xy}\,) = x$. Because $\max\{\pi(R), x\} = x$ and $x \leq y$, (1) holds. Because $\min\{\pi(R), y\} = \pi(R)$ and $\pi(R) \leq x$, (2) holds.

If $\pi(R) \geq y$, then $\tau(R, \overline{xy}\,) = y$. Because $\max\{\pi(R), x\} = \pi(R) \leq y$, (1) holds. Because $\min\{\pi(R), y\} = y \geq x$, (2) holds.

The characterization of the median in (3) follows immediately from either (1) or (2). □

Thus, when S is a closed interval $\overline{xy}$ of A and R is single-peaked, the top set $\tau(R, S)$ consists of a single alternative: x, y, or $\pi(R)$. It also follows from Lemma 4 that the median of three numbers can be computed using a combination of minimization and maximization operations. If S is not an interval, $\tau(R, S)$ may include two alternatives, but not more.

An immediate implication of Lemma 4 is that the top set of a single-peaked preference on a closed interval of A only depends on the preference peak and not on how non-peak alternatives are ordered.[6]

Lemma 5. *For any closed interval $\overline{xy}$ of A and any $R, R' \in \mathcal{S}$, if $\pi(R) = \pi(R')$, then $\tau(R, \overline{xy}\,) = \tau(R', \overline{xy}\,)$.*

[6] All preferences in $\mathcal{S}$ with the same peak order alternatives on the same side of the peak in the same way, but they may order alternatives on opposite sides of the peak differently.

If a social choice function takes account of all of the information contained in a preference profile, then, in general, it is manipulable. One way to restrict the usable information in a preference profile is for the social choice function to be sensitive only to the top sets of the individual preferences, what is known as the *tops-only property.*

Definition. A social choice function f satisfies the *tops-only property* if for all $\mathbf{R}, \mathbf{R}' \in \mathcal{D}^n$ for which $\tau(R_i, A) = \tau(R_i', A)$ for all $i \in N$, $f(\mathbf{R}) = f(\mathbf{R}')$.

For a single-peaked preference R, $\tau(R, A)$ is simply the preference peak. Hence, a social choice function f with domain $\mathcal{S}^n$ satisfies the tops-only property if the social choice only depends on the peaks of the individual preferences. Moulin (1980) restricts attention from the outset to social choice functions that satisfy the tops-only property. For a strategy-proof social choice function f with domain $\mathcal{S}^n$, Theorem 1 demonstrates that the tops-only property is equivalent to the range of f being an interval of A.[7]

Theorem 1. *A strategy-proof social choice function* $f\colon \mathcal{S}^n \to A$ (a) *has a range* A^f *which is an interval of* A *if and only if* (b) *it satisfies the tops-only property.*

Proof. For $n \geq 2$, Weymark (2008, Theorem 1) has shown that for a strategy-proof social choice function $f\colon \mathcal{S}^n \to A$, if A^f is an interval of A, then $f(\mathbf{R}) = f(\mathbf{R}')$ for any two profiles $\mathbf{R}, \mathbf{R}' \in \mathcal{D}^n$ for which $\tau(R_i, A^f) = \tau(R_i', A^f)$ for all $i \in N$.[8] That is, the same alternative is chosen whenever two profiles have the same individual peaks on the range of A. The same concluson holds for $n = 1$ by Lemma 1. By Lemma 5, for any $R, R' \in \mathcal{S}$, $\tau(R, A^f) = \tau(R', A^f)$ if $\tau(R, A) = \tau(R', A)$. Hence, f satisfies the tops-only property if A^f is an interval of A.

Now suppose that $f\colon \mathcal{S}^n \to A$ is strategy-proof and satisfies the tops-only property, but that A^f is not an interval of A. Because A^f is closed by Lemma 2, there therefore exist alternatives $x, y, z \in A$ with $x < y < z$ such that $x, z \in A^f$, but $(x, z) \cap A^f = \varnothing$. Let $R^x \in \mathcal{S}$ be such that $\tau(R^x, A) = \{y\}$ and $\tau(R^x, A^f) = \{x\}$. Similarly, let $R^z \in \mathcal{S}$ be such that $\tau(R^z, A) = \{y\}$ and $\tau(R^z, A^f) = \{z\}$. Clearly, such preferences exist. By Lemma 1, $f(R^x, \ldots, R^x) = x$ and $f(R^z, \ldots, R^z) = z$, which contradicts the assumption that f satisfies the tops-only property. $\square$

If a social choice function $f\colon \mathcal{S}^n \to A$ is Pareto efficient, then for any $x \in A$, x is chosen if everybody has a preference with peak at x. Hence, the range

[7] Barberà and Jackson (1994, Theorem 1) have shown that the restriction of $f\colon \mathcal{S}^n \to A$ to the domain of preference profiles that are single-peaked on the range of A satisfies the tops-only property if it is strategy-proof.

[8] Ching (1997, pp. 485–486) has shown that if A is a closed interval of $\mathbb{R}$ and $A^f = A$, then strategy-proofness implies the tops-only property when $\mathcal{D} = \mathcal{S}$. Sprumont (1995, Lemma 2) has established the same implication for the case in which A is a closed interval of $\mathbb{R}$ and the range is a closed interval of A.

of f is all of A and, by Theorem 1, any strategy-proof social choice function with domain $\mathcal{S}^n$ that is Pareto efficient satisfies the tops-only property.

Proposition 1. *If a strategy-proof social choice function* $f\colon \mathcal{S}^n \to A$ *is Pareto efficient, then it satisfies the tops-only property.*

4 Min-Max Social Choice Functions

A *voting scheme* is a function $v\colon \mathcal{A}^n \to A$, where $\mathcal{A} \subseteq A$. The domain of v is *unrestricted* if $\mathcal{A} = A$. Let $\mathbf{x} = (x_1, \ldots, x_n)$ for any $(x_1, \ldots, x_n) \in \mathbb{R}^n$. If $f\colon \mathcal{D}^n \to A$ is a social choice function that satisfies the tops-only property, then f can be identified with the voting scheme $v^f\colon \mathcal{A}^n \to A$, where $\mathcal{A} = \{x \in A \mid x = \tau(R, A) \text{ for some } R \in \mathcal{D}\}$, by setting $v^f(\mathbf{x}) = f(R_1, \ldots, R_n)$ for any profile $\mathbf{R} \in \mathcal{D}^n$ for which $x_i = \tau(R_i, A)$ for all $i \in N$. If $\mathcal{D} = \mathcal{S}$ in this definition, then $\mathcal{A} = A$ and, hence, v^f has an unrestricted domain. For a social choice function f that satisfies the tops-only property, (i) f is anonymous if and only if v^f is symmetric (i.e., the value of v^f is invariant to a permutation of its arguments) and (ii) if $\mathcal{D} = \mathcal{S}$, f is Pareto efficient if and only if $\min_{i \in N}\{x_i\} \leq v^f(\mathbf{x}) \leq \max_{i \in N}\{x_i\}$ for all $\mathbf{x} \in A^n$.

