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An Analysis of the Importance of Both Destruction and Creation to Economic Growth

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March, 2019

Abstract

A growth model is studied in which the destruction (or exit) and creative (or research) decisions are decoupled. This approach emphasizes that *different* agents make these interrelated decisions. The growth rate equals the *product* of a measure of the destruction and creation rates. The determinants of income mobility, income inequality, the lifespan of a firm, and the growth rate are studied. The equilibrium can either yield too high or low a level of innovation, but the destruction rate may also be too high or low. A non-linear tax/subsidy scheme, which alters the innovation and exit decisions, can improve welfare.

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1. Introduction

An novel growth model is studied in which there are autonomous, endogenous processes for both the creation and destruction of technologies. These processes are separate in that they are the result of decisions made by different agents, although both are influenced by equilibrium market forces. While in much of the existing literature the destructive process appears to be a (regrettable) consequence, or secondary effect, of the innovative activity, here the destructive process is of *equal* importance to that of innovation, and if the former were to cease, then so would the latter. This model will permit the study of how these autonomous decisions interact to produce an equilibrium growth rate and enables the study of why each of these decisions may not be made optimally.

Important contributions to the literature on economic growth have been made by the study of models that capture the notion of "Creative Destruction". However, in many of these models the "creative" mechanism is indistinct from the "destructive" mechanism, in that they are really the same process. It is apparent that such models do not capture the true nature of the "destructive process" in market economies, wherein products or firms are purged due to the change in factor or product prices, which ultimately reduce the profitability of older technologies.

As an example, consider the novel growth model of Aghion and Howitt 1992, in which there are innovations in the technology for producing an intermediate good. In their benchmark model innovators are given a monopoly, which lasts until some other producer develops a lower-cost technology. The incumbent is then displaced from the market. In this sense, the creative and destructive channels are really indistinguishable.[1] Actual markets rarely

[1]There are many other papers that have a similar linkage between the entry and exit of firms or technologies, such as that of Grossman and Helpman 1991b, or Klette and Kortum 2004. Grossman and Helpman study a model in which the incumbents are not necessarily driven out of the market completely, but instead they are forced into making zero profits. Aghion and Howitt also consider this case. In this instance, there are at most two participants in the market, so it is not quite a monopoly. But again, in these frameworks the innovative and destructive processes are essentially the indistinguishable. In the paper by Klette and Kortum, firms can produce a multitude of goods, but if another firm successfully innovates in producing an

function in this manner. Furthermore, this approach does not capture the notion that these entry and exit decisions are generally made by *different* agents or firms, and that one person's (or firm's) innovation does not necessarily compel the incumbent to leave. It is important to understand and model the exit decision properly because this exodus must inevitably influence the innovation decisions, and vice versa.[2]

In this paper, there will be separate endogenous creation and destruction processes.[3] The development of new technologies is influenced by expected future destruction or exit, while destruction is influenced by expected future innovation and the change in factor prices. However, in equilibrium the development channel makes existing technologies more costly to operate, and therefore reduces the incentive to keep them operational. Therefore, the number of operational technologies (or firms) will be determined endogenously. In addition, the separate destruction or exit decision by an incumbent is characterized as an optimal-stopping problem, and is then the result of that firm-owner behaving optimally.

The uncoupling of the creative (or innovative) and destructive (or exit) decisions is also important because it is then possible to build these autonomous decisions into a planning problem, and to compare these separate optimization conditions that result from such a problem with those that might arise from an equilibrium. It is then possible to assess why there might be too much, or too little innovation, as well as whether there is the proper

existing good, then the incumbent automatically loses the right or ability to produce that good. Once again, the incumbent must exit the market when another firm innovates.

[2] There are other papers in which incumbent firms exit an industry, while newer firms enter. For example, Luttmer 2007 presents a model that is used to characterize the size distribution of firms. In his paper, firms face exogenous variations in productivity, which eventually leads to exit from the market when they can no longer cover their costs. However, Luttmer does not study many of the issues addressed here, such as why the equilibrium exit decision may not be socially optimal manner, or how this decision affects the incentives for innovation, or how government policies might alter this decision to achieve a better outcome. There are other models such as firms exit at a random, exogenous rate (Jones and Kim 2018).

[3] It may be worthwhile before proceeding to establish the terminology that will be employed. In the context of the present discussion, the term "destruction" refers to the voluntary shutdown of a firm due to low productivity, or the voluntary withdrawal of a product from production due to low profitability. That is, the destruction is a result of market forces. What is *not* meant by this term is the shutdown of a firm or the termination of production due to government intervention or regulation, or of a competing firm encourage government authorities to target a firm.

degree of destruction of older technologies.

In much of the existing literature, it seems that the creative or innovation activity is viewed as beneficial, while the destructive process is seen as an unfortunate by-product of innovation. However, by separating the creative and destructive processes, it is possible to show that these activities, though interrelated, have a more complex relationship. It will be shown that the Creative forces have both a negative and a positive consequence, while the same can be said for the Destructive process. The Creative Process has a natural positive impact because it results in more productive technologies. However, it also has a *negative* consequence because it raises the cost of resource inputs to existing firms which makes these existing technologies less profitable. Similarly, the Destructive Process has a negative effect because it results in older firms shutting down, and resources moving on to existing firms. Nevertheless, this process also has a *positive* effect because it results in reduced growth of resource factor prices, which in turn makes existing firms more profitable. This latter effect raises the incentives to innovation, which raises the future growth rate.

