STRATEGY-PROOFNESS AND THE TOPS-ONLY PROPERTY

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Abstract. A social choice function satisfies the tops-only property if the chosen alternative only depends on each person's report of his most-preferred alternatives on the range of this function. On many domains, strategy-proofness implies the tops-only property provided that the range of the social choice function satisfies some regularity condition. The existing proofs of this result are model specific. In this article, a general proof strategy is proposed for showing that a strategy-proof social choice function satisfies the tops-only property when everyone has the same set of admissible preferences.

Keywords and Phrases: Social choice; strategy-proofness; option sets; tops-only property.

JEL Classification Numbers: D71, D82.

1. Introduction

One of the main lessons of the literature on strategy-proof social choice is that a social decision procedure must ignore most of the information about individual preferences, otherwise it is sometimes possible for someone to manipulate the outcome by misreporting his preferences.[1] A social choice function satisfies the tops-only property if the chosen alternative only depends on each person's report of his most-preferred alternatives on the range of this function. On many domains, strategy-proofness implies the tops-only property provided that the range of the social choice function satisfies some regularity condition. Sprumont (1995, p. 77) has noted that: "Proving this fact constitutes a key step in many papers in the literature. Unfortunately, the proofs remain model specific and are often quite complicated."

In this article, I propose a proof strategy for showing that a strategy-proof social choice function satisfies the tops-only property when the set of admissible preference profiles is the Cartesian product of a common set of individual preferences, what Barberà (2006) calls a common preference domain. This domain assumption would not be satisfied if there are private goods and individuals only care about their own consumption. My methodology also assumes that for each admissible preference, there is a unique most-preferred alternative on the range of the social choice function. When this condition is not satisfied, it is nevertheless possible in some circumstances to characterize the structure of a strategy-proof social choice function by first restricting attention to a subdomain in which this property is satisfied. See, for example, Barberà and Peleg (1990) and Le Breton and Weymark (1999).

My proof strategy is based on the strategy employed by Le Breton and Weymark (1999) to establish their tops-only result. Their proofs make use of arguments first developed by Barberà and Peleg (1990) and Barberà and Jackson (1994). These articles employ the option-set methodology introduced by Barberà (1983), Laffond (1980), and Satterthwaite and Sonnenschein (1981). The option set facing a subgroup of the population is the set of alternatives that are feasible for a social choice function given the reported preferences of the rest of the population.

I illustrate my proof strategy with two examples. In my first example, the set of alternatives is a subset of the real line, the domain of the social choice function consists of all the single-peaked preferences on this set, and the

[1] For surveys of this literature, see Barberà (2006) and Sprumont (1995).

range of the social choice function is an interval. The analysis of strategy-proofness on the domain of single-peaked preferences was initiated by the seminal article of Moulin (1980). Versions of the tops-only result for this domain have been established by Barberà and Jackson (1994), Barberà, Gul, and Stacchetti (1993), and Ching (1997), but their proofs differ from mine in a number of respects.[2] In my second example, the set of alternatives is a metric space and the domain is the set of continuous preferences with unique best alternatives on the range of the social choice function. This is the domain considered by Barberà and Peleg (1990). They do not explicitly prove a tops-only result, but tops-onlyness is implied by their main theorem.

My proof strategy is best understood with a concrete application in mind. Consequently, rather than starting with an abstract statement of this strategy, I instead first use it to prove tops-onlyness in the single-peaked preferences example. Following each step in the argument, I then summarize the essential features of the proof so as to identify the general structure of the proof strategy.

I do not claim that my approach to establishing the tops-only property applies for all common preference domains in which preferences have unique best alternatives on the range of the social choice function. Indeed, it is clear that the domain must be reasonably rich in order for my arguments to apply. However, my proof strategy does work on a number of other domains that have been considered in the literature, some of which are described in my concluding comments. Thus, the proof strategy proposed here provides a unified perspective from which to view many of the existing tops-only results. Hopefully, it will also facilitate the development of tops-only theorems on domains that have not yet been considered.

In the next section, I set out the notation and basic definitions that are employed in subsequent sections. In Section 3, I review a number of theorems that identify some properties that strategy-proof social choice functions must exhibit when there is a common preference domain. My proof strategy is used in Section 4 to prove a tops-only theorem for single-peaked preferences. The analogous theorem for continuous preferences with unique best alternatives on the range of the social choice function is established in Section 5. Some concluding remarks are provided in Section 6.

[2] A tops-only theorem for the related problem of allocating a divisible private good among individuals with single-peaked preferences over own consumption had previously been established by Sprumont (1991).

2. Notation and basic definitions

The set of *individuals* is $N = \{1, \ldots, n\}$, where n is finite. The set of *alternatives* is A, where $|A| \geq 2$.[3] Further assumptions about A will be made in subsequent sections.

An *ordering* R on A is a reflexive, complete, and transitive binary relation. The corresponding strict preference and indifference relations are denoted by P and I, respectively. The set of all orderings of A is $\mathcal{R}$. Preferences may be restricted *a priori*. The common *domain* of individual preferences is $\mathcal{D} \subseteq \mathcal{R}$. Each individual $h \in N$ is assumed to have a preference ordering $R^h \in \mathcal{D}$. A *profile* is an n-tuple of individual preference orderings $\mathbf{R} = (R^1, \ldots, R^n)$. The *set of admissible profiles* is $\mathcal{D}^n$. The assumption that there is a common preference domain is used in a fundamental way in establishing most of the theorems presented here.

A *social choice function* is a function $f \colon \mathcal{D}^n \to A$. The *range* of f is

$$A^f = \{x \in A \mid f(\mathbf{R}) = x \text{ for some } \mathbf{R} \in \mathcal{D}^n\}.$$

The $|H|$-tuple $\mathbf{R}^H = (R^h)_{h \in H}$ denotes the *subprofile* of preferences of the individuals in H, where $\varnothing \subset H \subset N$.[4] A profile is sometimes written as $\mathbf{R} = (\mathbf{R}^H; \mathbf{R}^{-H})$, where $-H$ is the complement of H. For the social choice function f, the *option set* generated by $\mathbf{R}^H$ is

$$O^f_{-H}(\mathbf{R}^H) = \{x \in A \mid x = f(\mathbf{R}^H; \mathbf{R}^{-H}) \text{ for some } \mathbf{R}^{-H} \in \mathcal{D}^{n-|H|}\}.$$

$O^f_{-H}(\mathbf{R}^H)$ is the set of alternatives that are attainable given that the individuals in H have reported the subprofile $\mathbf{R}^H$.

Given a subprofile $\mathbf{R}^H \in \mathcal{D}^{|H|}$, we can use f to define an $(n - |H|)$-person social choice function $g \colon \mathcal{D}^{n-|H|} \to A$ by setting

$$g(\mathbf{R}^{-H}) = f(\mathbf{R}^H; \mathbf{R}^{-H}) \text{ for all } \mathbf{R}^{-H} \in \mathcal{D}^{n-|H|}. \qquad (1)$$

The option set $O^f_{-H}(\mathbf{R}^H)$ is simply the range of g. If $|H| = n - 1$, g is a one-person social choice function. Although there is no social choice problem if there is only one individual, in order to identify some of the properties of option sets generated by the preferences of all but one individual, it is

[3]For any set S, its cardinality is denoted by $|S|$.
[4]If $H = \{h\}$, the braces are omitted.

necessary to consider one-person social choice functions. Except when this is the case, it is assumed that $n \geq 2$.