Moulin (1980) introduced a class of voting schemes known as *min-max voting schemes.*

Definition. For $\mathcal{A} \subseteq A \subseteq \mathbb{R}$, a voting scheme $v\colon \mathcal{A}^n \to A$ is a *min-max voting scheme* if for all $S \subseteq N$ (including $S = \varnothing$), there exists an $a_S \in A^*$ with (i) $a_T \leq a_S$ if $S \subseteq T \subseteq N$, (ii) $a_N \neq a_+$ if $a_+ \notin A$, and (iii) $a_\varnothing \neq a_-$ if $a_- \notin A$ such that for all $\mathbf{x} \in \mathcal{A}^n$,

$$v(\mathbf{x}) = \min_{S \subseteq N} \left[\max_{i \in S}\{x_i, a_S\} \right].^9 \tag{4}$$

Note that a min-max voting scheme is nondecreasing in its arguments and that it is characterized by 2^n parameters drawn from A^*, one for each subset of N.

For a domain of profiles of single-peaked preferences, the corresponding class of *min-max social choice functions* is defined as follows.

Definition. A social choice function $f\colon \mathcal{D}^n \to A$ for which $\mathcal{D} \subseteq \mathcal{S}$ is a *min-max social choice function* if for all $\mathbf{R} \in \mathcal{D}^n$,

$$f(\mathbf{R}) = v^f(\pi(R_1, \ldots, \pi(R_n)), \tag{5}$$

for some min-max voting scheme $v^f\colon \mathcal{A}^n \to A$, where $\mathcal{A} = \{x \in A \mid x = \tau(R, A) \text{ for some } R \in \mathcal{D}\}$.

[9] This definition extends the definition in Moulin (1980) for $A = \mathbb{R}$ to an arbitrary $A \subseteq \mathbb{R}$. By convention, $\max_{i \in \varnothing}\{x_i, a_\varnothing\} = a_\varnothing$.

By definition, a min-max social choice function satisfies the tops-only property.

Without conditions (ii) and (iii) in the definition of a min-max voting scheme, the function v defined in (4) need not be a voting scheme. If $a_+ \notin A$ and a_N is permitted to equal a_+ (and, hence, for a_S to equal a_+ for all S), then $v(\mathbf{x}) = a_+$ for all $\mathbf{x} \in \mathcal{A}^n$, in which case v is not a voting scheme because $a_+ \notin A$. Similarly, if $a_- \notin A$ and $a_\varnothing$ is permitted to equal a_- (and, hence, for a_S to equal a_- for all S), then $v(\mathbf{x}) = a_-$ for all $\mathbf{x} \in \mathcal{A}^n$ (because $a_\varnothing = a_-$) and again v is not a voting scheme.

To show that the function v defined in (4) is in fact a voting scheme, it is necessary to confirm that $v(\mathbf{x}) \in A$ for all $\mathbf{x} \in \mathcal{A}^n$. By definition, $v(\mathbf{x}) \in A^*$. There are four cases to consider.

(i) Suppose that $a_- \in A$ and $a_+ \in A$. Then, $a_S \in A$ for all $S \subseteq N$. In this case, $v(\mathbf{x})$ must be in A because each x_i and a_S is then in A.

(ii) Suppose that $a_- \in A$ and $a_+ \notin A$. Because $a_N \neq a_+$, $\max_{i \in N}\{x_i, a_N\} \in A$ and, therefore, by (4), $v(\mathbf{x}) \neq a_+$. For any $S \subset N$, either $a_S = a_+$ or $a_S \in A$. Hence, by (4), $v(\mathbf{x}) \in A$.

(iii) Suppose that $a_- \notin A$ and $a_+ \in A$. Because $a_\varnothing \neq a_-$, $\max_{i \in S}\{x_i, a_S\} \in A$ for all $S \subseteq N$. Hence, by (4), $v(\mathbf{x}) \in A$.

(iv) Suppose that $a_- \notin A$ and $a_+ \notin A$. Then, reasoning as in cases (ii) and (iii), it follows that $v(\mathbf{x}) \neq a_-$ and $v(\mathbf{x}) \neq a_+$, which implies that $v(\mathbf{x}) \in A$.

If $n = 1$, then (4) simplifies to

$$v(x_1) = \min\{a_\varnothing, \max\{x_1, a_{\{1\}}\}\}.$$

Note that $v(x_1)$ is the alternative that maximizes a single-peaked preference with peak at x_1 on the interval $\overline{a_{\{1\}}a_\varnothing}$ (of A). Hence, as has already been observed, $v(x_1)$ is also the median of x_1, $a_{\{1\}}$, and $a_\varnothing$.

For later reference, the formula for $v(\mathbf{x})$ in (4) when $n = 2$ is written out in full. In this case,

$$v(x_1, x_2) = \min\{a_\varnothing, \max\{x_1, a_{\{1\}}\}, \max\{x_2, a_{\{2\}}\}, \max\{x_1, x_2, a_{\{1,2\}}\}\}. \quad (6)$$

In the characterization theorems for various classes of strategy-proof social choice functions on the domain $\mathcal{S}^n$, I only need to consider voting schemes with an unrestricted domain. For simplicity, I henceforth restrict attention to such voting schemes.

Consider a voting scheme $v\colon A^n \to A$. For all $S \subseteq N$, let $\mathbf{x}^S$ be the n-vector defined by setting

$$x_i^S = \begin{cases} a_-, & \text{if } i \in S, \\ a_+, & \text{if } i \notin S. \end{cases}$$

In Proposition 2, I show that when both a_- and a_+ are in A, for all $S \subseteq N$, the parameter a_S that appears in the definition of a min-max voting scheme v is $v(\mathbf{x}^S)$. When either a_- or a_+ are not in A, then $\mathbf{x}^S$ is not in the domain of v. However, as I shall also show in Proposition 2, in such cases, a_S is the limiting value of $v(\mathbf{x})$ as $\mathbf{x}$ approaches $\mathbf{x}^S$. In order to establish these results, I first need some additional notation.

For all $S \subseteq N$ and all $\lambda, \mu \in A$, define the n-vector $\mathbf{x}^S(\lambda, \mu)$ by setting

$$x^S(\lambda,\mu)_i = \begin{cases} \lambda, & \text{if } i \in S, \\ \mu, & \text{if } i \notin S, \end{cases}$$

the n-vector $\mathbf{x}^S(\lambda)$ by setting

$$x^S(\lambda)_i = \begin{cases} \lambda, & \text{if } i \in S, \\ a_+, & \text{if } i \notin S, \end{cases}$$

and the n-vector $\mathbf{x}^S(\mu)$ by setting

$$x^S(\mu)_i = \begin{cases} a_-, & \text{if } i \in S, \\ \mu, & \text{if } i \notin S. \end{cases}$$

Note that

$$\lim_{\substack{\lambda \to a_- \\ \mu \to a_+}} \mathbf{x}^S(\lambda,\mu) = \lim_{\lambda \to a_-} \mathbf{x}^S(\lambda) = \lim_{\mu \to a_+} \mathbf{x}^S(\mu) = \mathbf{x}^S.$$

Proposition 2. *Let $v\colon A^n \to A$ be a min-max voting scheme. Then, for all $S \subseteq N$,*

(i) *if $a_- \in A$ and $a_+ \in A$, then*

$$v(\mathbf{x}^S) = a_S; \tag{7}$$

(ii) *if $a_- \in A$ and $a_+ \notin A$, then*

$$\lim_{\mu \to a_+} v(\mathbf{x}^S(\mu)) = a_S; \tag{8}$$

(iii) *if $a_- \notin A$ and $a_+ \in A$, then*

$$\lim_{\lambda \to a_-} v(\mathbf{x}^S(\lambda)) = a_S; \tag{9}$$

(iv) *if $a_- \notin A$ and $a_+ \notin A$, then*

$$\lim_{\substack{\lambda \to a_- \\ \mu \to a_+}} v(\mathbf{x}^S(\lambda,\mu)) = a_S. \tag{10}$$

Proof. Each of the four cases is considered in turn.