The model studied here has other novel features. First, in contrast with most representative agent models that are reticent on such topics as income mobility and inequality, here it is possible to characterize a measure of income inequality, as well as the Gini Coefficient. Secondly, in contrast to many other extant models, this one does not rely on market power (i.e. such as monopolists) to generate innovation or growth. Therefore, any distortions in the model will not result from non-competitive forces. Third, in many existing models the presence of an intertemporal spillover (or externality) will imply that there will be too little innovation or growth. In contrast, the model studied below *will* have an intertemporal spillover, but nevertheless this economy may produce either too high or low a level of innovation or growth. Fourth, by severing the direct linkage between the creative and destructive decisions, this permits the study of how government policies might influence these processes independently. For example, it is possible to study the impact of a policy that subsidizes the creation of new technologies, while simultaneously taxing the destruction of old technologies. Such a policy would seem impossible to study within the context of most extant models.

The model studied below has many features in common with Jaimovich and Rebelo 2017, even though the two models are quite different, and focus on quite different issues. Both models have agents segregate into workers and researchers, both yield a non-linear relationship between the growth rate and parameters such as the tax rate, and can produce an equilibrium in which the growth rate is relatively unresponsive to changes in the tax rate.

2. Description of the Model

Time is assumed to be continuous, and there is no aggregate uncertainty. There are a continuum of agents and the population size is normalized to unity. In the steady-state there will be N agents who are workers while $(1-N)$ who will be termed firm-owners or managers, and these quantities will be determined endogenously, since the agents will choose whether they work, or manage a firm. There will be a dynamic evolution of agents from workers to business (or firm) owners, and this movement will accompany and be related to the growth rate.[4] Workers supply one unit of labor, and the managers will use their unit of time to manage the firm. The analysis will initially presume that there in an internal solution for the optimum, but later there will be some analysis of equilibria at corner solutions.

2.1. The (static) problem of the firm

Each firm-owner has access to a production function $\lambda\left(n_t^{\alpha}\right)$, $\alpha \in (0,1)$, for producing the generic consumption good, with labor as an input. The variable $\lambda > 0$ denotes the technology parameter for a particular firm-owner, which is fixed while this firm is in operation. At any date t, there is a firm with the leading, or best technology, which will be labelled $\bar{\lambda}_t$. It will be supposed that there is a distribution of technologies, which will be denoted $G_t(\lambda)$, which

[4] The evolution of agents between operating a firm and entering the labor force is similar to that in Luttmer 2012.

is defined over some interval $\Lambda_t \equiv \left[\underline{\lambda}_t, \overline{\lambda}_t\right]$.

The firm-owner can hire labor in a competitive market at a price of w_t, and this price *will* change over time. The owner of a firm maximizes profits, which are written as follows:

$$\pi_t = \max_{n_t} \left\{ \lambda \left(n_t^{\alpha} \right) - w_t n_t \right\}.$$

The resulting profit-maximizing condition results in the following demand for labor: $n_t = \left(\frac{\lambda\alpha}{w_t}\right)^{\frac{1}{1-\alpha}}$. The indirect profit function is then written as

$$\pi_t = (\lambda)^{\frac{1}{1-\alpha}} (\alpha)^{\frac{\alpha}{1-\alpha}} (w_t)^{\frac{\alpha}{\alpha-1}} (1-\alpha). \tag{1}$$

For a particular firm, since the technology parameter λ is fixed, the following relationship must hold: $\frac{\dot{\pi}}{\pi} = \frac{\alpha}{\alpha-1}\left(\frac{\dot{w}}{w}\right) < 0$. It will be seen that if this economy is growing at a constant rate, the wage will then exhibit growth at this rate, which in turn implies that the profitability of each firm will be falling. The profit will continue to fall until the firm shuts down.

It must be that the quantity of labor available equals the quantity demanded by all firms. Note again that N is the amount of labor available. Let $G_t(\lambda_t)$ denote the distribution of technologies in period t. Equating the aggregate demand for labor to the supply (N) then results in the following equation which determines the date t wage:

$$w_t^{\frac{1}{1-\alpha}} = \frac{1}{N} \int_{\Lambda_t} (\lambda_t \alpha)^{\frac{1}{1-\alpha}} \, dG_t(\lambda_t). \tag{2}$$

Note that the wage is *homogeneous of degree one in all* λ_t. That is, if all the technologies of all firms in the economy were to be scaled up by some factor, then this would also be the case for the wage as well. The equilibrium below will be one in which $\overline{\lambda}_t$ is proportional to $\underline{\lambda}_t$, and in this case $(\dot{w}/w) = \left(\dot{\bar{\lambda}}_t/\bar{\lambda}_t\right)$.

2.2. The Distribution of Technologies

It will be convenient to put structure on the distribution of the technologies of the firms. Henceforth, we will let $\theta_t \equiv \left(\lambda/\bar{\lambda}_t\right)$ denote the "relative technology" of a particular firm, which possesses technology parameter λ, when the best, or *frontier*, technology is $\bar{\lambda}_t$ at that date. Obviously θ_t ranges between $\underline{\theta}= \left(\underline{\lambda}_t/\bar{\lambda}_t\right)$ and unity. On a balanced growth path, the distribution of θ_t will assumed to be time-invariant. It can then be shown, through the use of the Kolmogorov forward equation that the density must satisfy $f_\theta = (1/\theta)$ over the interval $[\underline{\theta}, 1]$.[5] This implies that the distribution $G_t(\lambda)$ will be a truncated reciprocal distribution.[6]

Since there are $1 - N$ firms, and their relative technologies are distributed with density $f_\theta = (1/\theta)$, over the interval $[\underline{\theta}, 1]$, it then follows that

$$1 - N = \int_{\underline{\theta}}^{1} \left(\frac{1}{\theta}\right) d\theta = -\ln\left(\underline{\theta}\right). \tag{3}$$

Since N can range from zero to unity, it follows that $\underline{\theta}$ can range from e^{-1} to unity. Because a high value of N implies that there are few firms, it seems that N can also be interpreted as one possible measure of firm destruction.

Along a balanced growth path it the frontier technology $\bar{\lambda}_t$ will grow at some rate g. Therefore, for a firm with a fixed technology λ, it must be that $\frac{\dot{\theta}}{\theta} = -g$.