A social choice function f is *manipulable* by person $h \in N$ at the profile $\mathbf{R} \in \mathcal{D}^n$ via $\bar{R}^h \in \mathcal{D}$ if $f(R^1, \ldots, R^{h-1}, \bar{R}^h, R^{h+1}, \ldots, R^n) P^h f(\mathbf{R})$. If there is no individual $h \in N$, no profile $\mathbf{R} \in \mathcal{D}^n$, and no preference $\bar{R}^h \in \mathcal{D}$ such that f is manipulable by person h at $\mathbf{R}$ via $\bar{R}^h$, then f is *strategy-proof*.

For any $R \in \mathcal{R}$ and any nonempty set $S \subseteq A$, the *top set* of R in S is

$$\tau(R, S) = \{x \in S \mid xRy \text{ for all } y \in S\}.$$

A social choice function f has the *tops-only property* if $f(\mathbf{R}) = f(\bar{\mathbf{R}})$ for all $\mathbf{R}, \bar{\mathbf{R}} \in \mathcal{D}^n$ for which $\tau(R^h, A^f) = \tau(\bar{R}^h, A^f)$ for all $h \in N$. That is, the only information about preferences that f is sensitive to is each person's top set on the range of f.

In some of the literature, the tops-only property is defined in terms of the top sets for A, not A^f. However, this alternative definition has typically been adopted for problems in which the two definitions coincide. This will be the case if either (i) $A = A^f$ or (ii) $A \neq A^f$ and the top set on A^f is uniquely determined by the top set on A for all admissible preferences. In the former case, the social choice function is *surjective*. The latter case applies when, for example, preferences are single-peaked and the range is a closed interval.

3. General results

In this section, I review a number of basic properties of strategy-proof social choice functions that hold quite generally for common preference domains. My discussion is based on Section 2 of Le Breton and Weymark (1999). Throughout this section, it is assumed that $f : \mathcal{D}^n \to A$ is a strategy-proof social choice function. The first four propositions make no assumptions about the structure of the set of alternatives and the domain of the social choice function other than that the preference domain is common.

Proposition 1 establishes that any admissible preference must have a well-defined maximum on the range.

Proposition 1. *Suppose that $n \geq 1$. If a social choice function $f : \mathcal{D}^n \to A$ is strategy-proof, then for all $R \in \mathcal{D}$, $\tau(R, A^f) \neq \varnothing$.*

The next proposition is a unanimity result. It shows that an alternative in S must be chosen if everybody agrees that S is the top set on the range.

Proposition 2. *Suppose that* $n \geq 1$ *and that* $f\colon \mathcal{D}^n \to A$ *is a strategy-proof social choice function. If there exists a set* $S \subseteq A^f$ *such that* $\tau(R^h, A^f) = S$ *for all* $h \in N$*, then* $f(\mathbf{R}) \in S$.

Recall that the option set $O^f_{-H}(\mathbf{R}^H)$ is the range of the social choice function g defined in (1). If f is strategy-proof, then so is g. By applying Proposition 2 to g, it follows that if the individuals not in H agree that S is the top set for this option set, then the chosen alternative must be in S.

Proposition 3. *Suppose that* $n \geq 2$ *and that* $f\colon \mathcal{D}^n \to A$ *is a strategy-proof social choice function. For all nonempty* $H \subset N$ *and all* $\mathbf{R}^H \in \mathcal{D}^{|H|}$*, if* $\tau(R^h, O^f_{-H}(\mathbf{R}^H)) = S$ *for all* $h \notin H$*, then* $f(\mathbf{R}) \in S$.

Another implication of Proposition 2 is that if a subgroup agrees that x is uniquely best on the range, then x must be in the option set that their preferences generate.

Proposition 4. *Suppose that* $n \geq 2$ *and that* $f\colon \mathcal{D}^n \to A$ *is a strategy-proof social choice function. For all nonempty* $H \subset N$*, if* $\tau(R^h, A^f) = \{x\}$ *for all* $h \in H$*, then* $x \in O^f_{-H}(\mathbf{R}^H)$.

For the final two propositions in this section, it is assumed that A is a subset of a metric space.[5] In many applications of these results, A is in fact a subset of a Euclidean space. It is also assumed that the domain of f satisfies a regularity condition. A social choice function $f\colon \mathcal{D}^n \to A$ has a *regular domain* if for all $x \in \mathrm{cl}(A^f)$, there exists a continuous preference $R \in \mathcal{D}$ such that $\tau(R, \mathrm{cl}(A^f)) = \{x\}$, where for any set $S \subseteq A$, $\mathrm{cl}(S)$ is the closure of S.[6]

This domain regularity condition is satisfied in the problems considered in Sections 4 and 5, as well as in many other problems considered in the literature. However, there are problems of economic interest that satisfy my other domain restrictions without being regular. For example, suppose that A is the set of lotteries generated by three or more certain alternatives and that $\mathcal{D}$ is the set of von Neumann–Morgenstern preferences on A. If the social choice function is surjective, then the domain is not regular because

[5] These results hold more generally for any first-countable topological space.

[6] The universal set is A, so closure is defined relative to A. In some applications, A may be embedded in a larger set, in which case $\mathrm{cl}(S)$ is the closure relative to A. A preference $R \in \mathcal{R}$ is *continuous* if for all $x \in A$, $\{y \in A \mid yRx\}$ and $\{y \in A \mid xRy\}$ are both closed sets (relative to A).

only a vertex of A can be a unique maximizer of a von Neumann–Morgenstern preference on A.[7]

Proposition 5 demonstrates that the range of f must be a closed set whenever A is a subset of a metric space provided that the domain regularity condition is satisfied.

Proposition 5. *Suppose that $n \geq 1$ and that A is a subset of a metric space. If $f\colon \mathcal{D}^n \to A$ is a strategy-proof social choice function with a regular domain, then A^f is closed.*

This result is easily established by observing that if A^f is not closed, then for any $x \in \text{cl}(A^f)\backslash A^f$, any $y \in A^f$, and any continuous preference $R \in \mathcal{D}$ for which $\tau(R, \text{cl}(A^f)) = \{x\}$, there must be alternatives close to x in A that are strictly preferred to y by R. This implies that the top set of R on A^f is empty, contradicting Proposition 1.

Because the option set $O^f_{-H}(\mathbf{R}^H)$ is the range of the social choice function g defined in (1) and because strategy-proofness and regularity are properties that g inherits from f, it follows from Proposition 5 that option sets are closed when A is a subset of a metric space and the domain is regular.