(i) In this case, (4) is used to compute the value of $v(\mathbf{x}^S)$. For any $T \subseteq S$, $\max_{i \in T}\{x_i^S, a_T\} = a_T$ because $a_T \geq a_- = x_i^S$ for all $i \in T$. For any $T \not\subseteq S$, $\max_{i \in T}\{x_i^S, a_T\} = a_+$ because $a_T \leq a_+$ and there exists a $j \in T$ with $a_j^S =$

a_+. Hence, $v(\mathbf{x}^S) = \min_{T \subseteq S} a_T$. Because $a_S \leq a_T$ if $T \subseteq S$, it follows that $v(\mathbf{x}^S) = a_S$, which establishes (7).

(ii) For all $\mu \in A$ and all $T \subseteq S$, as in case (i), $\max_{i \in T}\{\mathbf{x}^S(\mu)_i, a_T\} = a_T$. For any $T \not\subseteq S$, $\max_{i \in T}\{\mathbf{x}^S(\mu)_i, a_T\} = a_+$. Because $a_T \leq a_+$ if $T \subseteq S$, $\lim_{\mu \to a_+} v(\mathbf{x}^S(\mu)) = \min_{T \subseteq S} a_T = a_S$, which establishes (8).

(iii) For all $\lambda \in A$ and all $T \not\subseteq S$, as in case (i), $\max_{i \in T}\{\mathbf{x}^S(\lambda)_i, a_T\} = a_+$. For all $T \subseteq S$, $\lim_{\lambda \to a_-} \max_{i \in T}\{\mathbf{x}^S(\lambda)_i, a_T\} = a_T$ because $a_T \geq a_-$. Hence, $\lim_{\lambda \to a_-} v(\mathbf{x}^S(\lambda)) = \min_{T \subseteq S} a_T = a_S$, which establishes (9).

(iv) Applying the argument in (ii) for the upper limit and the argument in (iii) for the lower limit yields (10). □

I now show that a min-max voting scheme v on an unrestricted domain satisfies a kind of unanimity property. Specifically, for any alternative $x \in \overline{a_N a_\varnothing}$, $v(x, \ldots, x) = x$. Moreover, I also show that for any other $x \in A$, $v(x, \ldots, x)$ is the closest alternative to x in $\overline{a_N a_\varnothing}$. Thus, $v(x, \ldots, x)$ is the median of x, a_N, and $a_\varnothing$

Proposition 3. *Let $v \colon A^n \to A$ be a min-max voting scheme. Then, for all $x \in A$,* (i) *$v(x, \ldots, x) = x$ if $x \in \overline{a_N a_\varnothing}$,* (ii) *$v(x, \ldots, x) = a_N$ if $x < a_N$, and* (iii) *$v(x, \ldots, x) = a_\varnothing$ if $x > a_\varnothing$.*

Proof. (i) For any $x \in \overline{a_N a_\varnothing}$, $\max\{x, a_N\} = x$. Because $a_S \geq a_N$ for all $S \subseteq N$, it then follows from (4) that $v(x, \ldots, x) = \min\{x, a_\varnothing\}$. But, by assumption, $x \leq a_\varnothing$. Hence, $v(x, \ldots, x) = x$.

(ii) The conclusion in this case follows from (4) and the assumption that $a_N \leq a_S$ for all $S \subseteq N$.

(iii) The conclusion in this case follows from trivially from (4). □

In Proposition 4, I show that the range of a min-max voting scheme on an unrestricted domain is the closed interval of A defined by the parameters a_N and $a_\varnothing$. Hence, by Proposition 3, $v(x, \ldots, x)$ is the closest alternative to x in the range of v.

Proposition 4. *The range of a min-max voting scheme $v \colon A^n \to A$ is $\overline{a_N a_\varnothing}$.*

Proof. Because $a_N \leq a_S$ for all $S \subseteq N$, it follows from (4) that $v(\mathbf{x}) \geq a_N$ for all $\mathbf{x} \in A^n$. Because $a_\varnothing$ is one of the values to which the minimization operator in (4) is applied, $v(\mathbf{x}) \leq a_\varnothing$ for all $\mathbf{x} \in A^n$. Hence, the range of v is contained in $\overline{a_N a_\varnothing}$. For any $x \in \overline{a_N a_\varnothing}$, by Proposition 3, $v(x, \ldots, x) = x$. Thus, every $x \in \overline{a_N a_\varnothing}$ is in the range of v. □

Definition. A voting scheme $v \colon A^n \to A$ is *uncompromising* if for all $i \in N$ and all $\mathbf{x}, \mathbf{x}' \in A^n$ for which $x_j = x'_j$ for all $j \neq i$, (i) $v(\mathbf{x}) = v(\mathbf{x}')$ if $x'_i > x_i > v(\mathbf{x})$ and (ii) $v(\mathbf{x}) = v(\mathbf{x}')$ if $x'_i < x_i < v(\mathbf{x})$.

Unompromisingness was first considered by Border and Jordan (1983). Informally, v is uncompromising if for any individual i and any vector $\mathbf{x}$, raising (resp. lowering) x_i does not affect what is chosen whenever x_i is larger (resp. smaller) than $v(\mathbf{x})$. Min-max voting schemes are uncompromising.

Proposition 5. *A min-max voting scheme* $v\colon A^n \to A$ *is uncompromising.*

Proof. Consider any $i \in N$ and any $\mathbf{x}, \mathbf{x}' \in A^n$ for which $x_j = x'_j$ for all $j \neq i$.

(i) Suppose that $x'_i > x_i > v(\mathbf{x})$. For any $S \subseteq N$, increasing x_i to x'_i cannot decrease the value of $\max_{k \in S}\{a_k, a_S\}$ and it can only increase this value if $i \in S$. By the definition of a min-max voting scheme in (4), there exists an $S^* \subseteq N$ such that $v(\mathbf{x}) = \max_{k \in S^*}\{a_k, a_{S^*}\}$. Because $x_i > v(\mathbf{x})$, this implies that $i \notin S^*$. Hence, by (4), $v(\mathbf{x}') = \max_{k \in S^*}\{a_k, a_{S^*}\} = v(\mathbf{x})$.

(ii) Now suppose that $x'_i < x_i < v(\mathbf{x})$. Because v is nondecreasing in its arguments, $v(\mathbf{x}') \leq v(\mathbf{x})$. Contrary to what is to be shown, suppose that $v(\mathbf{x}') < v(\mathbf{x})$. This is only possible if there exists an $\bar{S} \subseteq N$ with $i \in \bar{S}$ such that both $v(\mathbf{x}) = \max_{k \in \bar{S}}\{a_k, a_{\bar{S}}\}$ and $\max\{x'_i, \max_{k \in \bar{S}\setminus\{i\}} x_k, a_{\bar{S}}\} < v(\mathbf{x})$. But this implies that $x_i = \max_{k \in \bar{S}}\{a_k, a_{\bar{S}}\} = v(\mathbf{x})$, a contradiction to the assumption that $x_i < v(\mathbf{x})$. □

I now provide some examples of min-max voting schemes. In each of these examples, the voting scheme v has the domain A^n.

