MARKET GAMES AND CLUBS

by

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Working Paper No. 09-W19

October 2009

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*A version of this paper has appeared in the *Encyclopedia of Complexity and Systems Sciences.* The curent version version has some small updates. See the author's web page for citation information to the published paper. Comments are most welcome.

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Keywords: Markets, games, market games, clubs, core, market-game equivalence, Shapley value, price taking equilibrium, small group effectiveness, inessentiality of large groups, per capita boundedness, competitive equilibrium, games with side payments, balanced games, totally balanced games, local public goods, core convergence, equal treatment property, equal treatment core, approximate core, strong epsilon core, weak epsilon core, cooperative game, asymptotic negligibility

1 Glossary of terms

Following are short definitions of the keywords for this entry. These definitions are slightly informal; formal definitions for most of the terms appear in the body of the entry.

game A (cooperative) game (in characteristic form) is defined simply as a finite set of players and a function or correspondence ascribing a worth (a nonnegative real number, interpreted as an idealized money) to each nonempty subset of players, called a group or coalition.
payoff vector A payoff vector is a vector listing a payoff (an amount of utility or money) for each player in the game.
core The core of a game is the set (possibly empty) of feasible outcomes – divisions of the worths arising from coalition formation among the players of the game – that cannot be improved upon by any coalition of players. core
totally balanced game A game is totally balanced if the game and every subgame of the game (a game with player set taken as some subset of players of the initially given game) has a nonempty core.
market A market is defined as a private goods economy in which all participants have utility functions that are linear in (at least) one commodity (money).
Shapley value The Shapley value of a game is feasible outcome of a game in which all players are assigned their expected marginal contribution to a coalition when all orders of coalition formation are equally likely.
pregame A pair, consisting of a set of player types (attributes or characteristics) and a function mapping finite lists of characteristics (repetitions allowed) into the real numbers. In interpretation, the pregame function ascribes a worth to every possible finite group of players, where the worth of a group depends on the numbers of players with each characteristic in the group. A pregame is used to generate games with arbitrary numbers of players.
small group effectiveness A pregame satisfies small group effectiveness if almost all gains to collective activities can be realized by cooperation only within arbitrarily small groups (coalitions) of players.
per capita boundedness A pregame satisfies per capita boundedness if the supremum of the average worth of any possible group of players (the per capita payoff) is finite.
asymptotic negligibility A pregame satisfies asymptotic negligibility if vanishingly small groups can have only negligible effects on per capita payoffs.

market games A market game is a game derived from a market. Given a market and a group of agents we can determine the total utility (measured in money) that the group can achieve using only the endowments belonging to the group members, thus determining a game.
club A club is a group of agents or players that forms for the purpose of carrying out come activity, such as providing a local public good.
an economy We use the term 'economy' to describe any economic setting, including economies with clubs, where the worth of club members may depend on the characteristics of members of the club, economies with pure public goods, local public goods (public goods subject to crowding and/or congestion), economies with production where what can be produced and the costs of production may depend on the characteristics of the individuals involved in production, and so on. A **large economy** has many participants.
price taking equilibrium A price taking equilibrium for a market is a set of prices, one for each commodity, and an allocation of commodities to agents so that each agent can afford his part of the allocation, given the value of his endowment.

2 Definition of the subject

The equivalence of markets and games concerns the relationship between two sorts of structures that appear fundamentally different – markets and games. Shapley and Shubik (1969) demonstrates that: (1) games derived from markets with concave utility functions generate totally balanced games where the players in the game are the participants in the economy and (2) every totally balanced game generates a market with concave utility functions. A particular form of such a market is one where the commodities are the participants themselves, a labor market for example.

But markets are very special structures, more so when it is required that utility functions be concave. Participants may also get utility from belonging to groups, such as marriages, or clubs, or productive coalitions. It may be that participants in an economy even derive utility (or disutility) from engaging in processes that lead to the eventual exchange of commodities. The question is when are such economic structures equivalent to markets with concave utility functions.

This paper summarizes research showing that a broad class of large economies generate balanced market games. The economies include, for example, economies with clubs where individuals may have memberships in multiple clubs, with indivisible commodities, with nonconvexities and with non-monotonicities. The main assumption are: (1) that an option open to any group of players is to break into smaller groups and realize the sum of the worths of these groups, that is, essential superadditivity is satisfied and :(2) relatively small groups of participants can realize almost all gains to coalition formation.

The equivalence of games with many players and markets with many participants indicates that relationships obtained for markets with concave utility functions and many participants will also hold for diverse social and economic situations with many players. These relationships include: (a) equivalence of the core and the set of competitive outcomes; (b) the Shapley value is contained in the core or approximate cores; (c) the equal treatment property holds – that is, both market equilibrium and the core treat similar players similarly. These results can be applied to diverse economic models to obtain the equivalence of cooperative outcomes and competitive, price taking outcomes in economies with many participants and indicate that such results hold in yet more generality.

3 Introduction

One of the subjects that has long intrigued economists and game theorists is the relationship between games, both cooperative and noncooperative, and economies. Seminal works making such relationships include Shubik (1959), Debreu and Scarf (1963), Aumann (1964), Shapley and Shubik (1969,1975) and Aumann and Shapley (1974), all connecting outcomes of price-taking behavior in large economies with cores of games. See also Shapley and Shubik (1977) and an ongoing stream of papers connecting strategic behavior to market behavior. Our primary concern here, however, is not with the equivalence of outcomes of solution concepts for economies, as is Debreu and Scarf (1963) or Aumann (1964) for example, but rather with equivalences of the *structures* of markets and games. Solution concepts play some role, however, in establishing these equivalences and in understanding the meaning of the equivalence of markets and games.

In this entry, following Shapley and Shubik (1969), we focus on markets in which utility functions of participants are quasi-linear, that is, the utility function u of a participant can be written as $u(x, \xi) = \widehat{u}(x) + \xi$ where $x \in \mathbb{R}^L_+$ is a commodity bundle, $\xi \in \mathbb{R}$ is interpreted as money and $\widehat{u}$ is a continuous function. Each participant in an economy has an endowment of commodities and, without any substantive loss of generality, it is assumed that no money is initially endowed. The price of money is assumed equal to one. A price taking equilibrium for a market then consists of a price vector $p \in \mathbb{R}^L$ for the commodities and an assignment of commodities to participants such that: the total amounts of commodities assigned to participants equals the total amount of commodities with which participants are endowed and; given prices, each participant can afford his assignment of commodities and no participant, subject to his budget constraint, can afford a preferred commodity bundle.

We also treat games with side payments, alternatively called games with transferable utility or, in brief, TU games. Such a game consists of a finite set N of players and a worth function that assigns to each group of players $S \subset N$ a real number $v(S) \in \mathbb{R}_+$, called the worth of the group. In interpretation, $v(S)$ is the total payoff that a group of players can realize by cooperation. A central game-theoretic concept for the study of games is the core. The core consists of those divisions of the maximal total worth achievable by cooperation among the players in N so that each group of players is assigned at least its worth. A game is balanced if it has a nonempty core and totally balanced if all subgames of the game have nonempty cores. A subgame of a game is simply a group of players $S \subset N$ and the worth function restricted to that group and the smaller groups that it contains.

Given an economy any feasible state of the economy generates a total worth of each group of participants. The worth of a group of participants (viewed as players of a game) is the maximal total utility achievable by the members of the group using only those resources they control. In this way a market generates a game – a set of players (the participants in the economy) and a worth for each group of players.

Shapley and Shubik (1969) demonstrate that any market where all participants have concave, monotonic increasing utility functions generates a totally balanced game and that any totally balanced game generates a market, thus establishing an equivalence between a class of markets and totally balanced cooperative games. A particular sort of market is canonical; one where each participant in the market is endowed with one unit of a commodity, his "type". Intuitively, one might think of the market as one where each participant owns one unit of himself or of his labor.

In the last twenty years or so there has been substantial interest in broader classes of economies, including those with indivisibilities, nonmonotonicities, local public goods or clubs, where the worth of a group depends not only on the private goods endowed to members of the group but also on the characteristics of the group members. For example, the success of the marriage of a man and a woman depends on their characteristics and on whether their characteristics are complementary. Similarly, the output of a machine and a worker using the machine depends on the quality and capabilities of the machine and how well the abilities of the worker fit with the characteristics of the machine – a concert pianist fits well with an high quality piano but perhaps not so well with a sewing machine. Or how well a research team functions depends not only on the members of the team but also on how well they interact. For simplicity, we shall refer to these economies as club economies. Such economies can be modelled as cooperative games.

In this entry we discuss and summarize literature showing that economies with many participants are approximated by markets where all participants have the same concave utility function and for which the core of the game is equivalent to the set of price-taking economic equilibrium payoffs. The research presented is primarily from Shubik and Wooders (1982), Wooders (1994) and earlier papers due to this author. For the most recent results in this line of research we refer the reader to Wooders (2007a,b, 2008). We also discuss other related works throughout the course of the entry. The models and results are set in a broader context in the conclusions.