Proposition 6. *Suppose that $n \geq 2$ and that A is a subset of a metric space. If $f\colon \mathcal{D}^n \to A$ is a strategy-proof social choice function with a regular domain, then for all nonempty $H \subset N$ and all $\mathbf{R}^H \in \mathcal{D}^{|H|}$, $O^f_{-H}(\mathbf{R}^H)$ is closed.*

[7] Gibbard (1977), Ehlers, Peters, and Storcken (2002), and Dutta, Peters, and Sen (2002), among others, model probabilistic social choice using a probabilistic decision scheme, rather than a social choice function. A *probabilistic decision scheme* assigns a probability measure over the set of certain outcomes A to each admissible profile of preferences on A. Ehlers, Peters, and Storcken (2002) and Dutta, Peters, and Sen (2002) have established tops-only theorems using this framework. A probabilistic decision scheme p can be reformulated as a social choice function f for the set of probability measures $\Delta(A)$ on A if the domain of f is permitted to include profiles of incomplete preference relations on $\Delta(A)$. The function f associated with p is defined by ordinally extending each admissible preference R on A to an incomplete preference $Q(R)$ on $\Delta(A)$ using first-degree stochastic dominance with respect to the upper contour sets of R. The top of $Q(R)$ on $\Delta(A)$ is the probability measure that assigns the top of R on A probability 1. Thus, the domain of f cannot be regular if, as in the articles cited above, random dictatorships are in the range of the social choice function and the tops of the preferences on A are unique.

4. Single-Peaked Preferences

In this section, A is assumed to be a subset of the real line $\mathbb{R}$ and preferences are assumed to be single-peaked on this set. Depending on the application, A could be a discrete or a connected set. For example, A could be a finite set of candidates for election, arrayed on a left-right ideological spectrum. Alternatively, A could be the set of nonnegative reals $\mathbb{R}_+$, with $x \in A$ interpreted as being the quantity of some public good.

It is useful to have a definition of an interval of A that applies whether or not A is itself an interval of $\mathbb{R}$. A subset S of $A \subseteq \mathbb{R}$ is an *interval of* A if $[x, y] \cap A \subseteq S$ whenever $x, y \in S$. In other words, for any two points in S, all the points in A lying between x and y are also in S. The closed interval of A containing all points between x and y is denoted $\overline{xy}$.

A preference R on A is *single-peaked* if there exists an alternative $\pi(R) \in A$ such that $\pi(R)PxPy$ whenever $x, y \in A$ and $y < x < \pi(R)$ or $\pi(R) < x < y$. The alternative $\pi(R)$ is the *peak* of the preference R. The set of single-peaked preferences on A is $\mathcal{S}$.

It is assumed that the social choice function f is defined on the set of all profiles of single-peaked preferences $\mathcal{S}^n$. It is also assumed that the range A^f is an interval of A. Note that f has a regular domain, so by Propositions 5 and 6, the range and all option sets of f must be closed if f is strategy-proof.

For all $R \in \mathcal{S}$, the top set $\tau(R, S)$ is uniquely defined by the peak $\pi(R)$ if S is a closed interval of A. Suppose that x (resp. y) is the smallest (resp. largest) point in S. We then have three cases. (i) If $\pi(R) \in S$, then $\tau(R, S) = \{\pi(R)\}$. (ii) If $\pi(R) < x$, then $\tau(R, S) = \{x\}$. (iii) If $\pi(R) > y$, then $\tau(R, S) = \{y\}$. Thus, if two preferences in $\mathcal{S}$ have the same peak on A, they also have the same top set on the range A^f if the range is a closed interval of A.

To show that strategy-proofness implies the tops-only property on the domain of single-peaked preferences when the range of the social choice function is an interval of A, I first establish a number of intermediate results. Lemma 1 shows that when a set of individuals H agrees that some alternative is uniquely best on the range of the social choice function, then how these individuals rank the other alternatives is irrelevant in determining the option set that their preferences generate. In effect, Lemma 1 shows that option sets satisfy a tops-only property when there is agreement among the individuals in H as to which alternative is best on the range. In the particular case in which there is only one person in H, Lemma 1 says that the option set generated by this person's preference only depends on his most-preferred alternative on the range provided that there is only one such alternative.

Lemma 1. *Suppose that* $n \geq 2$, $A \subseteq \mathbb{R}$, *and* A^f *is an interval of* A. *If* $f \colon \mathcal{S}^n \to A$ *is a strategy-proof social choice function, then for all nonempty* $H \subset N$ *and all* $\mathbf{R}^H, \bar{\mathbf{R}}^H \in \mathcal{S}^{|H|}$ *for which* $\tau(R^h, A^f) = \tau(\bar{R}^k, A^f)$ *for all* $h, k \in H$, $O^f_{-H}(\mathbf{R}^H) = O^f_{-H}(\bar{\mathbf{R}}^H)$.

Proof. The lemma is trivial if $|A^f| = 1$, so suppose that $|A^f| \geq 2$. Without loss of generality, we can suppose that $H = \{1, \ldots, k\}$ with $1 \leq k < n$. Consider the sequence of subprofiles:

$$\begin{aligned}
\mathbf{R}^{H,0} &= (R^1, \ldots, R^k) = \mathbf{R}^H, \\
\mathbf{R}^{H,1} &= (\bar{R}^1, R^2, \ldots, R^k), \\
&\vdots \\
\mathbf{R}^{H,k-1} &= (\bar{R}^1, \ldots, \bar{R}^{k-1}, R^k), \\
\mathbf{R}^{H,k} &= (\bar{R}^1, \ldots, \bar{R}^k) = \bar{\mathbf{R}}^H.
\end{aligned}$$

To establish the lemma, it is sufficient to show that

$$O^f_{-H}(\mathbf{R}^{H,h-1}) = O^f_{-H}(\mathbf{R}^{H,h}) \tag{2}$$

for all $h \in H$.

On the contrary, suppose that there exists an $\bar{h} \in H$ such that (2) does not hold for $h = \bar{h}$. Without loss of generality, we can suppose that there exists a $y \in O^f_{-H}(\mathbf{R}^{H,\bar{h}-1})$ with $y \notin O^f_{-H}(\mathbf{R}^{H,\bar{h}})$.

By assumption, $\tau(R^h, A^f) = \tau(\bar{R}^k, A^f)$ for all $h, k \in H$. Call this common value x. By Proposition 4, $x \in O^f_{-H}(\mathbf{R}^{H,\bar{h}-1}) \cap O^f_{-H}(\mathbf{R}^{H,\bar{h}})$. Without loss of generality, we can suppose that $x < y$. By Proposition 6, $O^f_{-H}(\mathbf{R}^{H,\bar{h}})$ is a closed set. Hence, in $\overline{xy}$ there is a unique closest point to y in $O^f_{-H}(\mathbf{R}^{H,\bar{h}})$. Call this point z.[8] This construction is illustrated in Figure 1 for the case in which $A = \mathbb{R}$. Because there is an open neighbourhood of y not contained in $O^f_{-H}(\mathbf{R}^{H,\bar{h}})$, we can find a preference R^* in $\mathcal{S}$ with peak at y that is maximized at z in $O^f_{-H}(\mathbf{R}^{H,\bar{h}})$. By Proposition 3, we have $f(\mathbf{R}^{H,\bar{h}-1}; R^*, \ldots, R^*) = y$ and $f(\mathbf{R}^{H,\bar{h}}; R^*, \ldots, R^*) = z$. But $zP^{\bar{h}}y$ because z lies between x (which is $\bar{h}$'s top alternative on the range) and y. Hence, $\bar{h}$ can manipulate f at $(\mathbf{R}^{H,\bar{h}-1}; R^*, \ldots, R^*)$ via $\bar{R}^{\bar{h}}$, contradicting strategy-proofness. $\square$

[8] If $|A^f| = 2$, z must be equal to x because x is the only alternative in $O^f_{-H}(\mathbf{R}^{H,\bar{h}})$.