2.3. Workers and Firm-Owners

All individuals are risk-neutral, and so merely wish to consume their income. Their preferences are a function of the discounted stream of consumption $(c_t,\, t \geq 0)$ [7]

$$\int_0^\infty e^{-rt}\left[c_t - h\left(z_t, \bar{\lambda}_t\right)\right] dt, \tag{4}$$

[5] I am indebted to a referee for pointing this out.

[6] The reciprocal distribution is limit of the Pareto distribution, as the latter's shape parameter approaches zero. Fortunately, there is some empirical support for this feature. Luttmer 2011, 2007 finds that the size distribution of firms can be closely approximated by the Pareto distribution. This has led researchers to construct growth models which give rise to such a distribution (for example, Acemoglu and Cao 2015, and Luttmer 2012). Obviously the "truncated" nature of the distribution employed here is a simplification used to characterize the distribution in a convenient manner. Similarly, estimates of income distribution also imply a Pareto distribution, at least at the upper tail, which is similar to that produced by the model (see Cao and Luo 2017, as well as Jones and Kim 2018).

[7] There is an alternative interpretation of the model in which each new firm produces a new commodity that provides more services than previous ones, and so there is creation and destruction of commodities. This approach is similar to that employed by Grossman and Helpman 1991a.

where r is the rate of time preference.[8] At any date there are two types of individuals. There are workers, who supply their unit of labor inelastically and earn the market wage, which is the consumed $c_t = w_t$.[9] Additionally, there are firm-owners, or managers, who use their time to manage their firm. These firms hire labor at the market wage, in order to maximize profit (π_t). The firm-owner has proprietary ownership over his technology (λ), and so owners of inferior technologies cannot costlessly upgrade or steal superior technologies.

Workers are also permitted to use some additional time or effort (z_t) to attempt to discover a new technology, which may eventually permit them to become a firm-owner, or manager. It seems appropriate to identify this as time spent in the pursuit of research or innovation. This activity is successful with some probability $\mu(\cdot)$, but also has disutility $-h\left(z_t, \bar{\lambda}_t\right)$.[10] This is the basis of the "creative process" in the economy. A worker who is successful in inventing a new technology suddenly possesses the frontier technology $(\bar{\lambda}_t)$, but this is at the frontier only momentarily. Firm-owners cannot engage in this activity, and so for them $z = 0$ (and $h(0, \lambda) = 0$). One could interpret this "research sector" as being an informal, or non-market, sector within which all innovation conducted.[11] The amount of

[8] The use of linear preferences simplifies the model but the analysis could also be conducted for any of the CRRA preferences, with a suitable modification of the $h(\cdot)$ function. One advantage of the present approach is that when making welfare comparisons there is no benefit from redistributing output across agents.

[9] The reader will realize that there is nothing intrinsic to the model that necessarily means that this factor must be "labor". It could alternatively be given any other name. It is merely important that there be some factor of production, which is in limited supply, that is owned by individuals, which is *mobile* across firms or technologies, and that this factor be priced and allocated through a competitive market.

[10] The rationale for having this function depend up on $\bar{\lambda}_t$ is that as the leading technology rises, the benefits of innovation are increased, but so are the costs.

[11] One interpretation would be that workers work for a wage, and then spend *extra* time, informally puttering around, and there is some prospect this activity will be very profitable. This is certainly motivated by economic history. Many momentous inventions were produced by individuals who were not employed in research labs, or universities, but instead were people tinkering around in their leisure. For example, the Wright brothers were merely two capable mechanics who had bicycle shop but who, in their spare time, loved to play around with things that might fly. This is also (or perhaps especially) true of the electronic revolution over the past century. Issacson 2015 describes the multitude of inventions that have given rise to electronic, computer, internet, and IT revolutions. Issacson repeatedly refers to people discovering things in their garage in their spare time. The word "garage" arises recurrently in this narrative, especially so when talking about the history of Silicon Valley. Reading this history one gets the impression that most of the discoveries were made by people, many of whom would never graduate college, working long hours in their garages, and that the company offices or laboratories were merely places where these inventors congregated the next day to brief others on the progress of their research effort.

effort expended by an agent in discovering a new technology (z) cannot be observed by other agents, and so it is not possible to engage in contracts contingent on the amount of effort (z), or the outcome from such effort. The cost and benefit of this innovative process is fully internalized by the individual alone.

One can imagine a multitude of factors that might influence the function $\mu(\cdot)$. Clearly it should be increasing in the of level of z, and so frequently below the shorthand notation of $\mu(z)$ will be used. However, one could envisage more complicated formulations that capture the ability of some economies to obtain newer technologies from more advanced economies.

It is assumed that firm-owners spend all their effort managing their firm, and cannot upgrade their technology parameter (λ). Firm-owners always have the option of disposing of their technology (i.e. shutting down their firm) and becoming a worker at the market wage.[12] This will be part of the "destruction process" of older technologies. However, only workers have the opportunity to develop or invent a new technology. This activity requires effort or disutility. When new technologies or firms are developed, this raises the demand for labor which increases the equilibrium wage. This increases the costs and reduces the profits of existing firms. At some juncture an owner of an older firm will find his profit sufficiently eroded that he will elect to shut down the firm, and to become a laborer. At this point he can begin to seek to obtain a new technology, which will give rise to a new firm in the future. There will then be a churning of workers and firms as this economy grow.

2.3.1. The Optimization Problem for a Worker

With a slight abuse of notation, let W_t and V_t denote the date-t value functions for a representative worker and firm-owner, respectively. These functions are implicitly a function of the distribution of technologies of operational firms, but given that distribution, the leading technology $(\bar{\lambda}_t)$ is a sufficient state variable for these value functions.

[12] All workers and firm-owners always have the option of using one of their old technologies to re-start an old firm. However, for reasons that will become clear, this is an option that they will never utilize.