The importance of the equivalence of markets and games with many players relates to the hypothesis of perfect competition, that large numbers of participants leads to price-taking behavior, or behavior "as if" participants took prices as given. Von Neumann and Morgenstern perceived that even though individuals are unable to influence market prices and cannot benefit from strategic behavior in large markets, large "coalitions" might form. Von Neumann and Morgenstern write:

> It is neither certain nor probable that a mere increase in the number of participants might lead *in fine* to the conditions of free competition. The classical definitions of free competition all involve further postulates besides this number. E.g., it is clear that if certain great groups of individuals will – for any reason whatsoever– act together, then the great number of participants may not become effective; the decisive exchanges may take place directly between large "coalitions," few in number and not between individuals, many in number acting independently. ... Any satisfactory theory ...will have to explain when such big coalitions will or will not be formed –i.e., when the large numbers of participants will become effective and lead to more or less free competition.

The assumption that small groups of individuals cannot affect market aggregates, virtually taken for granted by von Neumann and Morgenstern, lies behind the answer to the question they pose. The results presented in this entry suggest that the great number of participants will become effective and lead to more or less free competition when *small* groups of participants cannot significantly affect market outcomes. Since all or almost all gains to collective activities can be captured by relatively small groups, large groups gain no market power from size; in other words, large groups are inessential. That large groups are inessential is equivalent to small group effectiveness (Wooders (1992b). A remarkable feature of the results discussed in this essay is they are independent of any particular economic structure.

4 Transferable utility games; some standard definitions

Let (N,ν) be a pair consisting of a finite set N, called a *player set*, and a function v, called a *worth function*, from subsets of N to the real numbers $\mathbb{R}$ with $v(\phi) = 0$. The pair (N,ν) is a *TU game* (also called a game with side payments). Nonempty subsets S of N are called *groups* (of players) and the number of members of the group S is given by $|S|$. Following is a simple example.

Example 1. A glove game: Suppose that we can partition a player set N into two groups, say N_1 and N_2. In interpretation, a member of N_1 is endowed with a right-hand (RH) glove and a member of N_2 is endowed with a left-hand (LH) glove. The worth of a pair of gloves is \$1, and thus the worth of a group of players consisting of player $i \in N_1$ and player $j \in N_2$ is \$1. The worth of a single glove and hence of a one-player group is \$0. The worth of a group $S \subset N$ is given by $v(S) = \min\{|S \cap N_1|, |S \cap N_2|\}$. The pair (N,ν) is a game.

A *payoff vector* for a game (N,ν) is a vector $\overline{u}\in\mathbb{R}^N$. We regard vectors in finite dimensional Euclidean space $\mathbb{R}^T$ as functions from T to $\mathbb{R}$, and write $\overline{u}_i$ for the i^{th} component of $\overline{u}$, etc. If $S\subset T$ and $\overline{u}\in\mathbb{R}^T$, we shall write $\overline{u}_S := (\overline{u}_i : i\in S)$ for the restriction of $\overline{u}$ to S. We write 1_S for the element of $\mathbb{R}^S$ all of whose coordinates are 1 (or simply 1 if no confusion can arise.) A payoff vector $\overline{u}$ is *feasible for a group* $S\subset N$ if

$$\overline{u}(S)\overset{\text{def}}{=}\sum_{i\in S}\overline{u}^i\leq\sum_{k=1}^{K}v(S^k) \tag{1}$$

for some partition $\{S^1,...,S^K\}$ of S.

Given $\varepsilon\geq 0$, a payoff vector $\overline{u}\in\mathbb{R}^N$ is in the *weak ε-core* of the game (N,ν) if it is feasible and if there is a group of players $N^0\subset N$ such that

$$\frac{\left|N\backslash N^0\right|}{|N|}\leq\varepsilon \tag{2}$$

and, for all groups $S\subset N^0$,

$$\overline{u}(S)\geq v(S)-\varepsilon|S| \tag{3}$$

where $|S|$ is the cardinality of the set S. (It would be possible to use two different values for epsilon in expressions (2) and (3). For simplicity, we have chosen to take the same value for epsilon in both expressions.) A payoff vector $\overline{u}$ is in the *uniform ε-core* (or simply in the *ε-core*) if if is feasible and if (3) holds for *all* groups $S\subset N$. When $\varepsilon=0$, then both notions of ε-cores will be called simply the *core*.

Example 1 continued. The glove game (N,ν) described in Example 1 has the happy feature that the core is always nonempty. For the game to be of interest, we will suppose that there is least one player of each type (that is, there is at least one player with a RH glove and one player with a LH glove). If $|N_1|=|N_2|$ any payoff vector assigning the same share of a dollar to each player with a LH glove and the remaining share of a dollar to each player with a RH glove is in the core. If there are more players of one type, say $|N_1|>|N_2|$ for specificity, then any payoff vector in the core assigns \$1 to each player of the scarce type; that is, players with a RH glove each receive 0 while players with a LH glove each receive \$1.

Not all games have nonempty cores, as the following example illustrates.

Example 2. *A simple majority game with an empty core.* Let $N=\{1,2,3\}$ and define the function v as follows:

$$v(S)=\begin{cases}0\text{ if }|S|=1,\\1\text{ otherwise.}\end{cases}$$

It is easy to see that the core of the game is empty. For if a payoff vector $\overline{u}$ were in the core, then it must hold that for any $i \in N$, $\overline{u}_i \geq 0$ and for any $i, j \in N$, $\overline{u}_i + \overline{u}_j \geq 1$. Moreover, feasibility dictates that $\overline{u}_1 + \overline{u}_2 + \overline{u}_3 \leq 1$. This is impossible; thus, the core is empty.

Before leaving this example, let us ask whether it would be possible to subsidize the players by increasing the payoff to the total player set N and, by doing so, ensure that the core of the game with a subsidy is nonempty. We leave it to the reader to verify that if $v(N)$ were increased to \$3/2 (or more), the new game would have a nonempty core.

Let (N, ν) be a game and let $i, j \in N$. Then players i and j are *substitutes* if, for all groups $S \subset N$ with $i, j \notin S$ it holds that

$$v(S \cup \{i\}) = v(S \cup \{j\}).$$

Let (N, ν) be a game and let $\overline{u} \in \mathbb{R}^N$ be a payoff vector for the game. If for all players i and j who are substitutes it holds that $\overline{u}_i = \overline{u}_j$ then $\overline{u}$ has the *equal treatment property.* Note that if there is a partition of N into T subsets, say $N_1, ..., N_T$, where all players in each subset N_t are substitutes for each other, then we can *represent* $\overline{u}$ by a vector $\overline{\overline{u}} \in \mathbb{R}^T$ where, for each t, it holds that $\overline{\overline{u}}_t = \overline{u}_i$ for all $i \in N_t$.

4.0.1 Essential superadditivity

We wish to treat games where the worth of a group of players is independent of the total player set in which it is embedded and an option open to the members of a group is to partition themselves into smaller groups; that is, we treat games that are *essentially superadditive.* This is built into our the definition of feasibility above, (1). An alternative approach, which would still allow us to treat situations where it is optimal for players to form groups smaller than the total player set, would be to assume that v is the "superadditive cover" of some other worth function v'. Given a not-necessarily-superadditive function v', for each group S define $v(S)$ by:

$$v(S) = \max \sum v'(S^k) \tag{4}$$

where the maximum is taken over all partitions $\{S^k\}$ of S; the function v is the *superadditive cover* of v'. Then the notion of feasibility requiring that a payoff vector $\overline{u}$ is feasible only if

$$\overline{u}(N) \leq v(N), \tag{5}$$

gives an equivalent set of feasible payoff vectors to those of the game (N, v') with the definition of feasibility given by (1).

The following Proposition may be well known and is easily proven. This result was already well understood in Gillies (1953) and applications have appeared in a number of papers in the theoretical literature of game theory; see, for example (for $\varepsilon = 0$) Aumann and Dreze (1974) and Kaneko and Wooders (1982). It is also well known in club theory and the theory of economies with many players and local public goods.

Proposition 1: *Given $\varepsilon \geq 0$, let (N, v') be a game. A payoff vector $\overline{u} \in R^N$ is in the weak, respectively uniform, ε-core of (N, v') if and only if it is in the weak, respectively uniform, ε-core of the superadditive cover game, say (N, ν), where v is defined by (4).*

5 A market

In this section we introduce the definition, from Shapley and Shubik (1969), of a market. Unlike Shapley and Shubik, however, we do not assume concavity of utility functions. A *market* is taken to be an economy where all participants have continuous utility functions over a finite set of commodities that are all linear in one commodity, thought of as an "idealized" money. Money can be consumed in any amount, possibly negative. For later convenience we will consider an economy where there is a finite set of types of participants in the economy and all participants of the same type have the same endowments and preferences.

Consider an economy with $T + 1$ types of commodities. Denote the set of participants by

$$N = \{(t, q) : t = 1, \cdots, T, \text{ and } q = 1, \cdots, n_t\}.$$

Assume that all participants of the same type, (t, q), $q = 1, ..., n_t$ have the same utility functions given by

$$\widehat{u}_t(y, \xi) = u_t(y) + \xi$$

where $y \in \mathbb{R}^T_+$ and $\xi \in \mathbb{R}$. Let $a^{tq} \in \mathbb{R}^T_+$ be the *endowment* of the $(t, q)^{th}$ player of the first T commodities. The *total endowment* is given by $\sum_{(t,q)\in N} a^{tq}$. For simplicity and without loss of generality, we can assume that no participant is endowed with any nonzero amount of the $(T+1)^{th}$ good, the "money" or medium of exchange. One might think of utilities as being measured in money. It is because of the transferability of money that utilities are called "transferable".