In order to see how the strategy used to prove Lemma 1 can be applied to other domains, it is instructive to summarize the proof strategy without the details that depend on the specific domain used in the lemma. To establish the lemma, there is no loss of generality in supposing that only the preference of one person, say person $\bar{h}$, differs in the two subprofiles $\mathbf{R}^H$ and $\bar{\mathbf{R}}^H$.

By assumption, x is the uniquely-best alternative on the range of the social choice function f for $\bar{h}$ with either of his two preferences. Furthermore, everyone else in H agrees that x is best on the range. Hence, by Proposition 4, x is in the option set facing the rest of the population with either of $\bar{h}$'s preferences. Because the domain is regular, we know from Proposition 6 that the option set $O^f_{-H}(\mathbf{R}^{H,\bar{h}})$ is closed. Contrary to the lemma (and without loss of generality), it is supposed that $O^f_{-H}(\mathbf{R}^{H,\bar{h}-1}) \setminus O^f_{-H}(\mathbf{R}^{H,\bar{h}}) \neq \varnothing$.

The next step in the argument shows that there exists an alternative $y \in O^f_{-H}(\mathbf{R}^{H,\bar{h}-1}) \setminus O^f_{-H}(\mathbf{R}^{H,\bar{h}})$ and a preference R^* in the domain that satisfies two properties: (i) it is maximized at y on A and (ii) every alternative in the top set $\tau(R^*, O^f_{-H}(\mathbf{R}^{H,\bar{h}})$ is strictly preferred to y by the preference $R^{\bar{h}}$. How such a preference is identified is domain specific. For the domain considered in this section, a preference R^* with a unique maximum on $O^f_{-H}(\mathbf{R}^{H,\bar{h}})$ satisfying properties (i) and (ii) can be constructed for any $y \in O^f_{-H}(\mathbf{R}^{H,\bar{h}-1}) \setminus O^f_{-H}(\mathbf{R}^{H,\bar{h}})$. However, on other domains, such a preference may only exist for some $y \in O^f_{-H}(\mathbf{R}^{H,\bar{h}-1}) \setminus O^f_{-H}(\mathbf{R}^{H,\bar{h}})$ and its top set on $O^f_{-H}(\mathbf{R}^{H,\bar{h}})$ may contain more than one alternative.

The proof is completed by supposing that everyone not in H has the preference R^*. Because these individuals have the same top set $\tau(R^*, O^f_{-H}(\mathbf{R}^{H,\bar{h}})$, Proposition 3 implies that some z in this top set must be chosen when $\bar{h}$ reports $\bar{R}^{\bar{h}}$. Similarly, because y is their best alternative in the option set $O^f_{-H}(\mathbf{R}^{H,\bar{h}-1})$, Proposition 3 implies that y must be chosen when $\bar{h}$ reports $R^{\bar{h}}$. Which option set applies is determined by $\bar{h}$'s report. By property (ii), we have $zP^{\bar{h}}y$. Hence, $\bar{h}$ can manipulate the outcome by reporting $\bar{R}^{\bar{h}}$ when his true preference is $R^{\bar{h}}$, contradicting strategy-proofness. Therefore, when a group of individuals H agrees that some alternative x is uniquely best on the range, the option set they generate cannot depend on their rankings of the other alternatives.

Note that the only features of the domain that have been used in the preceding argument is that it is regular and there is a preference in the domain satisfying properties (i) and (ii) for some $y \in O^f_{-H}(\mathbf{R}^{H,\bar{h}-1}) \setminus O^f_{-H}(\mathbf{R}^{H,\bar{h}})$.

The proof strategy used to establish Lemma 1 is based on the proof used by Barberà and Peleg (1990) to prove a version of this lemma in which $|H| = 1$ and the domain is the one considered in the next section. In their proof, the preference R^* is chosen so that property (ii) is satisfied with $\tau(R^*, O^f_{-H}(\mathbf{R}^{H,\bar{h}}) = \{x\}$ (the best alternative on the range for $R^{\bar{h}}$). However, $\bar{h}$'s top on the range need not be in this top set when preferences are single-peaked.

By assumption, the range is an interval. Lemma 2 demonstrates that this property of the range is inherited by any option set generated by a single individual's preference when the domain is the set of single-peaked preferences and the social choice function is strategy-proof.

Lemma 2. *Suppose that $n \geq 2$, $A \subseteq \mathbb{R}$, and A^f is an interval of A. If $f : \mathcal{S}^n \to A$ is a strategy-proof social choice function, then for all $h \in N$ and all $R^h \in \mathcal{S}$, $O^f_{-h}(R^h)$ is a closed interval of A.*

Proof. The proof is trival if $|A^f| \leq 2$, so suppose that $|A^f| \geq 3$. By Proposition 6, $O^f_{-h}(R^h)$ is a closed set. Contrary to the lemma, suppose that there exists an $h \in H$ and an $R^h \in \mathcal{S}$ such that this option set is not an interval. Hence, there exist $\alpha, \beta, \gamma \in A^f$ with $\alpha < \beta < \gamma$ such that $\alpha, \gamma \in O^f_{-h}(R^h)$ and $(\alpha, \gamma) \cap O^f_{-h}(R^h) = \varnothing$. See Figure 2 for the case in which $A = \mathbb{R}$. Because A^f is a closed interval, $\tau(R^h, A^f)$ contains a single alternative, say x. By Proposition 4, $x \in O^f_{-h}(R^h)$. Without loss of generality, we can suppose that $x \geq \gamma$ (otherwise, the roles of α and γ can be reversed). Because R^h is single-peaked, we thus have $\gamma P^h \alpha$.

We can construct a preference $R^\alpha \in \mathcal{S}$ with peak at β that is maximized on $O^f_{-h}(R^h)$ at α. Similarly, we can find a preference $R^\gamma \in \mathcal{S}$ with peak at β that is maximized on $O^f_{-h}(R^h)$ at γ. By Proposition 3, we have $f(R^h; R^\alpha, \ldots, R^\alpha) = \alpha$ and $f(R^h; R^\gamma, \ldots, R^\gamma) = \gamma$. By Lemma 1, $O^f_h(R^\alpha, \ldots, R^\alpha) = O^f_h(R^\gamma, \ldots, R^\gamma)$. Strategy-proofness implies that R^h is maximized at α on $O^f_h(R^\alpha, \ldots, R^\alpha)$ and at γ on $O^f_h(R^\gamma, \ldots, R^\gamma)$. Therefore, $\alpha I^h \gamma$, contradicting our observation that $\gamma P^h \alpha$. □

Lemma 2 shows that there is a structural property of the range that is inherited by the option sets generated by a single person's preference when the social choice function is strategy-proof. In this section, this structural property is that the range is an interval. The regularity of the domain ensures that this interval is closed. Note that any single-peaked preference has a unique best alternative x on any closed interval of A and the only information about this preference that is needed to identify x is its peak.