Example 1. Let $a_S = \bar{x}$ for all $S \subseteq N$. Note that the restrictions on the parameters of v imply that $\bar{x} \in A$. By Proposition 4, the range of v is $\{\bar{x}\}$. Hence, v is imposed. For the corresponding min-max social choice function, $\bar{x}$ is chosen regardless of what the individual preferences are.

Example 2. Consider any $k \in N$. For all $S \subseteq N$, let $a_S = a_+$ if $|S| < k$ and let $a_S = a_-$ otherwise. For any $\mathbf{x} \in A^n$, let $\tilde{\mathbf{x}}$ be a permutation of $\mathbf{x}$ for which $\tilde{x}_1 \leq \tilde{x}_2 \leq \cdots \leq \tilde{x}_n$, with ties broken arbitrarily. Then, for all $\mathbf{x} \in A^n$, $v(\mathbf{x}) = \tilde{x}_k$, the kth smallest component of $\mathbf{x}$. The corresponding min-max social choice function always chooses the kth smallest preference peak.

Example 3. Suppose that $n = 2$. Let $a_\varnothing = a_+$, $a_N = a_-$, and $a_{\{1\}} = a_{\{2\}} = b$. Using (6), for all $\mathbf{x} \in A^2$,

$$\begin{aligned} v(\mathbf{x}) &= \min\{a_+, \max\{x_1, b\}, \max\{x_2, b\}, \max\{x_1, x_2, a_-\}\} \\ &= \min\{\max\{x_1, b\}, \max\{x_2, b\}, \max\{x_1, x_2\}\} \\ &= \operatorname{med}\{x_1, x_2, b\} \\ &= \operatorname{med}\{x_1, x_2, b, a_-, a_+\}. \end{aligned}$$

Thus, v first augments $\mathbf{x}$ with the parameter b and then chooses the median value. Equivalently, v first augments $\mathbf{x}$ with the parameters b, a_-, and a_+ and then chooses the median value. The corresponding min-max social choice function always chooses the median of the two preference peaks and b. Note that the median of x_1, x_2, and b can be computed by first identifying the maximum values in any two-element subset of these three alternatives and then choosing the smallest of them.

Example 4. Suppose that $n = 2$.

(i) Let $a_{\{2\}} = a_N = \alpha$ and $a_{\{1\}} = a_\varnothing = \beta$. Thus, the range of v is $\overline{\alpha\beta}$. Using (6), for all $\mathbf{x} \in A^2$,

$$\begin{aligned} v(\mathbf{x}) &= \min\{\beta, \max\{x_1, \beta\}, \max\{x_2, \alpha\}, \max\{x_1, x_2, \alpha\}\} \\ &= \min\{\beta, \max\{x_2, \alpha\}\}, \end{aligned}$$

which is the alternative in the range that is closest to x_2. That is, $v(\mathbf{x}) = \text{med}\{x_2, \alpha, \beta\}$.

(ii) Let $a_{\{1\}} = a_N = \alpha$ and $a_{\{2\}} = a_\varnothing = \beta$. Reasoning as in (i), for all $\mathbf{x} \in A^2$, $v(\mathbf{x})$ is the alternative in the range that is closest to x_1.. That is, $v(\mathbf{x}) = \text{med}\{x_1, \alpha, \beta\}$.

For the corresponding min-max social choice function, person 2 is a dictator on the range in case (i) and person 1 is a dictator on the range in case (ii).

There is a duality between minimization and maximization operators that allows one to rewrite the formula for the min-max voting scheme defined in (4) as a *max-min* voting scheme in which the order in which the minimization and maximization operations are employed is reversed. An illustration of this kind of role reversal is provided by the equivalence of (1) and (2) in Lemma 4.

Barberà, Gul, and Stacchetti (1993) have proposed an alternative, but equivalent, formulation of a min-max voting scheme v using left-winning coalition systems. A *left-winning coalition system* assigns to each alternative a (possibly empty) set of of subsets of N (the winning coalitions) satisfying a number of restrictions. The value of $v(\mathbf{x})$ is then the smallest alternative $\bar{x}$ for which the set $\{i \in N \mid x_i \leq \bar{x}\}$ is a winning coalition for $\bar{x}$. Using the duality between $\leq$ and $\geq$, this rule can also be expressed in terms of a *right-winning coalition system*. Formal definitions of left- and right-winning coalition systems may be found in Barberà, Gul, and Stacchetti (1993) when A is discrete and in Barberà, Massó, and Serizawa (1998) when A is a closed interval. Sprumont (1995), Barberà (2011), and Massó and Moreno de Barreda (2011) have provided interpretations of these rules that highlight how these constructions are related to the definition of min-max and generalized median voting schemes.

Yet another equivalent formulation of a min-max voting scheme was provided by Ching (1997). An *augmented median* voting scheme is characterized by 2^n parameters, one for each subset of the individuals, as in the definition of a min-max voting scheme. However, now the value of $v(\mathbf{x})$ is computed by taking the median of the components of $\mathbf{x}$ and $n + 1$ of the parameters, with the choice of the parameters depending on $\mathbf{x}$. See Sprumont (1995) and Ching (1997) for a formal definition and discussion of this class of rules.

5 Strategy-Proof Min-Max Social Choice Functions

Theorem 2 generalizes the characterization of min-max social choice functions in Moulin (1980, Proposition 3) by allowing A to be any subset of $\mathbb{R}$ containing at least two alternatives, not just $\mathbb{R}$ itself. It also replaces Moulin's assumption that f satisfies the tops-only property with the equivalent assumption that the range of f is an interval of A.

Theorem 2. *Let $f\colon \mathcal{S}^n \to A$ be a social choice function whose range A^f is an interval of A. Then,* (a) *f is strategy-proof if and only if* (b) *f is a min-max social choice function.*

Proof. (a) First, suppose that f is strategy-proof. By Lemma 2, A^f is closed. Hence $A^f = \overline{\alpha\beta}$ for some $\alpha, \beta \in A^*$ with $\alpha \leq \beta$. Furthermore, if $a_+ \notin A$, then $\alpha \neq a_+$ and if $a_- \notin A$, then $\beta \neq a_-$, for otherwise A^f would be empty. By Theorem 1, f satisfies the tops-only property and, hence, can be identified by its associated voting scheme v^f. For all $x \in A$, there exists an $R \in \mathcal{S}$ such that $\tau(R, A) = \{x\}$. Thus, v^f has the unrestricted domain A^n. We proceed by induction on n. Let f^n denote the social choice function when there are n individuals.

(i) Suppose that $n = 1$. By Lemma 1, $f^1(R_1) = \tau(R_1, \overline{\alpha\beta})$ for all $R_1 \in \mathcal{S}$. By (1), $\tau(R_1, \overline{\alpha\beta}) = \min\{\beta, \max\{\pi(R_1), \alpha\}\}$. Letting $a_\varnothing = \beta$ and $v_{\{1\}} = \alpha$, we conclude that f^1 is a min-max social choice function.