All workers are treated identically, irrespective of their history. Therefore, they will all devote the same amount of effort (z) in obtaining an idea or new technology (λ) which might become productive. As mentioned above, the effort that they expend in discovering a new technology is not observable by others.

It is assumed that workers have discoveries that arrive according to a Poisson arrival rate. Let $\mu(\cdot)$ be the probability of such innovations, and this rate $\mu(z)$, is solely a function of z.

At each instant the flow of utility for a worker is the wage (w_t) net of research effort expended $(h(z, \bar{\lambda}_t))$. In addition to the wage he receives the increased value of the job $(\dot{W}_t)$, plus with some probability $(\mu(z))$ he acquires a new technology so that he switches from being a worker, to managing a firm (with value function $V(\bar{\lambda}_t)$). Each worker takes the wage w_t, and the leading technology $(\bar{\lambda}_t)$ as given while expecting to receive a new technology $(\bar{\lambda}_t)$ for himself, should his research effort be successful. Therefore, the dynamic programming problem of worker is then written as following Hamilton-Jacobi-Bellman equation:[13]

$$rW_t = \max_z \left\{ w_t - h\left(z, \bar{\lambda}_t\right) + \dot{W}_t + \mu(z) \cdot \left[V\left(\bar{\lambda}_t\right) - W_t\right] \right\}. \tag{5}$$

The optimization condition, for an interior optimum, is written as follows:

$$h_1\left(z, \bar{\lambda}_t\right) = \mu'(z)\left[V\left(\bar{\lambda}_t\right) - W_t\right] > 0. \tag{6}$$

This condition determines the equilibrium amount of innovation (z). The right side of equation (6) is the relative benefit from engaging in research or innovation (z), while the left side is the marginal cost. Clearly, the greater is the benefit, as expressed by $\left(V\left(\bar{\lambda}_t\right) - W_t\right)$, the greater will be the amount innovation. But this reward $\left(V\left(\bar{\lambda}_t\right) - W_t\right)$ also reflects the amount of inequality in payoffs to the different agents. It follows that the amount of innovation is then likely to be linked to the degree of income inequality, and policies instituted to reduce this inequality are likely to reduce innovation.

[13] An alternative, but roughly equivalent formulation, is to assume that the individual gets to consume his wage, less some fraction (z) of this wage income that is spent on research. Consumption of the individual is then $w_t(1-z)$.

If it can be shown that equations (5) and (6) imply that if w_t, $h\left(z,\bar{\lambda}_t\right)$, and $V\left(\bar{\lambda}_t\right)$, are all homogeneous of degree 1 in all λ, then so will W_t, and $\dot{W}_t$. Therefore, it will be convenient to let $h\left(z,\bar{\lambda}_t\right) = h\left(z\right)\bar{\lambda}_t$, where $h\left(\cdot\right)$ is strictly convex and differentiable. This means that the utility cost of research becomes greater as $\bar{\lambda}_t$ increases.[14] This assumption implies that both sides of equation (5) are homogeneous of degree one in all λ, and this in turn makes both sides of equation (6) also homogeneous as well. This feature will be exploited below.

2.3.2. The Optimization Problem for the Owner of a Firm

Consider a specific firm-owner who has access to a fixed (i.e. unchanging) technology λ at date-t. This firm generates a flow of profit of π_t. Using some cryptic notation, the value function for this firm-owner is then written as $rV_t = \pi_t + \dot{V}_t$.

As wages grow, the value function for a worker (W_t) will be rising. But since $\dot{\pi} < 0$, because the technology for a firm is fixed, V_t will be *falling* over time. Hence, for an operational firm it must be that $V\left(\lambda_t\right) \geq W\left(w_t\right)$, and as soon as this equation holds with equality, the individual will shut down the firm and become a worker. Hence the HJB equation can then be written as follows:

$$rV_t = \max\left\{\pi_t + \dot{V}_t, rW_t\right\}. \tag{7}$$

This last equation characterizes the optimal stopping problem faced by a firm-owner, who must decide when to shut down his firm. Suppose that this shutdown date is denoted T. Then the solution to this equation is given by the following expression:

$$V_t = \int_t^T e^{-r(s-t)}\pi_s ds + e^{-r(T-t)}W_T. \tag{8}$$

Here the value (V_t) is actually the discounted value of the profit of the firm, plus an American put option. The put option entitles the owner of the firm to sell it (i.e. ownership of the

[14] Under the formulation suggested in the prior footnote this latter assumption would not be necessary, since research effort (z) would be proportional to the wage, which is homogeneous of degree one in all of the operational technologies.

profits), or really dispose of it, at any date for the value W_T. This equation satisfies the value matching condition ($V_T = W_T$) that insures that the welfare of a firm-owner is equal to that of a worker, when the former decides to become a worker.

It is shown in the Appendix that this expression also satisfies the smooth-pasting condition which would imply that $\dot{V}_T = \dot{W}_T$. The optimal shutdown, or exit date (T) of the firm is chosen optimally in equation (8), and this condition is also developed in the Appendix.

A sample path for the value functions for an individual is illustrated in Figure 1. Here the individual begins as a worker, and then at a random date he obtains a new frontier technology, and his value function jumps upward, but then falls and converges to the value function for a worker, at which time he then switches (shutters his firm) to become a worker again. Then the process repeats itself at random times in the future.