For other domains of preferences for which a version of Lemma 2 can be established, it is first necessary to identify what structural property to impose on the range of the social choice function f. The preceding discussion suggests that the range restriction must be such that each preference R in the domain has a unique maximum on any set of alternatives that satisfies the structural property assumed for the range, and this maximum must only depend on the top of R on the range. For example, when the set of alternatives is $\mathbb{R}^m$ for some $m \geq 2$, for the preference domains considered by Le Breton and Weymark (1999), the range can be any coordinate hyperplane. With a coordinate hyperplane, the values for some coordinates are unrestricted, whereas the values of the other components are fixed.

Lemma 2 is trivially true for any option set that only contains one alternative because any single point in the range is a closed interval of A. However, on other domains, an option set consisting of a single alternative may not satisfy the structural property assumed for the range of the social choice function. On such an option set, any preference necessarily has a unique maximum determined by its top on the range, and that is all that is needed to apply my proof strategy. As a consequence, on some domains, the analogue to Lemma 2 shows that each option set generated by a single person's preference either preserves some structural property of the range or it contains only a single alternative.

The basic proof strategy can be summarized as follows. Contrary to the lemma, assume that there is an individual h and a preference R^h in the domain for which the option set $O^f_{-h}(R^h)$ generated by R^h does not satisfy the structural property exhibited by the range, but which contains at least two alternatives. An implication of this assumption is that there is an alternative β satisfying certain properties that is not in the option set $O^f_{-h}(R^h)$, but is in the range A^f.[9] For example, in the proof of Lemma 2, β lies in the smallest interval containing the option set. In general, the location of β relative to the option set $O^f_{-h}(R^h)$ plays a role in constructing the preferences R^α and R^γ described below, as does the fact (implied by strategy-proofness) that the alternative x that maximizes R^h on the range is in this option set.

The proof proceeds by constructing two preferences R^α and R^γ in the domain that are both maximized on the range at β, but are uniquely maximized at distinct points α and γ in the option set $O^f_{-h}(R^h)$. The closure

[9]For the domain considered in Section 5, β can be any alternative not in the option set, but this need not be the case for other domains.

of the option set, which by Proposition 6 is implied by strategy-proofness and the regularity of the domain, is used in this step of the argument. The preferences R^α and R^γ are chosen so that the alternatives α and γ are not indifferent to each other according to the preference R^h. Without loss of generality, these alternatives can be labelled so that $\gamma P^h \alpha$. For example, in the proof of Lemma 2, because the peak of the single-peaked preference R^h on the range is x, if α and γ are chosen as in Figure 2, it is necessarily the case that R^h does not regard α and γ as being indifferent.

When everyone other than h shares the preference R^α, they agree that α is the uniquely-best alternative on the option set $O^f_{-h}(R^h)$. Proposition 3 then implies that α must be chosen for the profile $(R^h; R^\alpha, \ldots, R^\alpha)$. Similarly, γ must be chosen for the profile $(R^h; R^\gamma, \ldots, R^\gamma)$.

Consider a profile $\bar{\mathbf{R}}$ for which person h has the preference $\bar{R}^h$ and everyone else has the preference $\bar{R}$. Proposition 3 implies that the social choice $f(\bar{\mathbf{R}})$ must simultaneously (i) maximize $\bar{R}^h$ on the option set generated by the subprofile $\bar{\mathbf{R}}^{-h}$ and (ii) maximize $\bar{R}$ on the option set generated by $\bar{R}^h$. Thus, α must maximize R^h on the option set $O^f_h(R^\alpha, \ldots, R^\alpha)$ and γ must maximize R^h on the option set $O^f_h(R^\gamma, \ldots, R^\gamma)$.

Next, Lemma 1 is applied for a subgroup H consisting of everyone but person h. Because the tops of R^α and R^γ on the range are the same, the option sets $O^f_h(R^\alpha, \ldots, R^\alpha)$ and $O^f_h(R^\gamma, \ldots, R^\gamma)$ are identical. Because both α and γ maximize R^h on this set, we must have $\alpha I^h \gamma$, contradicting the fact that $\gamma P^h \alpha$, which completes the proof.

If it is not possible to find preferences R^α and R^γ such that $\neg(\alpha I^h \gamma)$, it is nevertheless possible to employ the same basic proof strategy if there is a preference $\tilde{R}^h$ in the domain with the same most-preferred alternative on the range as R^h for which such preferences exist. By Lemma 1, R^h and $\tilde{R}^h$ generate the same option set, so the preceding proof applies with $\tilde{R}^h$ substituting for R^h. This kind of substitution is employed by Le Breton and Sen (1995, Lemma 4).

The proof strategy used to establish Lemma 2 is based on the proof used by Le Breton and Weymark (1999) to prove the analogous lemma in their article. The main difficulties encountered in adapting this result to other domains are (i) identifying the structural property of the range that is preserved by non-singleton option sets generated by the preferences of single individuals and (ii) constructing the preferences R^α and R^γ. As Le Breton and Weymark's article demonstrates, the latter constructions might be quite involved.

Lemma 3 extends Lemma 2 by showing that any option set, not just those generated by a single individual's preference, must be an interval if the range satisfies this property, the domain is the set of single-peaked preferences, and the social choice function is strategy-proof.

Lemma 3. *Suppose that $n \geq 2$, $A \subseteq \mathbb{R}$, and A^f is an interval of A. If $f\colon \mathcal{S}^n \to A$ is a strategy-proof social choice function, then for all nonempty $H \subset N$ and all $\mathbf{R}^H \in \mathcal{S}^{|H|}$, $O^f_{-H}(R^H)$ is a closed interval of A.*

Proof. In view of Lemma 2, we only need to consider the case in which $|H| \geq 2$. First, assume that $|H| = 2$ and, without loss of generality, suppose that $H = \{1, 2\}$. For any $R^1 \in \mathcal{S}$, define the $(n-1)$-person social choice function $g_{R^1}\colon \mathcal{S}^{n-1} \to A$ by setting $g_{R^1}(R^2, \ldots, R^n) = f(R^1, \ldots, R^n)$ for all $(R^2, \ldots, R^n) \in \mathcal{S}^{n-1}$. Because f is strategy-proof, so is g_{R^1}. The range of g_{R^1} is $O^f_{-1}(R^1)$, which, by Lemma 2, is an interval of A. Applying Lemma 2 to g_{R^1}, it follows that the option set $O^{g_{R^1}}_{-H}(R^2) = O^f_{-H}(R^1, R^2)$ is an interval of $O^f_{-1}(R^1)$. Hence, because $O^f_{-1}(R^1)$ is an interval of A, $O^f_{-H}(R^1, R^2)$ is also an interval of A. Proceeding by induction on the number of individuals in H, the conclusion follows. □

Lemma 3 and its proof are due to Barberà and Jackson (1994). The reasoning used to establish this lemma applies whenever a structural property of the range A^f of a social choice function f is inherited by non-singleton option sets generated by a single person's preference. Fixing the preference R^h of individual h defines an $(n-1)$-person strategy-proof social choice function g_{R^h} whose arguments are the preferences of the other individuals. The range of g_{R^h} is simply the option set generated by R^h. By assumption, this option set exhibits the same structural property as A^f or it contains only one alternative. Applying the analogue of Lemma 2 for the domain under consideration to g_{R^h} implies that any option set of g_{R^h} generated by the preference of someone other than h must also either satisfy the same structural property or contain only one alternative. In terms of the original social choice function f, it then follows that any option set of f generated by the preferences of two individuals satisfies the structural property or it contains only one alternative. An induction argument on the number of individuals whose preferences are fixed completes the proof.