Remark 1: Instead of assuming that money can be consumed in negative amounts one might assume that endowments of money are sufficiently large so that no equilibrium allocates any participant a negative amount of money. For further discussion of transferable utility see, for example, Bergstrom and Varian (1985) or Kaneko and Wooders (2004).

Given a group $S \subset N$, a *S-allocation of commodities* is a set

$$\{(y^{tq}, \xi^{tq}) \in \mathbb{R}^T_+ \times \mathbb{R} : \sum_{(t,q)\in S} y^{tq} \leq \sum_{(t,q)\in S} a^{tq} \text{ and } \sum_{(t,q)\in S} \xi^{tq} \leq 0\};$$

that is, a S-allocation is a redistribution of the commodities owned by the members of S among themselves and monetary transfers adding up to no more than zero. When $S = N$, a S-allocation is called simply an *allocation*.

With the price of the $(T+1)^{th}$ commodity ξ set equal to 1, a *competitive outcome* is a price vector p in $\mathbb{R}^T$, listing prices for the first T commodities, and an allocation $\{(y^{tq}, \xi^{tq}) \in \mathbb{R}^T \times \mathbb{R} : (t,q) \in N\}$ for which

$$\begin{array}{l} \text{(a) } u_t(y^{tq}) - p \cdot (y^{tq} - a^{tq}) \geq u_t(\widehat{y}) - p \cdot (\widehat{y} - a^{tq}) \\ \text{for all } \widehat{y} \in \mathbb{R}^T_+,\ (t,q) \in N, \\ \\ \text{(b) } \sum_{(t,q)\in N} y^{tq} = \sum_{(t,q)} a^{tq} = \overline{y}, \\ \\ \text{(c) } \xi^{tq} = p \cdot (y^{tq} - a^{tq}) \text{ for all } (t,q) \in N \text{ and} \\ \\ \text{(d) } \sum_{(t,q)\in N} \xi^{tq} = 0. \end{array} \tag{6}$$

Given a competitive outcome with allocation $\{(y^{tq}, \xi^{tq}) \in \mathbb{R}^T_+ \times \mathbb{R} : (t,q) \in N\}$ and price vector p, the *competitive payoff to the* $(t,q)^{th}$ *participant* is $u(y^{tq}) - p \cdot (y^{tq} - a^{tq})$. A *competitive payoff vector* is given by

$$(u(y^{tq}) - p \cdot (y^{tq} - a^{tq}) : (t,q) \in N).$$

In the following we will assume that for each t, all participants of type t have the same endowment; that is, for each t, it holds that $a^{tq} = a^{tq'}$ for all $q, q' = 1, ..., n_t$. In this case, every competitive payoff has the equal treatment property;

$$u_t(y^{tq}) - p \cdot (y^{tq} - a^{tq}) = u_t(y^{tq'}) - p \cdot (y^{tq'} - a^{tq'})$$

for all q, q' and for each t. It follows that a competitive payoff vector can be represented by a vector in $\mathbb{R}^T$ with one component for each player type.

It is easy to generate a game from the data of an economy. For each group of participants $S \subset N$, define

$$v(S) = \max \sum_{tq \in S} u_t(y^{tq}, \xi^{tq})$$

where the maximum is taken over the set of S-allocations. Let (N, ν) denote a game derived from a market.

Under the assumption of concavity and monotonicity of the utility functions of the participants in an economy, Shapley and Shubik (1969) show that a competitive outcome for the market exists and that the competitive payoff vectors are in the core of the game. (Since Debreu and Scarf, 1963, such results have been obtained in substantially more general models of economies.)

6 Market-game equivalence

To facilitate exposition of the theory of games with many players and the equivalence of markets and games, we consider games derived from a common underlying structure and with a fixed number of types of players, where all players of the same type are substitutes for each other.

6.1 Pregames

Let T be a positive integer, to be interpreted as a number of player types. A *profile* $s = (s_1, \cdots, s_T) \in \mathbf{Z}_+^T$, where $\mathbf{Z}_+^T$ is the T-fold Cartesian product of the non-negative integers $\mathbf{Z}_+$, describes a group of players by the numbers of players of each type in the group. Given profile s, define the *norm* or *size* of s by

$$\|s\| \stackrel{\text{def}}{=} \sum_t s_t,$$

simply the total number of players in a group of players described by s. A *subprofile of a profile* $n \in \mathbf{Z}_+^T$ is a profile s satisfying $s \leq n$. A *partition of a profile* s is a collection of subprofiles $\{s^k\}$ of n, not all necessarily distinct, satisfying

$$\sum_k s^k = s.$$

A partition of a profile is analogous to a partition of a set except that all members of a partition of a set are distinct.

Let Ψ be a function from the set of profiles $\mathbf{Z}_+^T$ to $\mathbb{R}_+$ with $\Psi(0) = 0$. The value $\Psi(s)$ is interpreted as the total payoff a group of players with profile s can achieve from collective activities of the group membership and is called the *worth of the profile* s.

Given Ψ, define a worth function Ψ^*, called the *superadditive cover* of Ψ, by

$$\Psi^*(s) \stackrel{\text{def}}{=} \max \sum_k \Psi(s^k),$$

where the maximum is taken over the set of all partitions $\{s^k\}$ of s. The function Ψ is said to be *superadditive* if the worth functions Ψ and Ψ^* are equal.

We define a *pregame* as a pair (T, Ψ) where $\Psi : \mathbf{Z}_+^T \to \mathbb{R}_+$. As we will now discuss, a pregame can be used to generate multiple games. To generate a game from a pregame, it is only required to specify a total player set N and the numbers of players of each of T types in the set. Then the pregame can be used to assign a worth to every group of players contained in the total player set, thus creating a game.

A *game determined by the pregame* (T, Ψ), which we will typically call a *game* or a *game with side payments*, is a pair $[n; (T, \Psi)]$ where n is a profile. A *subgame* of a game $[n; (T, \Psi)]$ is a pair $[s; (T, \Psi)]$ where s is a subprofile of n.

With any game $[n; (T, \Psi)]$ we can associate a game (N, ν) in the form introduced earlier as follows: Let

$$N = \{(t, q) : t = 1, \cdots, T \text{ and } q = 1, \cdots, n_t\}$$

be a *player set* for the game. For each subset $S \subset N$ define the *profile of* S, denoted by *prof(S)*$\in \mathbf{Z}_+^T$, by its components

$$\mathit{prof(S)}_t \stackrel{\text{def}}{=} |\{S \cap \{(t', q) : t' = t \text{ and } q = 1, \cdots, n_t\}|$$

and define

$$v(S) \stackrel{\text{def}}{=} \Psi(\mathit{prof(S)}).$$

Then the pair (N, ν) satisfies the usual definition of a game with side payments. For any $S \subset N$, define

$$v^*(S) \stackrel{\text{def}}{=} \Psi^*(\mathit{prof(S)}).$$

The game (N, ν^*) is the *superadditive cover of* (N, ν).

A *payoff vector* for a game (N, ν) is a vector $\overline{u} \in \mathbb{R}^N$. For each nonempty subset S of N define

$$\overline{u}(S) \stackrel{\text{def}}{=} \sum_{(t,q) \in S} \overline{u}^{tq}.$$

A payoff vector $\overline{u}$ is *feasible for* S if

$$\overline{u}(S) \leq v^*(S) = \Psi^*(prof(S)).$$

If $S = N$ we simply say that the payoff vector $\overline{u}$ is *feasible* if

$$\overline{u}(N) \leq v^*(N) = \Psi^*(prof(N))$$

Note that our definition of feasibility is consistent with essential superadditivity; a group can realize at least as large a total payoff as it can achieve in any partition of the group and one way to achieve this payoff is by partitioning into smaller groups.

A payoff vector $\overline{u}$ satisfies the *equal-treatment property* if $\overline{u}^{tq} = \overline{u}^{tq'}$ for all $q, q' \in \{1, \cdots, n_t\}$ and for each $t = 1, \cdots, T$.

Let $[n, (T, \Psi)]$ be a game and let β be a collection of subprofiles of n. The collection is a *balanced collection of subprofiles of* n if there are positive real numbers γ_s for $s \in \beta$ such that $\sum\limits_{s \in \beta} \gamma_s s = n$. The numbers γ_s are called *balancing weights*. Given real number $\varepsilon \geq 0$, the game $[n; (T, \Psi)]$ is ε-*balanced* if for every balanced collection β of subprofiles of n it holds that

$$\Psi^*(n) \geq \sum_{s \in \beta} \gamma_s \left(\Psi(s) - \varepsilon \|s\|\right) \tag{7}$$

where the balancing weights for β are given by γ_s for $s \in \beta$. This definition extends that of Bondareva (1983) and Shapley (1967) to games with player types. Roughly, a game is (ε) balanced if allowing "part time" groups does not improve the total payoff (by more than ε per player). A game $[n; (T, \Psi)]$ is *totally balanced* if every subgame $[s; (T, \Psi)]$ is balanced.

The *balanced cover game* generated by a game $[n; (T, \Psi)]$ is a game $[n; (T, \Psi^b)]$ where

1. $\Psi^b(s) = \Psi(s)$ for all $s \neq n$ and

2. $\Psi^b(n) \geq \Psi(n)$ and $\Psi^b(n)$ is as small as possible consistent with the nonemptiness of the core of $[n; (T, \Psi^b)]$.