(ii) Now suppose that $n \geq 2$ and that (a) imples (b) when the number of individuals is less than or equal to n. Also suppose that f^{n+1} is strategy-proof. We need to show that f^{n+1} is a min-max social choice function.

Consider any $\mathbf{x} \in A^n$. Because f^{n+1} has the tops-only property, by Lemma 3, for all $(R_1, \ldots, R_n) \in \mathcal{S}^n$ for which $\pi(R_i) = x_i$ for all $i = 1, \ldots, n$, the option set $O^{\{n+1\}}(R_1, \ldots, R_n)$ is a closed interval $\overline{\alpha_{\mathbf{x}}\beta_{\mathbf{x}}}$. Because this option set is the range of the strategy-proof one-person social choice function obtained from f^{n+1} by fixing $(R_1, \ldots, R_n)$, by case (i) this one-person social choice function is a min-max social choice function. In terms of the voting scheme $v^{f^{n+1}}$ associated with f^{n+1}, we have

$$v^{f^{n+1}}(\mathbf{x}, x_{n+1}) = \min\{\beta_{\mathbf{x}}, \max\{x_{n+1}, \alpha_{\mathbf{x}}\}\} \tag{11}$$

for all $(\mathbf{x}, x_{n+1}) \in A^{n+1}$.

For any fixed $x_{n+1} \in A$, by the induction hypothesis,

$$v^{f^n}_{x_{n+1}}(\mathbf{x}) := v^{f^{n+1}}(\mathbf{x}, x_{n+1}) = \min_{S \subseteq \{1,\ldots,n\}} \left[\max_{i \in S}\{x_i, a_S(x_{n+1})\}\right], \tag{12}$$

for all $\mathbf{x} \in A^n$, where now the parameters $a_S(x_{n+1})$ of the n-person voting scheme $v^{f^n}_{x_{n+1}}$ are conditional on x_{n+1}.

Some of the details of the next part of the proof depend on whether a_- and a_+ are in A. We provide the argument for the case in which they are not. If either a_- or a_+ is in A, we instead use the relevant case in Proposition 2 to determine the analogue of (13). If both a_- and a_+ are in A, then no limiting arguments are needed.

By (12) and case (iv) of Proposition 2,

$$\lim_{\substack{\lambda\to a_-\\ \mu\to a_+}} v^{f^{n+1}}(\mathbf{x}^S(\lambda,\mu), x_{n+1})) = a_S(x_{n+1}), \tag{13}$$

for all $x_{n+1} \in A$ and all $S \subseteq \{1,\dots,n\}$. It then follows from (11) and (13) that for all $x_{n+1} \in A$ and all $S \subseteq \{1,\dots,n\}$,

$$a_S(x_{n+1}) = \lim_{\substack{\lambda\to a_-\\ \mu\to a_+}} \min\{\beta_{\mathbf{x}^S(\lambda,\mu)}, \max\{x_{n+1}, \alpha_{\mathbf{x}^S(\lambda,\mu)}\}\},$$

or, equivalently,

$$a_S(x_{n+1}) = \min\left\{\lim_{\substack{\lambda\to a_-\\ \mu\to a_+}} \beta_{\mathbf{x}^S(\lambda,\mu)}, \max\{x_{n+1}, \lim_{\substack{\lambda\to a_-\\ \mu\to a_+}} \alpha_{\mathbf{x}^S(\lambda,\mu)}\}\right\}. \tag{14}$$

For all $S \subseteq \{1,\dots,n\}$, let

$$\alpha_{\mathbf{x}^S} = \lim_{\substack{\lambda\to a_-\\ \mu\to a_+}} \alpha_{\mathbf{x}^S(\lambda,\mu)} \tag{15}$$

and

$$\beta_{\mathbf{x}^S} = \lim_{\substack{\lambda\to a_-\\ \mu\to a_+}} \beta_{\mathbf{x}^S(\lambda,\mu)}.^{10} \tag{16}$$

Substituting (15) and (16) into (14), we obtain

$$a_S(x_{n+1}) = \min\{\beta_{\mathbf{x}^S}, \max\{x_{n+1}, \alpha_{\mathbf{x}^S}\}\}, \tag{17}$$

for all $x_{n+1} \in A$ and all all $S \subseteq \{1,\dots,n\}$. Substituting (17) into (12), for all $(\mathbf{x}, x_{n+1}) \in A^{n+1}$,

$$\begin{aligned} v^{f^{n+1}}(\mathbf{x}, x_{n+1}) &= \min_{S\subseteq\{1,\dots,n\}} \left[\max_{i\in S}\{x_i, \min\{\beta_{\mathbf{x}^S}, \max\{x_{n+1}, \alpha_{\mathbf{x}^S}\}\}\}\right] \\ &= \min_{S\subseteq\{1,\dots,n\}} \left[\min\left\{\max_{i\in S}\{x_i, \beta_{\mathbf{x}^S}\}, \max_{i\in S}\{x_i, x_{n+1}, \alpha_{\mathbf{x}^S}\}\right\}\right] \end{aligned} \tag{18}$$

because $\max\{X, \min\{Y, Z\}\} = \min\{\max\{X, Y\}, \max\{X, Z\}\}$.

For all $S \subseteq \{1,\dots,n\}$, let $a_S = \beta_{\mathbf{x}^S}$ and $a_{S\cup\{n+1\}} = \alpha_{\mathbf{x}^S}$. Using these definitions in (18), we obtain for all $(\mathbf{x}, x_{n+1}) \in A^{n+1}$,

$$v^{f^{n+1}}(\mathbf{x}, x_{n+1}) = \min_{S\subseteq\{1,\dots,n\}} \left[\min\left\{\max_{i\in S}\{x_i, a_S\}, \max_{i\in S\cup\{n+1\}}\{x_i, a_{S\cup\{n+1\}}\}\right\}\right].$$

Hence, for all $(\mathbf{x}, x_{n+1}) \in A^{n+1}$,

[10] Note that if either a_- and a_+ are not in A, then $\mathbf{x}^S$ is not in A^n. However, if they are, then $\alpha_{\mathbf{x}^S}$ and $\beta_{\mathbf{x}^S}$ are the values used in (11) when $\mathbf{x} = \mathbf{x}^S$.

$$v^{f^{n+1}}(\mathbf{x}, x_{n+1}) = \min_{S \subseteq \{1,\ldots,n+1\}} \left[\max_{i \in S} \{x_i, a_S\} \right]. \tag{19}$$

Consider any $S \subset T \subseteq \{1, \ldots, n+1\}$. If $a_S < a_T$, then for all $(\mathbf{x}, x_{n+1}) \in A^{n+1}$, $\max_{i \in T}\{x_i, a_T\} \geq \max_{i \in S}\{x_i, a_S\}$. Reducing the value of a_T so that it equals a_S preserves this inequality and has no effect on the value of $v^{f^{n+1}}(\mathbf{x}, x_{n+1})$ in (19). Hence, it can be assumed that if $S \subset T \subseteq \{1, \ldots, n+1\}$, then $a_T \leq a_S$.

By Proposition 4, $a_{N \cup \{n+1\}} = \alpha$ and $a_\varnothing = \beta$, where $\overline{\alpha\beta}$ is the range of $v^{f^{n+1}}$. It has already been shown that $\alpha \neq a_+$ if $a_+ \notin A$ and $\beta \neq a_-$ if $a_- \notin A$. Therefore, $v^{f^{n+1}}$ is a min-max voting scheme and f is a min-max social choice function.