2.3.3. Characterizing the Steady-State Equilibrium

It will be convenient to characterize the steady-state behavior of the model, in which there is a balanced growth rate. From equation (2) it can be shown after some algebra that the wage can be written as $w_t = A_w \lambda_t$, where

$$A_w = \alpha \left[\frac{1}{N} \int_{\underline{\theta}}^{1} (\theta)^{\frac{1}{1-\alpha}} f_\theta(\theta)\, d\theta \right]^{1-\alpha} = \alpha \left[\left(\frac{1-\alpha}{N} \right) \left(1 - \underline{\theta}^{\frac{1}{1-\alpha}} \right) \right]^{1-\alpha}. \tag{9}$$

Aghion and Howitt term A_w the "productivity-adjusted wage". Similarly, for a firm with *relative technology* $\theta_t = \left(\lambda / \bar{\lambda}_t\right) \in (0, 1]$, using equations (2) and (1) it is possible to show that profit can be written as $\pi_t(\theta_t) = A_\pi \bar{\lambda}_t (\theta_t)^{\frac{1}{1-\alpha}}$, where

$$A_\pi = (1-\alpha) \left[\frac{1}{N} \int_{\underline{\theta}}^{1} (\theta_t)^{\frac{1}{1-\alpha}} f_\theta(\theta)\, d\theta \right]^{-\alpha} = (1-\alpha) \left[\left(\frac{1-\alpha}{N} \right) \left(1 - \underline{\theta}^{\frac{1}{1-\alpha}} \right) \right]^{-\alpha}. \tag{10}$$

It seems natural to refer to A_π as the "productivity-adjusted profit" for a firm at the technological frontier (i.e. $\theta = 1$). Similarly $A_\pi (\theta)^{\frac{1}{1-\alpha}}$ would be the "productivity-adjusted profit" for a firm with relative technology θ.

As mentioned above, the value functions, and the distribution of the firm productivities are characterized by the leading or frontier technology ($\bar{\lambda}_t$) at any date. The wage and the profit of all firms will be homogeneous of degree one in $(\bar{\lambda}_t)$. In the Appendix it is shown that since equation (8) is homogeneous in $(\bar{\lambda}_t)$ it is possible to re-write it as $\bar{\lambda}_t V(\theta_t)$, where $V(\cdot)$, henceforth referred to as the *normalized value function*, is given by the following:

$$V(\theta) = v_1 (\theta)^{\frac{1}{1-\alpha}} + v_2 (\theta)^{-(r/g)+1} \tag{11}$$

where

$$v_1 = \frac{A_\pi}{r + \left(\frac{\alpha g}{1-\alpha}\right)}, \text{ and } v_2 = \left[W - v_1 \left[(\underline{\theta})^{\left(\frac{1}{1-\alpha}\right)}\right]\right] (\underline{\theta})^{(1/g)(r-g)} > 0. \tag{12}$$

The first term in equation (11) represents the discounted value of the firm's profits, if the firm is operational forever. Since θ is falling over time, this term is also falling. The second term (involving v_2) reflects the fact that at some future date, when $\theta = \underline{\theta}$, it is advantageous for the firm-owner to cease operating the firm, and to become a worker. The term v_2 is then the discounted value of switching *at the optimal time*. Since the exponent $\left(\frac{-r+g}{g}\right) < 0$, this term is rising over time as θ falls. Note again that $V(\underline{\theta}) = W$.

The equation describing the worker's value function (5) can now be written as

$$rW_t = \left\{ A_w \bar{\lambda}_t - h(z^*) \bar{\lambda}_t + \dot{W}_t + \mu(z^*) \left[\bar{\lambda}_t V(1) - W_t\right] \right\}, \tag{13}$$

where z^* is the optimally-chosen value of research. Note that equation (13) is homogeneous of degree one in $\bar{\lambda}_t$. Also, the worker knows that in the event of obtaining an innovation, it will be right on the technological frontier $(\bar{\lambda}_t)$. As a result of the homogeneity, note that $\frac{\dot{W}}{W} = g$. Henceforth, the value functions for the worker and the firm-owner will be written as $\bar{\lambda}_t W$, and $\bar{\lambda}_t V(\theta)$, respectively, while W and $V(\theta)$ will be termed normalized value functions.

Therefore dividing equation (13) by $\bar{\lambda}_t$, allows this to be written as follows:

$$rW = A_w - h(z^*) + Wg + \mu(z^*) [V(1) - W], \tag{14}$$

where the latter equation has exploited the fact that an agent who discovers a frontier technology immediately has technology $\bar{\lambda}_t$.

It is shown in the Appendix the solution to the optimal stopping (or exit) problem faced by a firm with an existing relative technology θ, is given by[15]

$$A_{\pi}\left(\underline{\theta}\right)^{\left(\frac{1}{1-\alpha}\right)} = \left(r-g\right)W. \tag{15}$$

This means that a firm manager with technology parameter $\lambda = \underline{\theta}\bar{\lambda}_t$, (or technology $\underline{\theta}$ relative to the frontier) would be indifferent between being a firm-owner, earning profit $A_{\pi}\left(\underline{\theta}\right)^{\left(\frac{1}{1-\alpha}\right)}\bar{\lambda}_t$, or a worker at that instant. Since the frontier technology $(\bar{\lambda}_t)$ is continuously increasing, the firm-owner would then switch to being a worker at that point. Prior to this shutdown, or exit date, the profit from owning a firm is greater than the right side of equation (15).

The condition for optimal research is then given by

$$h'\left(z\right) = \mu'\left(z\right)\left[V\left(1\right) - W\right]. \tag{16}$$

Henceforth, z^* will denote the solution to this last equation. Equation (14) then yields the following expression for the normalized value function for a worker

$$W = \frac{A_w - h\left(z^*\right) + \mu\left(z^*\right)V\left(1\right)}{\left[r - g + \mu\left(z^*\right)\right]}. \tag{17}$$

It should be clear that the value functions of the two types of agents are interdependent. Factors that influence one of the programming problems will then influence the other. For example, a change in, say, the tax on wages, would then undoubtedly affect both value functions, and then also impinge on both optimization conditions, which are influenced by the size of these value functions.