From the Bondareva-Shapley Theorem it follows that $\Psi^b(n) = \Psi^*(n)$ if and only if the game $[n; (T, \Psi)]$ is balanced (ε-balanced, with $\varepsilon = 0$).

For later convenience, the notion of the balanced cover of a pregame is introduced. Let (T, Ψ) be a pregame. For each profile s, define

$$\Psi^b(s) \stackrel{\text{def}}{=} \max_{\beta} \sum_{g \in \beta} \gamma_g \Psi(g), \tag{8}$$

where the maximum is taken over all balanced collections β of subprofiles of s with weights γ_g for $g \in \beta$. The pair (T, Ψ^b) is called the *balanced cover pregame* of (T, Ψ). Since a partition of a profile is a balanced collection it is immediately clear that $\Psi^b(s) \geq \Psi^*(s)$ for every profile s.

6.2 Premarkets

In this section, we introduce the concept of a premarket and re-state results from Shapley and Shubik (1969) in the context of pregames and premarkets.

Let $L+1$ be a number of types of commodities and let $\{\widehat{u}_t(y, \xi) : t = 1, ..., T\}$ denote a finite number of functions, called *utility functions* of the form

$$\widehat{u}_t(y, \xi) = u_t(y) + \xi,$$

where $y \in \mathbb{R}^L_+$ and $\xi \in \mathbb{R}$. (Such functions, in the literature of economics, are commonly called *quasi-linear.*) Let $\{a^t \in \mathbb{R}^L_+ : t = 1, ..., T\}$ be interpreted as a set of *endowments.* We assume that $u_t(a^t) \geq 0$ for each t. For $t = 1, ..., T$ we define $c^t \stackrel{\text{def}}{=} (u_t(\cdot), a^t)$ as a *participant type* and let $\mathbb{C} = \{c^t : t = 1, ..., T\}$ be the set of participant types. Observe that from the data given by $\mathbb{C}$ we can construct a market by specifying a set of participants N and a function from N to $\mathbb{C}$ assigning endowments and utility functions – types – to each participant in N. A *premarket* is a pair $(T, \mathbb{C})$.

Let $(T, \mathbb{C})$ be a premarket and let $s = (s_1, ..., s_T) \in \mathbf{Z}^T_+$. We interpret s as representing a group of economic participants with s_t participants having utility functions and endowments given by c^t for $t = 1, ..., T$; for each t, that is, there are s_t participants in the group with type c^t. Observe that the data of a premarket gives us sufficient data to generate a pregame. In particular, given a profile $s = (s_1, ..., s_T)$ listing numbers of participants of each of T types, define

$$W(s) \stackrel{\text{def}}{=} \max \sum_t s_t u_t(y^t)$$

where the maximum is taken over the set $\{y^t \in \mathbb{R}^L_+ : t = 1, ..., T \text{ and } \sum_t s_t y^t = \sum_t a^t y^t\}$. Then the pair (T, W) is a *pregame generated by the premarket.*

The following Theorem is an extension to premarkets or a restatement of a result due to Shapley and Shubik (1969).

Theorem 1. Let $(T, \mathbb{C})$ be a premarket derived from economic data in which all utility functions are concave. Then the pregame generated by the premarket is totally balanced.

6.3 Direct markets and market-game equivalence

Shapley and Shubik (1969) introduced the notion of a direct market derived from a totally balanced game. In the direct market, each player is endowed with one unit of a commodity (himself) and all players in the economy have the same utility function. In interpretation, we might think of this as a labor market or as a market for productive factors, (as in Owen 1975, for example) where each player owns one unit of a commodity. For games with player types as in this essay, we take the player types of the game as the commodity types of a market and assign all players in the market the same utility function, derived from the worth function of the game.

Let (T, Ψ) be a pregame and let $[n; (T, \Psi)]$ be a derived game. Let $N = \{(t, q) : t = 1, ..., T \text{ and } q = 1, ..., n_t \text{ for each } t\}$ denote the set of players in the game where all participants $\{(t', q) : q = 1, ..., n_{t'}\}$ are of type t' for each $t' = 1, ..., T$. To construct the direct market generated by a derived game $[n; (T, \Psi)]$, we take the commodity space as $\mathbb{R}^T_+$ and suppose that each participant in the market of type t is endowed with one unit of the t^{th} commodity, and thus has endowment $\mathbf{1}_t = (0, \cdots, 0, 1, 0, \cdots, 0) \in \mathbb{R}^T_+$ where "1" is in the t^{th} position. The total endowment of the economy is then given by $\sum n_t \mathbf{1}_t = n$.

For any vector $y \in \mathbb{R}^T_+$ define

$$u(y) \stackrel{\text{def}}{=} \max \sum_{s \leq n} \gamma_s \Psi(s), \tag{9}$$

the maximum running over all $\{\gamma_s \geq 0 : s \in \mathbf{Z}^T_+, \ s \leq n\}$ satisfying

$$\sum_{s \leq n} \gamma_s s = y. \tag{10}$$

As noted by Shapley and Shubik (1969), but for our types case, it can be verified that the function u is concave and one-homogeneous. This does not depend on the balancedness of the game $[n; (T, \Psi)]$. Indeed, one may think of u as the "balanced cover of $[n; (T, \Psi)]$ extended to $\mathbb{R}^T_+$". Note also that u is superadditive, independent of whether the pregame (T, Ψ) is superadditive. We leave it to the interested reader to verify that if Ψ were not necessarily superadditive and Ψ^* is the superadditive cover of Ψ then it holds that $\max \sum_{s \leq n} \gamma_s \Psi(s) = \max \sum_{s \leq n} \gamma_s \Psi^*(s)$.

Taking the utility function u as the utility function of each player $(t, q) \in N$ where N is now interpreted as the set of participants in a market, we have generated a market, called the *direct market*, denoted by $[n, u; (T, \Psi)]$, from the game $[n; (T, \Psi)]$.

Again, the following extends a result of Shapley and Shubik (1969) to pregames.

Theorem 2. Let $[n, u; (T, \Psi)]$ denote the direct market generated by a game $[n; (T, \Psi)]$ and let $[n; (T, u)]$ denote the game derived from the direct market. Then, if $[n; (T, \Psi)]$ is a totally balanced game, it holds that $[n; (T, u)]$ and $[n; (T, \Psi)]$ are identical.

Remark 2. If the game $[n; (T, \Psi)]$ and every subgame $[s, (T, \Psi)]$ has a non-empty core – that is, if the game is 'totally balanced'– then the game $[n; (T, u)]$ generated by the direct market is the initially given game $[n; (T, \Psi)]$. If however the game $[n; (T, \Psi)]$ is not totally balanced then $u(s) \geq \Psi(s)$ for all profiles $s \leq n$. But, whether or not $[n; (T, \Psi)]$ is totally balanced, the game $[n; (T, u)]$ is totally balanced and coincides with the totally balanced cover of $[n; (T, \Psi)]$.

Remark 3. Another approach to the equivalence of markets and games is taken by Garratt and Qin (1997), who define a class of direct lottery markets. While a player can participate in only one coalition, both ownership of coalitions and participation in coalitions is determined randomly. Each player is endowed with one unit of probability, his own participation. Players can trade their endowments at market prices. The core of the game is equivalent to the equilibrium of the direct market lottery.

7 Equivalence of markets and games with many players

The requirement of Shapley and Shubik (1969) that utility functions be concave is restrictive. It rules out, for example situations such as economies with indivisible commodities. It also rules out club economies; for a given club structure of the set of players – in the simplest case, a partition of the total player set into groups where collective activities only occur within these groups – it may be that utility functions are concave over the set of alternatives available within each club, but utility functions need not be concave over all possible club structures. This rules out many examples; we provide a simple one below.

To obtain the result that with many players, games derived from pregames are market games, we need some further assumption on pregames. If there are many substitutes for each player, then the simple condition that *per capita payoffs are bounded* – that is, given a pregame (T, Ψ), that there exists some constant K such that $\frac{\Psi(s)}{\|s\|} < K$ for all profiles s – suffices. If, however, there may be 'scarce types', that is, players of some type(s) become negligible in the population, then a stronger assumption of 'small group effectiveness' is required. We discuss these two conditions in the next section.

7.1 Small group effectiveness and per capita boundedness

This section discusses conditions limiting gains to group size and their relationships. This definition was introduced in Wooders (1983), for NTU, as well as TU, games.

PCB A pregame (T, Ψ) satisfies *per capita boundedness (*PCB) if

$$PCB: \quad \sup_{s \in \mathbf{Z}_+^T} \frac{\Psi(s)}{\|s\|} \text{ is finite} \tag{11}$$

or equivalently,

$$\sup_{s \in \mathbf{Z}_+^T} \frac{\Psi^*(s)}{\|s\|} \text{ is finite.}$$

It is known that under the apparently mild conditions of PCB and essential superadditivity, in general games with many players of each of a finite number of player types and a fixed distribution of player types have nonempty approximate cores; Wooders (1977,1983). (Forms of these assumptions were subsequently also used in Shubik and Wooders, 1982,1983; Kaneko and Wooders, 1986; and Wooders 1992b,1994 among others.) Moreover, under the same conditions, approximate cores have the property that most players of the same type are treated approximately equally (Wooders 1977,1992b; see also Shubik and Wooders 1982). These results, however, either require some assumption ruling out 'scarce types' of players, for example, situations where there are only a few players of some particular type and these players can have great effects on total feasible payoffs. Following are two examples. The first illustrates that PCB does not control limiting properties of the per capita payoff function when some player types are scarce.