(b) Now suppose that f is a min-max social choice function and v^f is the corresponding min-max voting scheme. Consider any $\mathbf{R} \in \mathcal{S}^n$ and any $i \in N$. Let $\mathbf{x} = (\pi(R_1), \ldots, \pi(R_n))$.

(i) If $v^f(\mathbf{x}) = x_i$, then individual i obtains his most-preferred alternative and so cannot manipulate the outcome.

(ii) If $v^f(\mathbf{x}) < x_i$, because preferences are single-peaked, a necessary condition for individual i to be able to manipulate the outcome is that there exists a preference with peak x_i' that he could report that would increase what is chosen. By Proposition 5, v^f is uncompromising. Because v^f is also nondecreasing in its arguments, individual i cannot increase what is chosen.

(iii) If $v^f(\mathbf{x}) > x_i$, then individual i can only manipulate the outcome by reducing its value, which by the reasoning in (ii) is not possible.

Thus, it has been shown that f is strategy-proof. □

Moulin (1980) did not characterize the set of strategy-proof social choice functions that are also Pareto efficient. By Proposition 1, such functions also satisfy the tops-only property. Theorem 3 shows that the only restriction on a min-max social choice function that Pareto efficiency imposes is that the range must be all of A, which is ensured by setting $a_\varnothing = a_+$ and $a_N = a_-$.

Theorem 3. *A social choice function* $f \colon \mathcal{S}^n \to A$ *is* (a) *strategy-proof and Pareto efficient if and only if* (b) *it is a min-max social choice function with* $a_\varnothing = a_+$ *and* $a_N = a_-$.

Proof. (a) First, suppose that f is strategy-proof and Pareto efficient. By Proposition 1, f satisfies the tops-only property. Consider any $x \in A$ and any $R \in \mathcal{S}$ for which $\pi(R) = x$. Because f is Pareto efficient, $f(R, \ldots, R) = x$. Hence, $A^f = A$ and the range of f is trivially an interval of A. Theorem 2 then implies that f is a min-max social choice function. By Proposition 4, $A^f = \overline{a_N a_\varnothing}$. Therefore, $a_\varnothing = a_+$ and $a_N = a_-$.

(b) Now suppose that f is is a min-max social choice function with $a_\varnothing = a_+$ and $a_N = a_-$ and let v^f be the corresponding min-max voting scheme. By Proposition 4, A^f is the interval $\overline{a_N a_\varnothing}$. Hence, by Theorem 2, f is strategy-proof.

Because $a_N = a_-$, $\max_{i \in N}\{x_i, a_N\} = \max_{i \in N}\{x_i\}$ for all $\mathbf{x} \in A^n$. Hence, by (4), $v^f(\mathbf{x}) \leq \max_{i \in N}\{x_i\}$ for all $\mathbf{x} \in A^n$. Because $a_\varnothing = a_+$, it follows from (4) that for all $\mathbf{x} \in A^n$ there exists an $\bar{S} \subseteq N$ with $\bar{S} \neq \varnothing$ such that $v^f(\mathbf{x}) = \max_{i \in \bar{S}}\{x_i, a_{\bar{S}}\}$. Hence, $v^f(\mathbf{x}) \geq x_j$ for all $j \in \bar{S}$, which implies that $v^f(\mathbf{x}) \geq \min_{i \in N}\{x_i\}$. We have thus shown that for all $\mathbf{x} \in A^n$, $\min_{i \in N}\{x_i\} \leq v^f(\mathbf{x}) \leq \max_{i \in N}\{x_i\}$. Because preferences are single-peaked, these inequalities imply that f is Pareto efficient. □

6 Generalized Median Social Choice Functions

The objective in this section is to show how Moulin's min-max theorem can be used to help establish his two characterization theorems for generalized median social choice functions. This demonstration makes use of two propositions about the computation of medians using a combination of minimization and maximization operations.

In Example 3, the computation of $v(\mathbf{x})$ exploited the fact that the median of three numbers can be determined by first identifying the maximum on any two-element subset of these three numbers and then choosing the smallest of these three maxima. Proposition 6 generalizes this observation. It shows that for any positive integer n and any collection of $2n + 1$ numbers, the median can be computing by first identifying the maximum on any $(n + 1)$-element subset of these numbers and then choosing the smallest of these maxima.

Proposition 6. *Let n be a positive integer, $Y = \{a_1, \ldots, a_{2n+1}\}$ where $a_i \in \mathbb{R}$ for all $i = 1, \ldots, 2n + 1$, and $\mathcal{Y}^{n+1} = \{X \subseteq Y \mid |X| = n + 1\}$. Then,*

$$\operatorname{med} Y = \min_{X \in \mathcal{Y}^{n+1}} [\max\{a_i \mid a_i \in X\}]. \tag{20}$$

Proof. Without loss of generality, we can relabel the alternatives in Y so that $a_1 \leq a_2 \leq \cdots \leq a_{2n+1}$. Then $\operatorname{med} Y = \max\{a_1, \ldots, a_{n+1}\} = a_{n+1}$. For any $X \in \mathcal{Y}^{n+1}$ with $X \neq \{a_1, \ldots, a_{n+1}\}$, $\max\{a_i \mid a_i \in X\} \geq a_{n+1}$. □

In Proposition 6, no order structure was placed on the elements of Y. Now suppose that $Y = \{a_1, \ldots, a_n, b_0, b_1, \ldots, b_n\}$, where $b_i \geq b_{i+1}$ for $i = 1, \ldots, n - 1$. With this structure on the elements of Y, there is an alternative way of characterizing the median alternative in Y. Before providing this characterization for an arbitrary n, the general result is first illustrated for the special cases in which $n = 1$ and $n = 2$.

When $n = 1$, from (20),

$$\operatorname{med}\{a_1, b_0, b_1\} = \min\{\max\{b_0, b_1\}, \max\{a_1, b_0\}, \max\{a_1, b_1\}\}. \tag{21}$$

Because $b_0 \geq b_1$, (21) can be simplified by replacing $\max\{b_0, b_1\}$ with b_0 and by replacing $\{\max\{a_1, b_0\}, \max\{a_1, b_1\}\}$ with the minimum of these two values, which is $\max\{a_1, b_0\}$. Hence,

$$\text{med}\{a_1, b_0, b_1\} = \min\{b_0, \max\{a_1, b_1\}\}. \tag{22}$$

When $n = 2$, from (20),

$$\begin{aligned}\text{med}\{a_1, a_2, b_0, b_1, b_2\} = \min\{&\max\{b_0, b_1, b_2\}, \max\{a_1, b_0, b_1\}, \\ &\max\{a_1, b_0, b_2\}, \max\{a_1, b_1, b_2\}, \max\{a_2, b_0, b_1\}, \\ &\max\{a_2, b_0, b_2\}, \max\{a_2, b_1, b_2\}, \max\{a_1, a_2, b_0\}, \\ &\max\{a_1, a_2, b_1\}, \max\{a_1, a_2, b_2\}\}.\end{aligned} \tag{23}$$

Because $b_0 \geq b_1 \geq b_2$,

(i) $\max\{b_0, b_1, b_2\} = b_0$;
(ii) $\min\{\max\{a_i, b_0, b_1\}, \max\{a_i, b_0, b_2\}, \max\{a_i, b_1, b_2\}\} = \max\{a_i, b_1\}$, for $i = 1, 2$;
(iii) $\min\{\max\{a_1, a_2, b_0\}, \max\{a_1, a_2, b_1\}, \max\{a_1, a_2, b_2\}\} = \max\{a_1, a_2, b_2\}$.