Lastly, the growth rate is a function of the number of people engaged in research (i.e. workers) and the rate at which they acquire the capability to become firm-owners. Therefore, it is consistent with the feature that the technologies are distributed as truncated reciprocal, that the growth rate will then be characterized in the following functional form:

$$g = \frac{\dot{\bar{\lambda}}_t}{\bar{\lambda}_t} = N\mu\left(z\right). \tag{18}$$

[15] An equivalent expression can be derived from choosing T optimally in equation (8), or maximizing equation (11) with respect to θ, and evaluating the result at $\theta = \underline{\theta}$.

This equation is important in that the growth rate is a function not just of the amount of research effort expended by each worker, but also by the size of the population engaged in this activity. Therefore, in response to some change in the environment, it is possible for research effort (z) to fall, but for the growth rate to rise, if N also rises. Note also that from equation (3), the values of N and $\underline{\theta}$ are closely linked, and the latter is the measure of firm destruction. Therefore, equation (18) shows that it is a salient feature of the model that the growth rate is the *product* of the rates of creation or innovation ($\mu(z)$) and a measure of destruction (N), and so both are equally important in contributing to the growth rate. The job and firm creation rate is positively related to the growth rate, which seems consistent with what we observe about these rates.[16] Also, the simple nature of equation (18) is a result of the fact that all innovation is conducted by new entrants, rather then by existing firms.[17]

2.3.4. Summary of the Equilibrium Conditions

A *competitive equilibrium* on a balanced growth path for this economy consists of time-invariant values for the eight variables $(A_w, A_\pi, V(1), W, \underline{\theta}, g, z, N)$ which satisfy the following equations (3), (9), (10), (11), (14), (15), (16), and (18). Equation (9) is the market clearing condition for labor while equation (3) equates the number of firm-owners to the number of operational firms. The general equilibrium structure of the model means that the growth rate (g), the level of innovation (z), and the rate of destruction ($\underline{\theta}$ or N), are

[16] While the firm dynamics of the model are certainly not identical to what we observe in all respects, they are model are broadly consistent with the documented behavior of firms. For example, in the model younger firms have unusually high innovation intensity, higher total factor productivity, and high employment growth. Decker, Haltiwanger, Jarmin and Miranda 2014 document that this is certainly what is observed in the US economy. They describe how young establishments tend to have substantially higher productivity than existing establishments. In addition, startups have a disproportionately large impact on net and gross job creation, which is certainly true in this model as well. One point of departure is that in the data, existing firms do continue to innovate. In order to preserve the simplicity of the model, this feature was not incorporated. It seems possible to build this feature into the model with some added complications. The model adopts an extreme view of the observation, documented by the Acemoglu and Cao 2015, that new entrants appear to engage in more radical innovation than do incumbents. Lastly, the model also predicts that any slowdown in innovation can be traced to new innovators or firms, which is one interpretation of what has taken place in recent years.

[17] This feature greatly simplifies the analysis of the model. This contrasts with models, such as Acemoglu and Cao 2015, where innovation is undertaken by both incumbents and entrants.

determined jointly with the wages for workers and the profit for firms. All firms and workers behave competitively, and maximize utility or profit while treating market prices parametrically.

Before proceeding it seems appropriate to note what the role that the reciprocal distribution for the technologies (λ) is purchasing. This feature simplifies the formulae in equations (9) and (10). This provides a convenient association, through equation (3), between the number of people operating firms, and the rate of firm destruction. Lastly, it simplifies equation (14) because the value function ($V(\cdot)$) for a person who discovers a new frontier technology is then proportional to the leading technology at that moment.

3. Analysis of the Model

Despite the simplicity of the model, because the general equilibrium, or feedback effects are so Byzantine, it is difficult to use analytical methods to establish how various parameter or policy changes influence such endogenous features, such as the growth rate. Nevertheless, it is possible to establish some important properties that will hold in such an equilibrium.

Proposition 1 *The function $V(\theta)$, from equation (11), consists of two terms, one of which is increasing in θ while the other is decreasing. This function has the property that $\frac{dV(\theta)}{d\theta} > 0$, for $\underline{\theta} < 1$, and $\frac{dV(\theta)}{d\theta} \to 0$, as $\underline{\theta} \to 1$.*

This means the normalized value function must be falling over time, even though there is the beneficial prospect that future wages are rising. The case in which $\underline{\theta} \to 1$ means that firms have an infinitesimal lifetime, and so the value of such a firm cannot decline excessively.

Proposition 2 *From equation (11) it is possible to establish the following:*

$$\frac{\partial V(1)}{\partial v_1}\frac{\partial v_1}{\partial g} < 0, \textit{and } \frac{\partial V(1)}{\partial g} > 0.$$

The first expression is the effect that destruction has on existing firm owners. The higher is the growth rate, the quicker the profit will deteriorate for these firm owners which makes them worse off. The second effect shows that increased growth can be better for firm owners for several reasons. First, the higher will be welfare of these agents when they subsequently terminate operations of their firm, and become a worker. Secondly, the higher is growth, the sooner the existing firms reach the shutdown threshold, and then choose to cease operating (holding the shutdown threshold ($\underline{\theta}$) constant). Third, higher growth will lower the shutdown threshold ($\underline{\theta}$), and therefore, the sooner the firm will reach it. It warrants repeating that while a marginal increase in the growth rate benefits workers, it can reduce the welfare of exiting firm owners.

A separate characterization of the value functions W and $V(1)$ is problematic because these two functions are interrelated. However, the following is a useful and intuitive result.

Proposition 3 *Equations (11) and (17) imply that*

$$\frac{\partial \left(V\left(1\right)-W\right)}{\partial A_{\pi}}>0,\frac{\partial \left(V\left(1\right)-W\right)}{\partial A_{w}}\leq 0,$$

with the latter derivative holding strictly when $\underline{\theta}<1$.

On the surface this seems obvious: raising the reward to firm-owners (workers) relative to workers (firm-owners) increases (decreases) the relative difference in the value function. But this also implies that a policy such as using a profit tax to fund lump-sum transfers will lower the size of $\left(V\left(1\right)-W\right)$. This will lower the reward to research, which is given on the right side of equation (16). This in turn will influence the growth rate.