Example 3. (Wooders 2008a) Let $T = 2$ and let (T, Ψ) be the pregame given by

$$\Psi(s_1, s_2) = \begin{cases} s_1 + s_2 & \text{when } s_1 > 0 \\ \\ 0 & \text{otherwise} \end{cases}$$

The function Ψ obviously satisfies PCB. But there is a problem in defining $\lim \frac{\Psi(s_1, s_2)}{s_1 + s_2}$ as $s_1 + s_2$ tends to infinity, since the limit depends on how it is approached. Consider the sequence (s_1^ν, s_2^ν) where $(s_1^\nu, s_2^\nu) = (0, \nu)$; then $\lim \frac{\Psi(s_1^\nu, s_2^\nu)}{s_1^\nu + s_2^\nu} = 0$. Now suppose in contrast that $(s_1^\nu, s_2^\nu) = (1, \nu)$; then $\lim \frac{\Psi(s_1^\nu, s_2^\nu)}{s_1^\nu + s_2^\nu} = 1$. This illustrates why, to obtain the result that games with many players are market games either it must be required that there are no scarce types or some some assumption limiting the effects of scarce types must be made. We return to this example in the next section.

The next example illustrates that, with only PCB, uniform approximate cores of games with many players derived from pregames may be empty.

Example 4 . (Wooders 2008a) Consider a pregame (T,Ψ) where $T=\{1,2\}$ and Ψ is the superadditive cover of the function Ψ' defined by:

$$\Psi'(s) \stackrel{\text{def}}{=} \begin{cases} |s| & \text{if } s_1 = 2, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, if a profile $s=(s_1,s_2)$ has $s_1=2$ then the worth of the profile according to Ψ' is equal to the total number of players it represents, s_1+s_2, while all other profiles s have worth of zero. In the superadditive cover game the worth of a profile s is 0 if $s_1<2$ and otherwise is equal to s_2 plus the largest even number less than or equal to s_1.

Now consider a sequence of profiles $(s^\nu)_\nu$ where $s_1^\nu=3$ and $s_2^\nu=\nu$ for all ν. Given $\varepsilon>0$, for all sufficiently large player sets the uniform ε-core is empty. Take, for example, $\varepsilon=1/4$. If the uniform ε-core were nonempty, it would have to contain an equal-treatment payoff vector.[1] For the purpose of demonstrating a contradiction, suppose that $u^\nu=(u_1^\nu,u_2^\nu)$ represents an equal treatment payoff vector in the uniform ε-core of $[s^\nu;(T,\Psi)]$. The following inequalities must hold:

$$\begin{gathered} 3u_1^\nu+\nu u_2^\nu \leq \nu+3, \\ 2u_1^\nu+\nu u_2^\nu \geq \nu+3, \text{ and} \\ u_1^\nu \geq \tfrac{3}{4}. \end{gathered}$$

which is impossible. A payoff vector which assigns each player zero is, however, in the weak ε-core for any $\varepsilon>\frac{3}{\nu+3}$. But it is not very appealing, in situations such as this, to ignore a relatively small group of players (in this case, the players of type 1) who can have a large effect on per capita payoffs. This leads us to the next concept.■

To treat the scarce types problem, Wooders (1992a,b,1994) introduced the condition of small group effectiveness (SGE). SGE is appealing technically since it resolves the scarce types problem. It is also economically intuitive and appealing; the condition defines a class of economies that, when there are many players, generate competitive markets. Informally, SGE dictates that *almost all* gains to collective activities can be realized by relatively small groups of players. Thus, SGE is exactly the sort of assumption required to ensure that multiple, relatively small coalitions, firms, jurisdictions, or clubs, for example, are optimal or near-optimal in large economies.

[1] It is well known and easily demonstrated that the uniform ε-core of a TU game is nonempty if and only if it contains an equal treatment payoff vector. This follows from the fact that the uniform ε-core is a convex set.

A pregame (T, Ψ) satisfies *small group effectiveness, SGE,* if:

$$SGE: \quad \begin{array}{c} \text{For each real number } \varepsilon > 0, \text{ there is an integer } \eta_0(\varepsilon) \text{ such that} \\ \text{for each profile } s, \text{ for some partition } \{s^k\} \text{ of } s \text{ with} \\ \|s^k\| \leq \eta_0(\varepsilon) \text{ for each subprofile } s^k, \text{ it holds that} \\ \Psi^*(s) - \sum_k \Psi(s^k) \leq \varepsilon \|s\|; \end{array} \tag{12}$$

given $\varepsilon > 0$ there is a group size $\eta_0(\varepsilon)$ such that the loss from restricting collective activities within groups to groups containing fewer that $\eta_0(\varepsilon)$ members is at most ε per capita (Wooders 1992a).[2]

SGE also has the desirable feature that if there are no 'scarce types' – types of players that appear in vanishingly small proportions– then SGE and PCB are equivalent.

Theorem 3 (Wooders 1994)**.** *With 'thickness,' SGE=PCB.*

(1) Let (T, Ψ) be a pregame satisfying SGE. Then the pregame satisfies PCB.

(2) Let (T, Ψ) be a pregame satisfying PCB. Then given any positive real number ρ, construct a new pregame (T, Ψ_ρ) where the domain of Ψ_ρ is restricted to profiles s where, for each $t = 1, \cdots, T$, either $\frac{s_t}{\|s\|} > \rho$ or $s_t = 0$. Then (T, Ψ_ρ) satisfies SGE on its domain.

It can also be shown that small groups are effective for the attainment of nearly all feasible outcomes, as in the above definition, if and only if small groups are effective for improvement – any payoff vector that can be significantly improved upon can be improved upon by a small group (see Proposition 3.8 in Wooders 1992b).

Remark 4: Under a stronger condition of *strict* small group effectiveness, which dictates that $\eta(\varepsilon)$ in the definition of small group effectiveness can be chosen independently of ε, stronger results can be obtained than those presented in this section and the next. We refer to Winter and Wooders (1990) for a treatment of this case.

Remark 5. *On the importance of taking into account scarce types.* Recall the quotation from von Neumann and Morgenstern and the discussion following the quotation. The assumption of per capita boundedness has significant consequences but is quite innocuous – ruling out the possibility of average utilities becoming infinite as economies grow large does not seem restrictive. But with only per capita boundedness, even the formation of small coalitions can have significant impacts on aggregate outcomes. With small group effectiveness, however, there is no problem of either large or small coalitions acting together – large coalitions cannot do significantly better then relatively small coalitions.

[2] Exactly the same definition applies to situations with a compact metric space of player types, c.f. Wooders (1988,1992a).

Roughly, the property of large games we next introduce is that relatively small groups of players make only "asymptotic negligible" contributions to per-capita payoffs of large groups. A pregame (Ω, Ψ) satisfies *asymptotic negligibility* if, for any sequence of profiles $\{f^\nu\}$ where

$$\begin{aligned} &\|f^\nu\| \to \infty \text{ as } \nu \to \infty, \\ &\sigma(f^\nu) = \sigma(f^{\nu'}) \text{ for all } \nu \text{ and } \nu' \text{ and} \\ &\lim_{\nu\to\infty} \tfrac{\Psi^*(f^\nu)}{\|f^\nu\|} \text{ exists,} \end{aligned} \tag{13}$$

then for any sequence of profiles $\{\ell^\nu\}$ with

$$\lim_{\nu\to\infty} \frac{\|\ell^\nu\|}{\|f^\nu\|} = 0, \tag{14}$$

it holds that

$$\begin{aligned} \lim_{\nu\to\infty} \tfrac{\Psi^*\|f^\nu+\ell^\nu\|}{\|f^\nu+\ell^\nu\|} &\text{ exists, and} \\ \lim_{\nu\to\infty} \tfrac{\Psi^*\|f^\nu+\ell^\nu\|}{\|f^\nu+\ell^\nu\|} &= \lim_{\nu\to\infty} \tfrac{\Psi^*(f^\nu)}{\|f^\nu\|}\ . \end{aligned} \tag{15}$$

Theorem 4 (Wooders (1992b, 2008b). A pregame (T, Ψ) satisfies SGE if and only if it satisfies PCB and asymptotic negligibility.

Intuitively, asymptotic negligibility ensures that vanishingly small percentages of players have vanishingly small effects on aggregate per-capita worths. It may seem paradoxical that SGE, which highlights the importance of relatively small groups, is equivalent to asymptotic negligibility. To gain some intuition, however, think of a marriage model where only two person marriages are allowed. Obviously two-person groups are (strictly) effective, but also, in large player sets, no two persons can have a substantial affect on aggregate per-capita payoffs.