With Lemmas 1, 2, and 3 in hand, it is now a simple matter to show that the tops-only property is satisfied by a strategy-proof social choice function on the domain of single-peaked preferences when the range is an interval.

Theorem 1. *Suppose that $n \geq 2$, $A \subseteq \mathbb{R}$, and A^f is an interval of A. If $f\colon \mathcal{S}^n \to A$ is a strategy-proof social choice function, then f satisfies the tops-only property.*

Proof. Consider any $\mathbf{R}, \bar{\mathbf{R}} \in \mathcal{S}^n$ for which $\tau(R^h, A^f) = \tau(\bar{R}^h, A^f)$ for all $h \in N$. For any $\bar{h} \in N$, it is sufficient to show that if $\tau(R^{\bar{h}}, A^f) = \tau(\bar{R}^{\bar{h}}, A^f)$ and $R^j = \bar{R}^j$ for all $j \neq \bar{h}$, then $f(\mathbf{R}) = f(\bar{\mathbf{R}})$. By Lemma 3 and Proposition 6, the option set $O^f_{\bar{h}}(R^{-\bar{h}})$ is a closed interval. Because $R^{\bar{h}}$ and $\bar{R}^{\bar{h}}$ are single-peaked with $\tau(R^{\bar{h}}, A^f) = \tau(\bar{R}^{\bar{h}}, A^f)$, they have the same unique maximum on $O^f_{\bar{h}}(R^{-\bar{h}})$. By Proposition 3, we thus have $f(\mathbf{R}) = f(\bar{\mathbf{R}})$. □

The proof strategy used to establish Theorem 1 is due to Le Breton and Weymark (1999). This strategy can be applied whenever it is the case that if two preferences in the domain have the same most-preferred alternative on the range of the social choice function, then they also have the same most-preferred alternative on any option set generated by the preferences of $n-1$ individuals. For a given domain, whether this property of a social choice function is satisfied depends on the structural properties of the option sets. For any domain for which a version of Theorem 1 applies, the analogue of Lemma 3 would be used to show that the relevant properties of an option set are satisfied. To establish the theorem, there is no loss of generality in supposing that only the preference of one person, say person $\bar{h}$, differs in the two profiles. Because these two preferences have the same top on the range, they must be maximized at the same point x in the option set generated by the other individuals' preferences. By Proposition 3, x must be chosen in both cases, from which the tops-only property immediately follows.

When the range is not an interval, a strategy-proof social choice function f defined on $\mathcal{S}^n$ need not satisfy the tops-only property. The problem is that a single-peaked preference on A need not be single-peaked on the range A^f. Single-peakedness on the range is a property of f that has been used to help prove that the tops-only property is implied by strategy-proofness. However, as shown by Barberà and Jackson (1994), if f is strategy-proof, on the subdomain of preferences that are single-peaked on A^f, f satisfies the tops-only property. Letting $\hat{f}$ denote the restriction of f to this subdomain, it follows from Proposition 2 that the range of $\hat{f}$ is the same as the range of f. Hence, by defining intervals relative to A^f (rather than with respect to A), the arguments used here provide an alternative proof of Barberà and Jackson's theorem. No explicit range assumption is required for $\hat{f}$ because A^f is trivally an interval of itself.

5. Continuous Preferences

Let $\mathcal{C}$ denote the set of continuous preferences on a metric space A. Consider a strategy-proof social choice function $f: \mathcal{C}^n \to A$. Barberà and Peleg (1990) have shown that f must be dictatorial provided that the range A^f of f contains at least three alternatives. When the domain is $\mathcal{C}$, a dictatorial social choice function need not satisfy the tops-only property. For example, suppose that everyone's preference, including the dictator's, is maximized on A^f at both x and y. In such a situation, information about preferences other than the tops on the range can be used to break this tie.

Let $\hat{\mathcal{C}}$ be the set of continuous preferences on A that have unique most-preferred alternatives on A^f and let $\hat{f}: \hat{\mathcal{C}}^n \to A$ be the restriction of f to $\hat{\mathcal{C}}^n$. Because f is strategy-proof, so is $\hat{f}$. To prove their theorem, Barberà and Peleg first show that $\hat{f}$ is dictatorial on $\hat{\mathcal{C}}^n$ when $|A^f| \geq 3$. On this subdomain, a dictatorial social choice function satisfies the tops-only property. In this section, I directly show that $\hat{f}$ has the tops-only property regardless of the number of alternatives in its range.

Lemma 4, which is due to Barberà and Peleg (1990), shows that the range $A^{\hat{f}}$ of $\hat{f}$ coincides with A^f.

Lemma 4. *Suppose that $n \geq 2$ and A is a metric space. If $f: \mathcal{C}^n \to A$ is a strategy-proof social choice function, then $A^{\hat{f}} = A^f$.*

Proof. Because $\hat{\mathcal{C}} \subseteq \mathcal{C}$, $A^{\hat{f}} \subseteq A^f$. Consider any $x \in A^f$. By the definition of $\hat{\mathcal{C}}$, there exists a preference $\bar{R} \in \hat{\mathcal{C}}$ such that $\tau(\bar{R}, A^f) = \{x\}$. By Proposition 2, $\hat{f}(\bar{R}, \ldots, \bar{R}) = x$. Thus, $x \in A^{\hat{f}}$. Hence, $A^f \subseteq A^{\hat{f}}$. □

For every $x \in A$, there is a preference $R \in \mathcal{C}$ for which $\tau(R, A) = \{x\}$. Hence, f has a regular domain and, by Proposition 5, A^f is closed. Thus, by Lemma 4, $A^{\hat{f}}$ is closed. Furthermore, $\hat{\mathcal{C}}$ is a regular domain for $\hat{f}$. Because $\hat{f}$ inherits strategy-proofness from f, it follows from Proposition 6 that all the option sets of $\hat{f}$ must be closed.

Lemma 5 demonstrates that if the social choice function f on the domain $\mathcal{C}$ is strategy-proof, then on the subdomain $\hat{\mathcal{C}}$, the option sets of its restriction $\hat{f}$ exhibit the limited form of a tops-only property found in Lemma 1.