Hence, (23) simplifies to

$$\begin{aligned}&\text{med}\{a_1, a_2, b_0, b_1, b_2\} = \\ &\qquad \min\{b_0, \max\{a_1, b_1\}, \max\{a_2, b_1\}, \max\{a_1, a_2, b_2\}\}.\end{aligned} \tag{24}$$

When $n = 1$ and $n = 2$, as can be seen from (22) and (24), the median is determined in two steps. First, for each $S \subseteq N$, $\max_{i \in S}\{a_i, b_{|S|}\}$ is computed. Second, the minimum of these values is chosen. In order for this procedure to identify the median, it is essential that $b_i \geq b_{i+1}$ for $i = 0, \ldots, n-1$. Proposition 7 shows that this is a general procedure for identifying the median of $Y = \{a_1, \ldots, a_n, b_0, b_1, \ldots, b_n\}$ when the b_i are ordered in this way.

Proposition 7. *Let n be a positive integer and $Y = \{a_1, \ldots, a_n, b_0, b_1, \ldots, b_n\}$ where $a_i \in \mathbb{R}$ for all $i = 1, \ldots, n$, $b_i \in \mathbb{R}$ for all $i = 0, \ldots, n$, and $b_i \geq b_{i+1}$ for $i = 0, \ldots, n-1$. Then,*

$$\text{med}\, Y = \min_{S \subseteq N} \left[\max_{i \in S}\{a_i, b_{|S|}\}\right]. \tag{25}$$

Proof. Define $\mathcal{Y}^{n+1}$ as in Proposition 6. Let $Y^A = \{a_1, \ldots, a_n\}$ and $Y^B = \{b_0, b_1, \ldots, b_n\}$. For $j = 1, \ldots, n+1$, let $\mathcal{Y}_j^{n+1} = \{X \in \mathcal{Y}^{n+1} \mid |X \cap Y^B| = j\}$. By Proposition 6,

$$\text{med}\, Y = \min_{j \in \{1, \ldots, n+1\}} \left\{\min_{X \in \mathcal{Y}_j^{n+1}} [\max X]\right\}. \tag{26}$$

For each $j = 1, \ldots, n+1$, because $b_0 \geq b_1 \geq \cdots \geq b_n$,

$$\min_{X \in \mathcal{Y}_j^{n+1}} [\max X] = \min_{\substack{S \subseteq N \\ |S| = n+1-j}} \left[\max_{i \in S}\{a_i, b_{|S|}\}\right]. \tag{27}$$

Combining (26) and (27) establishes (25). □

An $(n+1)$*-parameter generalized median voting scheme* v is characterized by $n+1$ parameters drawn from A^* with, for all $\mathbf{x}$ in the domain of v, the value of $v(\mathbf{x})$ given by the median of the components of $\mathbf{x}$ and these $n+1$ parameters.

Definition. For $\mathcal{A} \subseteq A \subseteq \mathbb{R}$, a voting scheme $v\colon \mathcal{A}^n \to A$ is an $(n+1)$*-parameter generalized median voting scheme* if there exist $b_i \in A^*$ for $i = 0, \ldots, n$ with (i) not all $b_i = a_+$ if $a_+ \notin A$ and (ii) not all $b_i = a_-$ if $a_- \notin A$ such that for all $\mathbf{x} \in \mathcal{A}^n$,

$$v(\mathbf{x}) = \text{med}\{x_1, \ldots, x_n, b_0, \ldots, b_n\}. \tag{28}$$

For a domain of profiles of single-peaked preferences, the corresponding class of $(n+1)$*-parameter generalized median social choice functions* is defined as follows.

Definition. A social choice function $f\colon \mathcal{D}^n \to A$ for which $\mathcal{D} \subseteq \mathcal{S}$ is an $(n+1)$*-parameter generalized median social choice function* if for all $\mathbf{R} \in \mathcal{D}^n$, (5) holds for some $(n+1)$-parameter generalized median voting scheme $v^f\colon \mathcal{A}^n \to A$, where $\mathcal{A} = \{x \in A \mid x = \tau(R, A) \text{ for some } R \in \mathcal{D}\}$.

Border and Jordan (1983) have interpreted the parameters in a generalized medan social choice function as being the fixed preference peaks of "phantom" voters. Examples 1, 2, and 3 are examples of such rules. In Example 1, $b_i = \bar{x}$ for all i. In Example 2, k of the b_i are set equal to a_+, with the rest of them set equal to a_-. In Example 3, the three parameter values are b, a_-, and a_+. If n is odd, then by setting half of the parameters equal to a_- and the other half to a_+, the median of $\mathbf{x}$ is always chosen, which is Black's median-voter rule when $\mathbf{x}$ is the profile of preference peaks.

Theorem 4 is my version of the characterization theorem for $(n+1)$-parameter generalized median social choice functions established by Moulin (1980, Proposition 2).

Theorem 4. *Let $f\colon \mathcal{S}^n \to A$ be a social choice function whose range A^f is an interval of A. Then,* (a) *f is strategy-proof and anonymous if and only if* (b) *f is an $(n+1)$-parameter generalized median social choice function.*

Proof. (a) First, suppose that f is strategy-proof and anonymous. By Theorem 2, f is a min-max social choice function. Let $v^f\colon A^n \to A$ be the corresponding min-max voting scheme.

It is first shown that $a_S = a_T$ for all $S, T \subseteq N$ with $|S| = |T|$. Note that because f is anonymous and, by Proposition 1, it satisfies the tops-only property, the value of v^f is invariant to a permutation of its arguments. Consider the case in which neither a_- nor a_+ are in A. Because $|S| = |T|$, for all $\lambda, \mu \in A$, $\mathbf{x}^S(\lambda, \mu)$ is a permutation of $\mathbf{x}^T(\lambda, \mu)$. Hence, by anonymity,

$$v^f(\mathbf{x}^S(\lambda, \mu)) = v^f(\mathbf{x}^T(\lambda, \mu)) \tag{29}$$

for all $\lambda, \mu \in A$. By (10), the limit of the left-hand side of (29) as λ goes to a_- is a_S and the corresponding limit for the right-hand side of (29) is a_T. By (29), these two limits must be the same and, hence, $a_S = a_T$. If either a_- or a_+ are in A, then the relevant case in Proposition 2 is used instead to determine the analogue of (29). If both a_- and a_+ are in A, the conclusion that $a_S = a_T$ follows without taking any limits.

For $j = 0, \ldots, n$, let $b_j = a_S$ for any $S \subseteq N$ for which $|S| = j$. By the preceding argument, b_j is well-defined. Substituting $b_{|S|}$ for a_S in (4),

$$v^f(\mathbf{x}) = \min_{S \subseteq N} \left[\max_{i \in S} \{x_i, b_{|S|}\} \right]. \tag{30}$$

Note that $b_0 \geq b_1 \geq \cdots \geq b_n$ because v^f is a min-max voting scheme and $a_S \geq a_T$ whenever $S \subseteq T$. Furthermore, (i) if $a_+ \notin A$, then $b_n \neq a_+$ and (ii) if $a_- \notin A$, then $b_0 \neq a_-$. By (30) and Proposition 7, it then follows that v^f is an $(n+1)$-parameter generalized median voting scheme and, hence, that f is an $(n+1)$-parameter generalized median social choice function.