Remark 6 Without some assumptions ensuring essential superadditivity, at least as incorporated into our definition of feasibility, nonemptiness of approximate cores of large games cannot be expected; superadditivity assumptions (or the close relative, essential superadditivity) are heavily relied upon in all papers on large games cited. In the context of economies, superadditivity is a sort of montonicity of preferences or production functions assumption, that is, superadditivity of Ψ implies that for all $s, s' \in \mathbf{Z}_+^T$, it holds that $\Psi(s+s') \geq \Psi(s)+\Psi(s')$. Our assumption of small group effectiveness, SGE, admits non-monotonicities. For example, suppose that 'two is company, three or more is a crowd,' by supposing there is only one commodity and by setting $\Psi(2) = 2$, $\Psi(n) = 0$ for $n \neq 2$. The reader can verify, however, that this example satisfies small group effectiveness since $\Psi^*(n) = n$ if n is even and $\Psi^*(n) = n - 1$ otherwise. Within the context of pregames, requiring the superadditive cover payoff to be approximately realizable by partitions of the total player set into relatively small groups is the weakest form of superadditivity required for the equivalence of games with many players and concave markets.

7.2 Derivation of markets from pregames satisfying SGE

With SGE and PCB in hand, we can now derive a premarket from a pregame and relate these concepts.

To construct a limiting direct premarket from a pregame, we first define an appropriate utility function. Let (T, Ψ) be a pregame satisfying SGE. For each vector x in $\mathbb{R}^T_+$ define

$$U(x) \stackrel{\text{def}}{=} \|x\| \lim_{\nu \to \infty} \frac{\Psi^*(f^\nu)}{\|f^\nu\|} \tag{16}$$

where the sequence $\{f^\nu\}$ satisfies

$$\begin{gathered} \lim_{\nu \to \infty} \tfrac{f^\nu}{\|f^\nu\|} = \tfrac{x}{\|x\|} \\ \text{and} \\ \|f^\nu\| \to \infty. \end{gathered} \tag{17}$$

Theorem 5. (Wooders 1988; 1994). Assume the pregame (T, Ψ) satisfies small group effectiveness. Then for any $x \in \mathbb{R}^T_+$ the limit (16) exists. Moreover, $U(\cdot)$ is well-defined, concave and 1-homogeneous and the convergence is uniform in the sense that, given $\varepsilon > 0$ there is an integer η such that for all profiles s with $\|s\| \leq \eta$ it holds that

$$\left| U(\frac{s}{\|s\|}) - \frac{\Psi^*(s)}{\|s\|} \right| \leq \varepsilon.$$

From Wooders (1994, Theorem 4), if arbitrarily small percentages of players of any type that appears in games generated by the pregame are ruled out, then the above result holds under per capita boundedness (Wooders 1994, Theorem 6). As noted in the introduction to this paper, for the TU case, the concavity of the limiting utility function, for the model of Wooders (1983) was first noted by Aumann (1987). The concavity is shown to hold with a compact metric space of player types in Wooders (1988) and is simplified to the finite types case in Wooders, (1994).

Theorem 5 follows from the facts that the function U is superadditive and 1-homogeneous on its domain. Since U is concave, it is continuous on the interior of its domain; this follows from PCB. Small group effectiveness ensures that the function U is continuous on its entire domain (Wooders 1994, Lemma 2).

Theorem 6. (Wooders 1994) Let (T, Ψ) be a pregame satisfying small group effectiveness and let (T, U) denote the derived direct market pregame. Then

(T, U) is a totally balanced market game. Moreover, U is one-homogeneous, that is, $U(\lambda x) = \lambda U(x)$ for any non-negative real number λ.

In interpretation, T denotes a number of types of players/commodities and U denotes a utility function on $\mathbb{R}^T_+$. Observe that when U is restricted to profiles (in $\mathbf{Z}^T_+$), the pair (T, U) is a pregame with the property that every game $[n; (T, U)]$ has a nonempty core; thus, we will call (T, U) the *premarket generated by the pregame* (T, Ψ). That every game derived from (T, U) has a nonempty core is a consequence of the Shapley and Shubik (1969) result that market games derived from markets with concave utility functions are totally balanced.

It is interesting to note that, as discussed in Wooders (1994, Section 6), if we restrict the number of commodities to equal the number of player types, then the utility function U is *uniquely* determined. (If one allowed more commodities then one would effectively have 'redundant assets'.) In contrast, for games and markets of fixed, finite size, as demonstrated in Shapley and Shubik (1975), even if we restrict the number of commodities to equal the number of player types, given any nonempty, compact, convex subset of payoff vectors in the core, it is possible to construct utility functions so that this subset coincides with the set of competitive payoffs. Thus, in the Shapley and Shubik approach, equivalence of the core and the set of price-taking competitive outcomes for the direct market is only an artifact of the method used there of constructing utility functions from the data of a game and is quite distinct from the equivalence of the core and the set of competitive payoff vectors as it is usually understood (that is, in the sense of Debreu and Scarf 1963 and Aumann 1964). See also Kalai and Zemel (1982a,b) which characterize the core in multi-commodity flow games.

8 Cores and approximate cores

The concept of the core clearly was important in the work of Shapley and Shubik (1966,1969,1975) and is also important for the equivalence of games with many players and market games. Thus, we discuss the related results of nonemptiness of approximate cores and convergence of approximate cores to the core of the 'limit' - the game where all players have utility functions derived from a pregame and large numbers of players. First, some terminology is required. A vector p is a *subgradient at* x of the concave function U if $U(y) - U(x) \leq p \cdot (y - x)$ for all y. One might think of a subgradient as a bounding hyperplane. To avoid any confusion it might be helpful to note that, as Mas-Colell (1985) remarks: "Strictly speaking, one should use the term *subgradient* for convex functions and *supergradient* for concave. But this is cumbersome", (Mas-Colell 1985, p.29-30.).

For ease of notation, equal-treatment payoff vectors for a game $[n; (T, \Psi)]$ will typically be represented as vectors in $\mathbb{R}^T$. An *equal-treatment payoff vector,* or simply a *payoff vector* when the meaning is clear, is a point $\overline{x}$ in $\mathbb{R}^T$. The t^{th} component of $\overline{x}$, $\overline{x}_t$, is interpreted as the payoff to each player of type t. The feasibility of an equal-treatment payoff vector $\overline{x} \in \mathbb{R}^T$ for the game $[n; (T, \Psi)]$

can be expressed as:

$$\Psi^*(n) \geq \overline{x} \cdot n.$$

Let $[n; (T, \Psi)]$ be a game determined by a pregame (T, Ψ), let ε be a non-negative real number, and let $\overline{x} \in \mathbb{R}^T$ be a (equal-treatment) payoff vector. Then $\overline{x}$ is in the *equal-treatment ε-core* of $[n; (T, \Psi)]$ or simply "in the ε-core" when the meaning is clear, if $\overline{x}$ is feasible for $[n; (T, \Psi)]$ and

$$\Psi(s) \leq \overline{x} \cdot s + \varepsilon \|s\| \text{ for all subprofiles } s \text{ of } n.$$

Thus, the equal-treatment ε-core is the set

$$\begin{aligned} C(n; \varepsilon) \stackrel{\text{def}}{=} \{&\overline{x} \in \mathbb{R}^T_+ : \Psi^*(n) \geq \overline{x} \cdot n \text{ and} \\ &\Psi(s) \leq \overline{x} \cdot s + \varepsilon \|s\| \text{ for all subprofiles } s \text{ of } n\}. \end{aligned} \tag{18}$$

It is well known that the ε-core of a game with transferable utility is nonempty if and only if the equal-treatment ε-core is nonempty.

Continuing with the notation above, for any $s \in \mathbb{R}^T_+$, let $\Pi(s)$ denote the set of subgradients to the function U at the point s;

$$\Pi(s) \stackrel{\text{def}}{=} \{\pi \in \mathbb{R}^T : \pi \cdot s = U(s) \text{ and } \pi \cdot s' \geq U(s') \text{ for all } s' \in \mathbb{R}^T_+\}. \tag{19}$$

The elements in $\Pi(s)$ can be interpreted as equal-treatment core payoffs to a limiting game with the mass of players of type t given by s_t. The core payoff to a player is simply the value of the one unit of a commodity (himself and all his attributes, including endowments of resources) that he owns in the direct market generated by a game. Thus $\Pi(\cdot)$ is called *the limiting core correspondence* for the pregame (T, Ψ). Of course $\Pi(\cdot)$ is also the limiting core correspondence for the pregame (T, U).

Let $\widehat{\Pi}(n) \subset \mathbb{R}^T$ denote *equal-treatment core of the market game* $[n; (T, u)]$:

$$\widehat{\Pi}(n) \stackrel{\text{def}}{=} \{\pi \in \mathbb{R}^T : \pi \cdot n = u(n) \text{ and } \pi \cdot s \geq u(s) \text{ for all } s \in \mathbf{Z}^T_+,\ s \leq n\}. \tag{20}$$

Given any player profile n and derived games $[n; (T, \Psi)]$ and $[n; (T, U)]$ it is interesting to observe the distinction between the equal-treatment core of the game $[n; (T, U)]$, denoted by $\widehat{\Pi}(n)$, defined by (20), and the set $\Pi(n)$ (that is, $\Pi(\overline{x})$ with $\overline{x} = n$). The definitions of $\Pi(n)$ and $\widehat{\Pi}(n)$ are the same except that the qualification "$s \leq n$" in the definition of $\widehat{\Pi}(n)$ does not appear in the definition of $\Pi(n)$. Since $\Pi(n)$ is the *limiting* core correspondence, it takes into account arbitrarily large coalitions. For this reason, for any $\overline{x} \in \Pi(n)$ and $\widehat{x} \in \widehat{\Pi}(n)$ it holds that $\overline{x} \cdot n \geq \widehat{x} \cdot n$. A simple example may be informative.