Lemma 5. *Suppose that $n \geq 2$ and A is a metric space. If $f: \mathcal{C}^n \to A$ is a strategy-proof social choice function, then for all nonempty $H \subset N$ and all $\mathbf{R}^H, \bar{\mathbf{R}}^H \in \hat{\mathcal{C}}^{|H|}$ for which $\tau(R^h, A^f) = \tau(\bar{R}^k, A^f)$ for all $h, k \in H$, $O^{\hat{f}}_{-H}(\mathbf{R}^H) = O^{\hat{f}}_{-H}(\bar{\mathbf{R}}^H)$.*

Proof. The lemma is trivial if $|A^f| = 1$, so suppose that $|A^f| \geq 2$. The proof is identical to the proof of Lemma 1 (with $\hat{f}$ substituted for f) up to the point at which it is assumed that there exists a $y \in O^f_{-H}(\mathbf{R}^{H,\bar{h}-1})$ with $y \notin O^f_{-H}(\mathbf{R}^{H,\bar{h}})$. As in the proof of Lemma 1, let $\{x\} = \tau(R^h, A^f) = \tau(\bar{R}^k, A^f)$ for all $h, k \in H$.

I now show that there exists a preference $R^* \in \hat{\mathcal{C}}$ that (i) has its unique top at y and (ii) is maximized on the option set $O^f_{-H}(\mathbf{R}^{H,\bar{h}})$ at a unique alternative z that is preferred to y by $R^{\bar{h}}$. Because the option set $O^f_{-H}(\mathbf{R}^{H,\bar{h}})$ is closed and $y \notin O^f_{-H}(\mathbf{R}^{H,\bar{h}})$, it is easy to construct such a preference. The preference R^* is chosen so that it has two local maxima. The first is a global maximum at y, which is strictly preferred to the second local maximum, which is located at x (the top of $R^{\bar{h}}$ on A^f). Because there exists an open neighbourhood containing y that has an empty intersection with $O^f_{-H}(\mathbf{R}^{H,\bar{h}})$, such a "double-peaked" preference R^* for which R^* is maximized on $O^f_{-H}(\mathbf{R}^{H,\bar{h}})$ at x necessarily exists.[10] Thus, property (ii) is satisfied with $z = x$. The rest of the proof is identical to the proof of Lemma 1. □

The only substantive difference in the proofs of Lemmas 1 and 5 is how the preference satisfying properties (i) and (ii) is identified, which is necessarily domain specific.

Next, I need to identify a structural property of the range of f (and, hence, of $\hat{f}$) that is preserved by the non-singleton option sets of the social choice function $\hat{f}$. For any closed strict subset S of A^f containing at least two alternatives and any alternative $x \in A^f \setminus S$, using constructions similar to the one employed in the proof of Lemma 5, one can find two preferences in $\hat{\mathcal{C}}$ with tops at x that are maximized at different points in S. Thus, the only non-singleton subset S of A^f for which every $R \in \hat{\mathcal{C}}$ has a unique maximum on S that only depends on the top of R on the range is A^f. This suggests that the required structural property is that of identity. Thus, the analogue to Lemma 2 for the domain $\hat{\mathcal{C}}$ shows that the option set generated by a single individual's preference is either all of the range or a single alternative.[11]

Lemma 6. *Suppose that $n \geq 2$ and A is a metric space. If $f\colon \mathcal{C}^n \to A$ is a strategy-proof social choice function, then for all $h \in N$ and all $R^h \in \hat{\mathcal{C}}$, $O^{\hat{f}}_{-h}(R^h)$ is either a singleton or all of A^f.*

[10] For a more formal discussion of the construction of R^*, see Barberà and Peleg (1990).

[11] In the latter case, it follows from Proposition 4 that the option set generated by R^h is h's most-preferred alternative on the range.

Proof. The proof is trival if $|A^f| \leq 2$, so suppose that $|A^f| \geq 3$. By Proposition 6, $O^{\hat{f}}_{-h}(R^h)$ is a closed set. Contrary to the lemma, suppose that there exists an $h \in H$ and an $R^h \in \mathcal{S}$ such that $O^{\hat{f}}_{-h}(R^h)$ is neither a singleton nor all of $A^f = A^{\hat{f}}$. By the definition of $\hat{\mathcal{C}}$, $\tau(R^h, A^f)$ contains a single alternative, say γ. By Proposition 4, $\gamma \in O^{\hat{f}}_{-h}(R^h)$. Hence, there exist $\alpha, \beta \in A^f$ such that $\alpha \in O^{\hat{f}}_{-h}(R^h)$, $\alpha \neq \gamma$, and $\beta \notin O^{f}_{-h}(R^h)$. Because $\tau(R^h, A^f) = \{\gamma\}$, we must have $\gamma P^h \alpha$.

As in the proof of Lemma 5, we can construct a "double-peaked" preference $R^\alpha \in \hat{\mathcal{C}}$ with peak at β that is maximized on $O^{\hat{f}}_{-h}(R^h)$ at α and a "double-peaked" preference $R^\gamma \in \hat{\mathcal{C}}$ with peak at β that is maximized on $O^{\hat{f}}_{-h}(R^h)$ at γ. By Proposition 3, we have $\hat{f}(R^h; R^\alpha, \dots, R^\alpha) = \alpha$ and $\hat{f}(R^h; R^\gamma, \dots, R^\gamma) = \gamma$. By Lemma 5, $O^{\hat{f}}_{h}(R^\alpha, \dots, R^\alpha) = O^{\hat{f}}_{h}(R^\gamma, \dots, R^\gamma)$. Strategy-proofness implies that R^h is maximized at α on $O^{\hat{f}}_{h}(R^\alpha, \dots, R^\alpha)$ and at γ on $O^{\hat{f}}_{h}(R^\gamma, \dots, R^\gamma)$. Therefore, $\alpha I^h \gamma$, contradicting our observation that $\gamma P^h \alpha$. □

The proof of Lemma 6 differs from that of Lemma 2 in only two respects. First, the alternative γ can be chosen to be equal to the top of R^h on the range of $\hat{f}$. Second, the construction of the preferences R^α and R^γ are domain specific.

Next, I show that all option sets, not just those generated by a single individual's preference, are either a singleton or all of the range A^f.

Lemma 7. *Suppose that $n \geq 2$ and A is a metric space. If $f : \mathcal{C}^n \to A$ is a strategy-proof social choice function, then for all nonempty $H \subset N$ and all $\mathbf{R}^H \in \hat{\mathcal{C}}^{|H|}$, $O^{\hat{f}}_{-H}(R^H)$ is either a singleton or all of A^f.*

Proof. In view of Lemma 6, we only need to consider the case in which $|H| \geq 2$. First, assume that $|H| = 2$ and, without loss of generality, suppose that $H = \{1, 2\}$. For any $R^1 \in \hat{\mathcal{C}}$, define the $(n-1)$-person strategy-proof social choice function $g_{R^1} : \hat{\mathcal{C}}^{n-1} \to A$ by setting $g_{R^1}(R^2, \dots, R^n) = \hat{f}(R^1, \dots, R^n)$ for all $(R^2, \dots, R^n) \in \hat{\mathcal{C}}^{n-1}$. The range of g_{R^1} is $O^{\hat{f}}_{-1}(R^1)$, which, by Lemma 6, is either a singleton or all of A^f. Applying Lemma 6 to g_{R^1}, it follows that the option set $O^{g_{R^1}}_{-H}(R^2) = O^{\hat{f}}_{-H}(R^1, R^2)$ is also either a singleton or all of A^f. If it is a singleton, then for any $\bar{H} \supseteq H$, $O^{\hat{f}}_{-\bar{H}}(R^{\bar{H}})$ must be as well, and the proof is complete. If it is not, then proceeding by induction on the number of individuals in H, the conclusion follows. □

The property that the option sets exhibit differs in Lemmas 3 and 7, but this does not affect the proof strategy used to establish these results.