(b) Now suppose that f is an $(n+1)$-parameter generalized median social choice function. Without loss of generality, suppose that $b_0 \geq b_1 \geq \cdots \geq b_n$. By letting $a_S = b_{|S|}$ for all $S \subseteq N$, it then follows from Proposition 7 that f is a min-max social choice function. Because $a_S = a_T$ if $|S| = |T|$, f is anonymous. □

Another class of generalized median rules can be obtained by using $n-1$ instead of $n+1$ parameters.

Definition. For $\mathcal{A} \subseteq A \subseteq \mathbb{R}$, a voting scheme $v \colon \mathcal{A}^n \to A$ is an $(n-1)$-*parameter generalized median voting scheme* if there exist $b_i \in A^*$ for $i = 1, \ldots, n-1$ such that for all $\mathbf{x} \in \mathcal{A}^n$,

$$v(\mathbf{x}) = \text{med}\{x_1, \ldots, x_n, b_1, \ldots, b_{n-1}\}. \tag{31}$$

Definition. A social choice function $f \colon \mathcal{D}^n \to A$ for which $\mathcal{D} \subseteq \mathcal{S}$ is an $(n-1)$-*parameter generalized median social choice function* if for all $\mathbf{R} \in \mathcal{D}^n$, (5) holds for some $(n-1)$-parameter generalized median voting scheme $v^f \colon \mathcal{A}^n \to A$, where $\mathcal{A} = \{x \in A \mid x = \tau(R, A) \text{ for some } R \in \mathcal{D}\}$.

Because there are fewer parameters than individuals, for any generalized median voting scheme v with $n-1$ parameters, $v(\mathbf{x}) \in A$ for all $\mathbf{x} \in A^n$ even if all $b_i = a_-$ or all $b_i = a_+$. Thus, there is no need to place any restrictions on the choice of these parameters other than that they are in A^*. Note that a generalized median voting scheme (resp. social choice function) with $n-1$ parameters can be rewritten as a generalized median voting scheme (resp. social choice function) with $n+1$ parameters by adding the parameters $b_0 = a_+$ and $b_n = a_-$ (and using (31) instead of (28)).

When $A = \mathbb{R}$, Moulin (1980, Theorem) has shown that the set of all social choice functions on the domain $\mathcal{S}^n$ that are strategy-proof, anonymous, Pareto efficient, and satisfy the tops-only property is the set of all $(n-1)$-parameter generalized median social choice functions. Because the range of a strategy-proof social choice function with domain $\mathcal{S}^n$ is all of A, by Theorem 1, the tops-only property is redundant in this characterization. Theorem 5 shows that Moulin's theorem (without assuming the tops-only property) is valid for any set $A \subseteq \mathbb{R}$ containing at least two alternatives.

Theorem 5. *A social choice function* $f\colon \mathcal{S}^n \to A$ *is* (a) *strategy-proof, anonymous, and Pareto efficient if and only if* (b) *it is an* $(n-1)$*-parameter generalized median social choice function.*

Proof. (a) First, suppose that f is strategy-proof, anonymous, and Pareto efficient. By the argument in the proof of Theorem 3, $A^f = A$, which is trivially an interval of A. Hence, by Theorem 4, f is a generalized median social choice function with $n+1$ parameters. In order for A^f to be equal to A, at least one of the parameters b_i in (28) must be equal to a_- and at least one of them must be equal to a_+. Without loss of generality, let $b_0 = a_-$ and $b_n = a_+$. But then, for all $\mathbf{x} \in A^n$, $\text{med}\{x_1, \ldots, x_n, b_0, \ldots, b_n\} = \text{med}\{x_1, \ldots, x_n, b_1, \ldots, b_{n-1}\}$. Hence, f is a generalized median social choice function with $n-1$ parameters.

(b) Now suppose that f is a generalized median social choice function with $n-1$ parameters and let v^f be the corresponding voting scheme. By adding the parameters $b_0 = a_-$ and $b_n = a_+$ when computing medians, f is a generalized median social choice function with $n+1$ parameters because not all of these $n+1$ parameters b_i can be equal to a_-, nor can they all be equal to a_+. Therefore, by Theorem 4, f is strategy-proof and anonymous.

Because $\mathcal{S} = \mathcal{S}$, f is Pareto efficient if and only if $\min_{i \in N}\{x_i\} \leq v^f(\mathbf{x}) \leq \max_{i \in N}\{x_i\}$ for all $\mathbf{x} \in A^n$. Because $\text{med}\{x_1, \ldots, x_n, b_1, \ldots, b_{n-1}\}$ is the nth smallest of these values (with ties broken arbitrarily) and because there are fewer parameters than components of $\mathbf{x}$, it follows that $\min_{i \in N}\{x_i\} \leq v^f(\mathbf{x}) \leq \max_{i \in N}\{x_i\}$ for all $\mathbf{x} \in A^n$. Hence, f is Pareto efficient. □

7 Concluding Remarks

Moulin (1980) has noted that his characterization theorems are also valid with strategy-proofness replaced by the stronger requirement of group strategy-proofness. A social choice function is *group strategy-proof* if for any profile of preferences in the domain, there is no subgroup of individuals who could manipulate the outcome in a way that would make them all better off by jointly reporting different preferences. Recently, Le Breton and Zaporozhets (2009) and Barberà, Berga, and Moreno (2010) have identified restrictions on the domain of a social choice function for which strategy-proofness is satisfied if and only group strategy-proofness is satisfied. The domain of single-peaked preference profiles satisfies their conditions. Hence, all of the theorems in this article that employ strategy-proofness could instead use group strategy-proofness.

A single-peaked preference is continuous on either side of its peak, but it need not be continuous on the whole set of alternatives. The arguments used here do not exploit possible discontinuities in preferences in any way. As a consequence, the characterization theorems also hold for the smaller domain of all profiles of continuous single-peaked preferences.

For the domain of all profiles of single-peaked preferences on the real line, Barberà and Jackson (1994) have provided a characterization of the class of all strategy-proof social choice functions when these functions are not *a priori* required to satisfy the tops-only property or, equivalently, for the range to be an interval. This characterization employs strategy-proof tie-breaking rules for selecting one of the alternatives from an individual's top set on the range when this set contains more than one alternative (recall that it can contain at most two). Their description of these tie-breaking rules is expressed in terms of a property that they must satisfy. For the subdomain of all profiles of Euclidean spatial preferences (i.e., single-peaked preferences that rank alternatives in reverse order of their distances from the peak), Massó and Moreno de Barreda (2011) have provided a characterization of the class of all strategy-proof social choice functions on a closed interval of the real line that explictly describes how these tie-breaking rules are constructed. This class consists of Moulin's min-max social choice functions agumented by additional social choice functions that are obtained by perturbing the min-max social choice functions so as to allow for specific kinds of discontinuities. The results presented here help explain why when Massó and Moreno de Barreda require anonymity in addition to strategy-proofness, the resulting class of rules are based on generalized median social choice functions with $n + 1$ parameters.

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