Example 5. Let (T, Ψ) be a pregame where $T = 1$ and $\Psi(n) = n - \frac{1}{n}$ for each $n \in \mathbb{Z}_+$. and let $[n; (T, \Psi)]$ be a derived game. Then $\Pi(n) = \{1\}$ while $\widehat{\Pi}(n) = \{(1 - \frac{1}{n^2})\}$.

The following Theorem extends a result due to Shapley and Shubik (1975) stated for games derived from pregames.

Theorem 7 (Shapley and Shubik, 1975): Let $[n;(T,\Psi)]$ be a game derived from a pregame and let $[n,u;(T,\Psi)]$ be the direct market generated by $[n;(T,\Psi)]$. Then the equal-treatment core $\widehat{\Pi}(n)$ of the game $[n;(T,u)]$ is nonempty and coincides with the set of competitive price vectors for the direct market $[n,u;(T,\Psi)]$.

Remark 7: Let (T,Ψ) be a pregame satisfying PCB. In the development of the theory of large games as models of competitive economies, the following function on the space of profiles plays an important role:

$$\lim_{r\to\infty}\frac{\Psi^*(rf)}{r};$$

see, for example, Wooders (1977) and Shubik and Wooders (1982). For the purposes of comparison, we introduce another definition of a limiting utility function. For each vector x in $\mathbb{R}^T_+$ with rational components let $r(x)$ be the smallest integer such that $r(x)x$ is a vector of integers. Therefore, for each rational vector x, we can define

$$\hat{U}(x)\stackrel{\text{def}}{=}\lim_{\nu\to\infty}\frac{\Psi^*(\nu r(x)x)}{\nu r(x)}.$$

Since Ψ^* is superadditive and satisfies per capita boundedness, the above limit exists and $\hat{U}(\cdot)$ is well-defined. Also, $\hat{U}(x)$ has a continuous extension to any closed subset strictly in the interior of $\mathbb{R}^T_+$. The function $\hat{U}(x)$, however, may be discontinuous at the boundaries of $\mathbb{R}^T_+$. For example, suppose that $T=2$ and

$$\Psi^*(k,n)=\begin{cases} k+n & \text{when } k>0\\ \\ 0 & \text{otherwise}\end{cases}$$

The function Ψ^* obviously satisfies PCB but does not satisfy SGE. To see the continuity problem, consider the sequences $\{x^\nu\}$ and $\{y^\nu\}$ of vectors in $\mathbb{R}^2_+$ where $x^\nu=(\frac{1}{\nu},\frac{\nu-1}{\nu})$ and $y^\nu=(0,\nu)$. Then $\lim_{\nu\to\infty}x^\nu=\lim_{\nu\to\infty}y^\nu=(0,1)$ but $\lim_{\nu\to\infty}\hat{U}(x^\nu)=1$ while $\lim_{\nu\to\infty}\hat{U}(y^\nu)=0$. SGE is precisely the condition required to avoid this sort of discontinuity, ensuring that the function U is continuous on the boundaries of $\mathbb{R}^T_+$.

Before turning to the next section, let us provide some additional interpretation for $\widehat{\Pi}(n)$. Suppose a game $[n;(T,\Psi)]$ is one generated by an economy, as in Shapley and Shubik (1966) or Owen (1975), for example. Players of different types may have different endowments of private goods. An element π in $\widehat{\Pi}(n)$ is an equal-treatment payoff vector in the core of the balanced cover game generated by $[n;(T,\Psi)]$ and can be interpreted as listing prices for player types where π_t is the price of a player of type t; this price is a price for the player himself, <u>including</u> his endowment of private goods.

9 Nonemptiness and convergence of approximate cores of large games.

The next Proposition is an immediate consequence of the convergence of games to markets shown in Wooders (1992b,1994) and can also be obtained as a consequence of Theorem 5 above.

Proposition 2: *Nonemptiness of approximate cores.* Let (T, Ψ) be a pregame satisfying SGE. Let ε be a positive real number. Then there is an integer $\eta_1(\varepsilon)$ such that any game $[n; (T, \Psi)]$ with $\|n\| \geq \eta_1(\varepsilon)$ has a nonempty ε-core.

[Note that no assumption of superadditivity is required but only because our definition of feasibility is equivalent to feasibility for superadditive covers.]

The following result was stated in Wooders (1992b). For more recent results see Wooders (2008a).

Theorem 8 (Wooders 1992b**)**: *Uniform closeness of (equal-treatment) approximate cores to the core of the limit game.* Let (T, Ψ) be a pregame satisfying SGE and let $\Pi(\cdot)$ be as defined above. Let $\delta > 0$ and $\rho > 0$ be positive real numbers. Then there is a real number ε^* with $0 < \varepsilon^*$ and an integer $\eta_0(\delta, \rho, \varepsilon^*)$ with the following property: for each positive $\varepsilon \in (0, \varepsilon^*]$ and each game $[f; (T, \Psi)]$ with $\|f\| > \eta_0(\delta, \rho, \varepsilon^*)$ and $\frac{f_t}{\|f\|} \geq \rho$ for each $t = 1, \cdots, T$, if $C(f; \varepsilon)$ is nonempty then both

$$\text{dist } [C(f;\varepsilon), \Pi(f)] < \delta \text{ and}$$
$$\text{dist } [C(f;\varepsilon), \widehat{\Pi}(f)] < \delta,$$

where 'dist' is the Hausdorff distance with respect to the sum norm on $\mathbb{R}^T$.

Note that this result applies to games derived from diverse economies, including economies with indivisibilities, nonmonotonicities, local public goods, clubs, and so on.

Theorem 8 motivates the question of whether approximate cores of games derived from pregames satisfying small group effectiveness treat players most of the same type nearly equally. The following result, from Wooders (1977,1992b, 2007) answers this question.

Theorem 9. Let (T, Ψ) be a pregame satisfying SGE. Then given any real numbers $\gamma > 0$ and $\lambda > 0$ there is a positive real number ε^* and an integer ρ such that for each $\varepsilon \in [0,\varepsilon^*]$ and for every profile $n \in \mathbb{Z}_+^T$ with $\|n\|_1 > \rho$, if $x \in \mathbb{R}^N$ is in the ε-core of the game $[n, \Psi]$ with player set

$$N = \{(t, q) : t = 1, ..., T \text{ and, for each } t,\ q = 1, ..., n_t\}$$

then, for each $t \in \{1, ..., T\}$ with $\frac{n_t}{\|n\|_1} \geq \frac{\lambda}{2}$ it holds that

$$|\{(t, q) : |x^{tq} - z_t| > \gamma\}| < \lambda n_t \ \},$$

where, for each $t = 1, ..., T$,

$$z_t = \frac{1}{n_t} \sum_{q=1}^{n_t} x^{tq},$$

the average payoff received by players of type t.

10 Shapley values of games with many players

Let (N, ν) be a game. The *Shapley value* of a superadditive game is the payoff vector whose i^{th} component is given by

$$SH(v,i) = \frac{1}{|N|} \sum_{J=0}^{|N|-1} \frac{1}{\left(\begin{array}{c} |N|-1 \\ J \end{array} \right)} \sum_{\substack{S \subset N \backslash \{i\} \\ |S|=J}} \left[v(S \cup \{i\}) - v(S) \right].$$

To state the next Theorem, we require one additional definition. Let (T, Ψ) be a pregame. The pregame satisfies *boundedness of marginal contributions* (BMC) if there is a constant M such that

$$|\Psi(s + 1_t) - \Psi(s)| \leq M$$

for all vectors $1_t = (0, ..., 0, 1_{t^{th} \text{ place}}, 0, ...0)$ for each $t = 1, ..., T$.Informally, this condition bounds marginal contributions while SGE bounds average contributions. That BMC implies SGE is shown in Wooders (1992b). The following result restricts the main Theorem of Wooders and Zame (1987) to the case of a finite number of types of players.

Theorem 10 (Wooders and Zame 1987). Let (T, Ψ) be a superadditive pregame satisfying boundedness of marginal contributions. For each $\varepsilon > 0$ there is a number $\delta(\varepsilon) > 0$ and an integer $\mu(\varepsilon)$ with the following property:

If $[n, (T, \Psi)]$ is a game derived from the pregame, for which $n_t > \mu(\varepsilon)$ for each t, then the Shapley value of the game is in the (weak) ε-core.

Similar results hold within the context of private goods exchange economies (cf., Shapley (1964), Shapley and Shubik (1969), Champsaur (1975), Mas-Colell (1977), Cheng (1981) and others. Some of these results are for economies without money but all treat private goods exchange economies with divisible goods and concave, monotone utility functions. Moreover, they all treat either replicated sequences of economies or convergent sequences of economies. That games satisfying SGE are asymptotically equivalent to balanced market games clarifies the contribution of the above result. In the context of the prior results developed in this paper, the major shortcoming of the Theorem is that it requires BMC. This author conjectures that the above result, or a close analogue, could be obtained with the milder condition of SGE, but this has not been demonstrated.