Finally, using the same basic proof strategy as was used to prove Theorem 1, I show that the restriction to the domain $\hat{\mathcal{C}}$ of a strategy-proof social choice function on the domain $\mathcal{C}$ satisfies the tops-only property.

Theorem 2. *Suppose that $n \geq 2$ and A is a metric space. If $f : \mathcal{C}^n \to A$ is a strategy-proof social choice function, then $\hat{f}$ satisfies the tops-only property.*

Proof. The proof is the same as the proof of Theorem 1 (with $\hat{f}$ substituted for f) with the following modifications. Lemma 7 is used to conclude that the option set $O^{\hat{f}}_{\bar{h}}(R^{-\bar{h}})$ is either a singleton or all of A^f. Because $R^{\bar{h}}$ and $\bar{R}^{\bar{h}}$ have the same top on the range, in either case it trivially follows that they are maximized at the same point in $O^{\hat{f}}_{\bar{h}}(R^{-\bar{h}})$. □

6. Concluding Remarks

The set of alternatives A is a *product set* if it can be written as $A = \prod_{i=1}^{m} A_i$ for some $m \geq 2$. For some domains of preferences that satisfy a separability assumption, the tops-only property has been shown to follow from strategy-proofness when A is a product set and the social choice function is surjective. Examples include: (i) the domain of continuous separable preferences with unique tops on A (Le Breton and Weymark, 1999), (ii) the domain of separable quadratic preferences on A when $A_i = \mathbb{R}$ for all i (Le Breton and Sen, 1995), (iii) the domain of separable star-shaped preferences on A when $A_i = \mathbb{R}$ for all i (Le Breton and Sen, 1995), (iv) the domain of multidimensional single-peaked preferences when A is a grid; i.e., each set A_i consists of a finite number of points with a fixed distance separating adjacent points (Barberà, Gul, and Stacchetti, 1993), and (v) the domain of separable preferences on A when $A_i = \{0, 1\}$ for all i (Barberà, Sonneneschein, and Zhou, 1991).[12] In the latter two examples, it is supposed that preferences are strict, so that no two alternatives are indifferent to each other. Le Breton and Sen's method of proof for the domains that they consider employs the same basic strategy used to prove the tops-only property in Le Breton and Weymark (1999) and, hence, conforms with the proof strategy proposed here. My proof strategy can also be used to establish tops-onlyness for domains (iv) and (v).

[12]Le Breton and Sen's tops-only theorems are used to provide relatively simple proofs of the main theorems in Border and Jordan (1983).

In each of these examples, the social choice function can be decomposed into the product of m social choice functions, with each of these functions only depending on the marginal preferences for one of the components of A. These social choice functions satisfies the tops-only property. Knowing that the social choice function satisfies the tops-only property facilitates proving that it is decomposable. See, for example, Le Breton and Sen (1995), Le Breton and Weymark (1999), and Weymark (1999).

If there are constraints that preclude some of the alternatives in the product set A from being feasible, then decisions made separately on each component of A may not result in an alternative that is in the feasible subset Z of A, at least if Z is not itself a product set.[13] Restrictions on the social choice function that ensure that this kind of decentralized decision-making is feasible have been identified by, for example, Barberà, Massó, and Neme (1997, 2005) and Barberà, Massó, and Serizawa (1998) when preferences are single-peaked in each component. The social choice functions that are characterized in these articles satisfy the tops-only property.

It may be difficult, perhaps impossible, to apply the methodology proposed here to provide direct proofs of these tops-only results because there does not appear to be any natural structural property of the range A^f that is inherited by the option sets in these problems. Nevertheless, there may be an indirect way of applying my proof strategy using an observation made by Barberà, Massó, and Neme (2005) and Svensson and Torstensson (2005). They note that, for the domains that they consider, there exists a number $\bar{m} \leq m$ ($\bar{m}$ could be 1) such that (a) it is possible to partition $\{1, \ldots, m\}$ into $\bar{m}$ sets such that $A^f = \prod_{j=1}^{\bar{m}} B_j$ (i.e., the range is a product set), where each B_j only includes alternatives from the sets A_i for which i is in the jth cell of this partition and (b) there is no finer partition with this property. Alternatives can be chosen independently from each of the sets B_j without violating feasibility. If it can be shown that the social choice function is decomposable relative to this partitioning of the components of A, I conjecture that my proof strategy can be applied to show that each of the resulting $\bar{m}$ social choice functions satisfies the tops-only property.

My proof strategy presupposes that every preference in the domain has a unique top on the range of the social choice function.[14] As has been noted

[13] Even if $Z = A$, this coordination problem arises if the range is not a product set. See Svensson and Torstensson (2005).

[14] This assumption is automatically satisfied if all preferences are asymmetric, as in Barberà, Sonneneschein, and Zhou (1991) and Barberà, Gul, and Stacchetti (1993).

above, when this is not the case, a strategy-proof social choice function need not satisfy the tops-only property. For each individual h, strategy-proofness requires that the chosen alternative maximizes h's preference on the option set he faces. If there is more than one alternative in this top set, then a strategy-proof tie-breaking rule must be employed to select among these alternatives. Barberà and Jackson (1994) have examined strategy-proof tie-beaking rules for the domain of single-peaked preferences considered in Section 4 when the range is not an interval of the real line.[15] They have shown that except for a limited and identifiable set of profiles, strategy-proof tie-breaking rules satisfy the tops-only property.

Some of the propositions in Section 3 use the assumption that there is a common preference domain in a fundamental way. As a consequence, my proof strategy does not apply if there are personalized preference domains, as would be the case if there are private goods. Nevertheless, tops-only theorems have been established in models with private goods by, for example, Sprumont (1991) and Serizawa (1999), but the proofs are model specific.[16]

It is my hope that the proof strategy proposed here will facilitate progress in our understanding of strategy-proof social choice on domains other than the ones considered in this article. As the analysis of the examples in Sections 4 and 5 demonstrates, the proofs of tops-only theorems when there are no private components to preferences need not be model specific, nor need they be particularly complicated.

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[15]See also the related discussion in Barberà and Jackson (1995).

[16]In Serizawa's article, private goods are used to finance the production of a public good.

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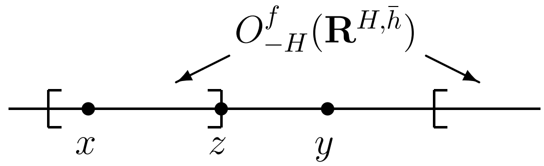

Figure 1

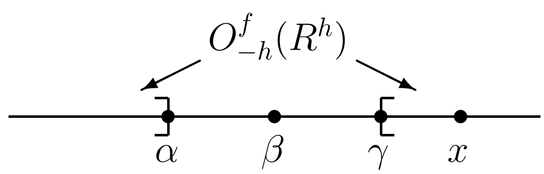

Figure 2