Additionally this result suggests that the growth rate can be decreasing in the profit tax but *increasing* in the labor tax. Jaimovich and Rebelo 2017 document that these exact effects can be found in some panel regressions.

The following are results regarding inequality, or relative incomes, in the model.

Proposition 4 *The ratio of incomes* (A_π/A_w) *is increasing in* N *(or equivalently* $\underline{\theta}$*). Additionally, along a balanced growth path,* $A_\pi\left(\underline{\theta}\right)^{\left(\frac{1}{1-\alpha}\right)} > A_w$*.*

The first statement says that the ratio of the highest income to the lowest income is positively related to a measure of the rate of firm destruction. This is a constructive result because it turns out that this measure of inequality is highly correlated with other measures, such as the Gini coefficient. The second statement asserts that at the time at which the firm shuts down, the firm's profit will exceed the market wage.

In an equilibrium, it may be that the growth rate is very low, but it should still be positive, as is shown in the following result:

Proposition 5 *If* $h\left(0\right) = h'\left(0\right) = 0$*, and* $\mu'\left(0\right) > 0$*, then on a balanced growth path* $g > 0$*.*

It is interesting is to investigate how this economy might display zero growth. One method of characterizing this situation is now described.

Proposition 6 *If* $h'\left(0\right) > h\left(0\right) = 0$*, or if* $\mu\left(0\right) = 0$*, then there may exist an equilibrium in which* $g = 0$*.*

In this case the marginal cost of engaging in research (z) can be greater than the benefit, and hence no research takes place $(z = 0)$, even if $V\left(1\right) - W > 0$, and so $g = 0$. In this case the system has 6 equations and unknowns .

This is a useful and important result. This shows that in economies where the relative costs to innovation or research are sufficiently high, there will be no growth. Any possible reduction in this cost, or an increase the returns (e.g. raise $\mu'\left(0\right)$) can facilitate the promotion of growth. Another important point that arises here is that, although equation (16) suggests that inequality, as reflected in $V\left(1\right) - W > 0$, is necessary for economic growth, it is not sufficient. In this case the firm owners will never shut down their firm, and therefore there will be neither creation nor destruction of new technologies. This shows that in economies in which the relative costs to innovation or research are not sufficiently high, there will be no growth.

Furthermore, this result illustrates why one might observe similar economies contemporaneously exhibiting different growth rates, even if they have the same interest rate. It can be that they have different values for the functions $h(z)$ or $\mu(z)$. That is, they have different cost or reward functions for the process of acquiring new technologies.

Corollary 7 *If, in addition to the conditions of this no-growth equilibrium, it is the case that* $A_\pi(\underline{\theta})^{\left(\frac{1}{1-\alpha}\right)} > rW$, *then there are a continuum of equilibria with* $g = z = 0$.

In this case the "marginal" firm, or owner of the firm with the worst technology, is receiving profit that is higher strictly greater than the equilibrium wage. The continuum results from the fact that it is possible to shift a few agents from being firm-owners to workers, and although this would marginally affect the equilibrium wages and profit (A_w and A_π), it would not change them sufficiently to initiate any growth.

The following establishes how to characterize the lifespan of a typical firm, a measure of the rate of firm destruction, as well as the degree of income mobility.

Proposition 8 *The length of time that a firm is operational is calculated as follows:* $\hat{T} = \frac{-\ln(\underline{\theta})}{g} = \frac{1-N}{g}$. *The average time it takes the worker to cycle through from initially becoming a worker, to becoming a firm-owner, and finally shutting it down, is* $T = \frac{1}{g}$.[18]

There is one final "non-result" that is of note. The characterization of the factors that influence the growth rate is not straightforward, despite the simplicity of equation (18). A beneficial alteration in the environment, such as an increase in the probability of an innovation $\mu()$ does not necessarily result in a higher growth rate. The reason is that although this would appear to increase the equilibrium amount of research (z), from equation

[18] Using equation (18) it is possible to see that $N\mu(z) = (1-N)/\hat{T}$, which equates the flow of new firms created to the flow of firms that cease production.

(16), and likely raise welfare (W), from equation (17), equation (15) also suggests that this could also raise the value of ($\underline{\theta}$), which means that N also rises. This effect would then *lower* the growth rate in equation (18). This is where the (endogenous) level of firm destruction, which is inherent in the level of ($\underline{\theta}$) or N, will influence the growth rate.

4. Further Characterization of an Equilibrium

To obtain further insights into the behavior of the model it is necessary to put more structure on to it, and then study specific examples. To this end, the following form will be used for the $h(\cdot)$ function

$$h(z) = \gamma \frac{z^{1+\omega}}{1+\omega} \tag{19}$$

where $\gamma, \omega > 0$. Much of the analysis below is only used to illustrate some features of the model, and is not intended to mimic any specific economy. Unless stated otherwise, the following parameter values will be used for the benchmark economy: $r = .07$, $\alpha = .65$, $\mu = .1$, $\gamma = 0.38$, $\omega = 1.0$. These values produce a resulting equilibrium growth rate of 3%. Some of these parameters (e.g. r, α) have usual justifications. For others, it is not clear how to arrive at an appropriate value. For example, normally the value of $(1/\omega)$ might be thought of as related to the labor elasticity, but some reflection would reveal that this is not the case here for several reasons. First, there is no intensive margin of employment. Secondly, the choice of z is not an employment decision, and in fact it is the opposite: The choice of z reflects the agent's *desire to exit the labor force*, and to manage a firm.[19]

In general it is problematic to employ such an explicit model to attempt to mimic an actual economy because models with linear preferences frequently give implausible results. In particular, these preferences imply an infinite intertemporal elasticity of substitution of

[19] Additionally, it is natural to suppose that the parameter α represents "labor's share" of income. However, as mentioned above (see footnote 9), a literal interpretation of this as labor may not be appropriate, and instead it may represent any finite resources that are mobile across alternative technologies. To the extent that resources are not mobile across various firms or industries, the parameter α may have to take on a much lower value.

consumption, and this in turn can imply an implausibly large change in the growth rate in response to a change in the after tax return.