11 Economies with clubs

By a club economy we mean an economy where participants in the economy form groups – called clubs– for the purposes of collective consumption and/or production collectively with the group members. The groups may possibly overlap. A club structure of the participants in the economy is a covering of the set of players by clubs. Providing utility functions are quasi-linear, such an economy generates a game of the sort discussed in this essay. The worth of a group of players is the maximum total worth that the group can achieve by forming clubs. The most general model of clubs in the literature at this point is Allouch and Wooders (2008). Yet, if one were to assume that utility functions were all quasi-linear and the set of possible types of participants were finite. the results of this paper would apply.

In the simplest case, the utility of an individual depends on the club profile (the numbers of participants of each type) in his club. The total worth of a group of players is the maximum that it can achieve by splitting into clubs. The results presented in this section immediately apply. When there are many participants, club economies can be represented as markets and the competitive payoff vectors for the market are approximated by equal-treatment payoff vectors in approximate cores. Approximate cores converge to equal treatment and competitive equilibrium payoffs. A more general model making these points is treated in Shubik and Wooders (1982). For recent reviews of the literature, see Conley and Smith (2005) and Kovalenkov and Wooders (2005).[3]

Coalition production economies may also be viewed as club economies. We refer the reader to Böhm (1974), Sondermann (1974), Shubik and Wooders (1983), and for a more recent treatment and further references, Sun, Trockel and Zang (2008).

Let us conclude this section with some historical notes. Club economies came to the attention of the economics profession with the publication of Buchanan (1965). The author pointed out that people care about the numbers of other people with whom they share facilities such as swimming pool clubs. Thus, there may be congestion, leading people to form multiple clubs. Interestingly, much of the recent literature on club economies with many participants and their competitive properties has roots in an older paper, Tiebout (1956). Tiebout conjectured that if public goods are 'local' – that is, subject to exclusion and possibly congestion – then large economies are 'market-like'. A first paper treating club economies with many participants was Pauly (1970), who showed that, when all players have the same preferred club size, then the core of economy is nonempty if and only if all participants in the economy can be partitioned into groups of the preferred size. Wooders (1978) modelled a club economy as one with local public goods and demonstrated that, when individuals within a club (jurisdiction) are required to pay the same share of the costs of public good provision, then outcomes in the core permit heterogeneous clubs if and only if

[3] Other approaches to economies with clubs/local public goods include Casella and Feinstein (2002), Demange (1994), and Haimanko, O., M. Le Breton and S. Weber (2004). Recent research has treated clubs as networks.

all types of participants in the same club have the same demands for local public goods and for congestion. Since these early results, the literature on clubs has grown substantially.

12 With a continuum of players

Since Aumann (1964), much work has been done on economies with a continuum of players. It is natural to question whether the asymptotic equivalence of markets and games reported in this article holds in a continuum setting. Some such results have been obtained.

First, let $N = [01]$ be the $0, 1$ interval with Lesbeque measure and suppose there is a partition of N into a finite set of subsets N_1,...,N_T where, in interpretation, a point in N_t represents a player of type t. Let Ψ be given. Observe that Ψ determines a payoff for any finite group of players, depending on the numbers of players of each type. If we can aggregate partitions of the total player set into finite coalitions then we have defined a game with a continuum of players and finite coalitions.

For a partition of the continuum into finite groups to 'make sense' economically, it must preserve the relative scarcities given by the measure. This was done in Kaneko and Wooders (1986). To illustrate their idea of measurement consistent partitions of the continuum into finite groups, think of a census form that requires each three-person household to label the players in the household, #1, #2, or #3. When checking the consistency of its figures, the census taker would expect the numbers of people labelled #1 in three-person households to equal the numbers labelled #2 and #3. For consistency, the census taker may also check that the number of first persons in three-person households in a particular state is equal to the number of second persons and third persons in three person households in that state. It is simple arithmetic. This consistency should also hold for k-person households for any k. Measurement consistency is the same idea with the work "number" replaced by "proportion" or "measure".

One can immediately apply results reported above to the special case of TU games of Kaneko-Wooders (1986) and conclude that games satisfying small group effectiveness and with a continuum of players have nonempty cores and that the payoff function for the game is one-homogenous. (We note that there have been a number of papers investigating cores of games with a continuum of players that have came to the conclusion that non-emptiness of exact cores does not hold, even with balancedness assumptions, cf., Weber 1979,1981). The results of Wooders, 1994, show that the continuum economy must be representable by one where all players have the same concave, continuous one-homogeneous utility functions. Market games with a continuum of players and a finite set of types are also investigated in Azriel and Lehrer 2007, who confirm these conclusions.)

13 Other related concepts and results.

In an unpublished 1972 paper due to Edward Zajac, which has motivated a large amount of literature on 'subsidy-free pricing', cost sharing, and related concepts, the author writes:

> "A fundamental idea of equity in pricing is that "no consumer group should pay higher prices than it would pay by itself."...If a particular group is paying a higher price than it would pay if it were severed from the total consumer population, the group feels that it is subsidizing the total population and demands a price reduction."

The "dual" of the cost allocation problem is the problem of surplus sharing and subsidy-free pricing.[4] Tauman (1987) provides a excellent survey. Some recent works treating cost allocation and subsidy free-pricing include Moulin (1988,1992). See also the recent notion of "Walras' core" in Qin, Shapley and Shimomura (2001).

Another related area of research has been into whether games with many players satisfy some notion of the Law of Demand of consumer theory (or the Law of Supply of producer theory). Since games with many players resemble market games, which have the property that an increase in the endowment of a commodity leads to a decrease in its price, such a result should be expected. Indeed, for games with many players, a Law of Scarcity holds – if the numbers of players of a particular type is increased, then core payoffs to players of that type do not increase and may decrease. (This result was observed by Scotchmer and Wooders 1988). See Kovalenkov and Wooders (2005,2006) for the most recent version of such results and a discussion of the literature. Laws of scarcity in economies with clubs are examined in Cartwright, Conley and Wooders (2006).

14 Some remarks on markets and more general classes of economies

Forms of the equivalence of outcomes of economies where individuals have concave utility functions but not necessarily linear in money. These include Billera (1974), Billera and Bixby (1974) and Mas-Colell (1975). A natural question is whether the results reported in this paper can extend to nontransferable utility games and economies where individuals have utility functions that are not necessarily liner in money. So far the results obtained are not entirely satisfactory. Nonemptiness of approximate cores of games with many players, however, holds in substantial generality; see Kovalenkov and Wooders (2003) and Wooders (2008).

[4] See, for example Moulin (1988,1992) for excellent discussions of these two problems.

15 Conclusions and Future Directions

The results of Shapley and Shubik (1969), showing equivalence of structures, rather than equivalence of outcomes of solution concepts in a fixed structure (as in Aumann 1964, for example) are remarkable. So far, this line of research has been relatively little explored. The results for games with many players have also not been fully explored, except for in the context of games, such as those derived from economies with clubs, and with utility functions that are linear in money.

Per capita boundedness seems to be about the mildest condition that one can impose on an economic structure and still have scarcity of per capita resources in economies with many participants. In economies with quasi-linear utilities (and here, I mean economies in a general sense, as in the glossary) satisfying per capita boundedness and where there are many substitutes for each type of participant, then as the number of participants grows, these economies resemble or (as if they) *are* market economies where individuals have continuous, and monotonic increasing utility functions. Large groups cannot influence outcomes away from outcomes in the core (and outcomes of free competition) since large groups are not significantly more effective than many small groups (from the equivalence, when each player has many close substitutes, between per capita boundedness and small group effectiveness).

But if there are not many substitutes for each participant, then, as we have seen, per capita boundedness allows small groups of participants to have large effects and free competition need not prevail (cores may be empty and price-taking equilibrium may not exist). The condition required to ensure free competition in economies with many participants, without assumptions of "thickness," is precisely small group effectiveness.

But the most complete results relating markets and games, outlined in this paper, deal with economies in which all participants have utility functions that are linear in money and in games with side payments, where the worth of a group can be divided in any way among the members of the group without any loss of total utility or worth. Nonemptiness of approximate cores of large games without side payments has been demonstrated; see Wooders (1983, 2008) and Kovalenkov and Wooders (2003). Moreover, it has been shown that when side payments are 'limited' then approximate cores of games without side payments treat similar players similarly; see Kovalenkov and Wooders (2001).

Results for *specific economic structures*, relating cores to price taking equilibrium treat can treat situations that are, in some respects, more general. A substantial body of literature shows that certain classes of club economies have nonempty cores and also investigates price-taking equilibrium in these situations. Fundamental results are provided by Gale and Shapley (1962), Shapley and Shubik (1972), and Crawford and Kelso (1982) and many more recent papers. We refer the reader to Roth and Sotomayor (1990) and to the entry "Two sided matching," by Ömer and Sotomayor in this encyclopaedia. A special feature of the models of these papers is that there are two sorts of players or two sides to the market; examples are (1) men and women, (2) workers and firms,

(3) interns and hospitals and so on.

Going beyond two-sided markets to clubs in general, however, one observes that the positive results on nonemptiness of cores and existence of price-taking equilibria only holds under restrictive conditions. A number of recent contributions however, provide specific economic models for which, when there are many participants in the economy, as in exchange economies it holds that price-taking equilibrium exists, cores are non-empty, and the set of outcomes of price-taking equilibrium are equivalent to the core. (see, for example, Wooders 1985,1997, Ellickson et al, 2001, Allouch and Wooders 2008 and Allouch, Conley and Wooders 2008).

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