4.1. Inequality and Taxation

It has been a long-standing research issue to investigate the relationship between the level of income inequality and the corresponding growth rate (see, for example, Greenwood and Jovanovic 1990, or Jones and Kim 2018). In many models, inequality is the result from growth, but here the inequality is *both the cause and the result of growth.* It is shown in Huffman 2018 that this model has a Gini coefficient that is straightforward to characterize, and this is useful for studying how various policies might have an impact on this measure of inequality. In particular, the levels of creation and destruction certainly influence how income is allocated across the population.

In general, it is the case that the Gini coefficients tend to be decreasing in the profit tax. However, the relationship between inequality and labor taxation is more complicated. An example of this is shown in Figure 2, for the benchmark model. In this case, the Gini coefficient is shown as a function of the tax rate, for both the labor and profit, and revenue is given back to individuals as a lump-sum transfer. As can be seen, it appears that inequality is decreasing in *both* taxes for this economy, but this effect is more pronounced for the profit tax. Raising the profit tax reduces inequality because this amounts to redistributing income from richer to poorer agents.

The effect of the labor tax on inequality may seem puzzling: How can a policy, that taxes relatively poor workers and transfers some revenue to richer firm-owners, reduce inequality? The general equilibrium effects dictate that the labor tax will cause N to fall, which implies business destruction falls. Essentially, an increase in the labor tax increases the incentive for workers to engage in research, and makes firm-owners want to keep their firms operating for longer. This implies there will be fewer workers and more firms in equilibrium, which results in marginally lower income inequality. This experiment illustrates the complicated factors that influence the determination inequality within such a model. Also, since both taxes raise inequality but have the opposite impact on growth, this illustrates the complicated relationship between growth and inequality.

As indicated earlier, some degree of inequality, as reflected in the size of $(V - W)$, is vital for growth to motivate individuals to engage in the research activity (z). There are other models in which greater inequality may accompany higher growth (see, for example, Greenwood and Jovanovic 1990). This is true here, but additionally some degree of inequality is requisite for growth. This effect is partially attenuated since it can be shown that $\frac{\partial \ln(V_t)}{\partial g} < \frac{\partial \ln(W_t)}{\partial g}$, and so a small change in the growth rate can also reduce inequality of welfare.[20]

4.2. Growth and Taxation

It has been recognized that in the US there seems to be very little relationship between the growth rate, and various measures of income taxation (e.g., see Jaimovich and Rebelo 2017, Stokey and Rebelo 1995). It is then somewhat of a test of any model to see if it can replicate this (non) relationship. Therefore, consider the benchmark model without taxes, in which the growth rate is 3.0%. If an income tax (i.e. on both labor and profit) of 30% is introduced, with the resulting revenue distributed in a lump-sum manner, the growth rate is only reduced to 2.46%. This is a reduction that is sufficiently small that it is unlikely to be detected in the data.[21] Raising the value of the parameter ω reduces the impact on growth even further, as $(1/\omega)$ seems to act like an elasticity of the growth with respect to the tax rate. In fact, for sufficiently large values of ω, raising the profit tax can result in a very modest increase in the growth rate. This effect will be further illustrated below.

As mentioned above, the model would seem to imply that while a profit tax would lower

[20]Again, it is not the case that the welfare of all firm-owners is elevated by a marginal increase in the growth rate.

[21]These reductions in the growth rate are of a similar magnitude, whether the government revenue is destroyed, or given back to individuals in a lump-sum manner.

growth, a labor tax would raise it. This is true for the benchmark economy. This is not a result that typically arises in growth models. Fortunately there is some empirical support for this. Jaimovich and Rebelo 2017 find that in some panel regressions, which include time and fixed effects, that the growth rate is positively related to the labor income tax rate, while negatively related to the capital tax rate, and these results are significant.[22]

Also, for the case in which research is paid out of worker's post-tax income, so that consumption equals $w_t(1-z)$ (see footnote 13) it can be shown analytically that equal labor and profit taxes of any magnitude will not affect the growth rate if the revenue is not transferred back to individuals.

4.3. Factors Influencing Firm Destruction

An innovative feature of this model is that gives rise to an endogenous level of firm exit, or destruction. It is then instructive to investigate how various factors influence this exit rate. First, it is essential to determine how to measure this feature. One approach is to let "N" denote an ordinal measure of destruction, since this is inversely related to the number of firms. An alternative measure of destruction is the inverse of the average time a new firm will spend being operational. This time-span is given by the variable $\hat{T} = \frac{1-N}{g}$.

Next, it is necessary to vary some feature of the model to study how this influences the level of destruction. Varying the tax rates seems like a natural candidate. Figure 3a shows how both N and $\left(1/\hat{T}\right)$ vary, as the labor tax rate changes, for the benchmark economy, and the resulting revenue is distributed in a lump-sum manner.[23] Increases in the tax rate lead to lower levels of N, and higher levels of $\hat{T}$, both of which indicate a lower level of business exit. Increased labor taxation results in more operational firms, and these firms produce for

[22] This result suggests yet another reason why it could be difficult to uncover any relationship between tax rates and growth rates in the data. Suppose that different economies employed different combinations of labor and capital taxes, in a world like that of the model in which the growth rate was increasing in the former but decreasing in the latter. Then it could be very difficult to find any relationship between growth rates and "income tax rates."

[23] In this figure the values of both N and $(1/\hat{T})$ are normalized to unity when the tax rate is zero.

a longer period of time.

Next, Figure 3b shows how both N and $\hat{T}$ vary in the steady-state, as the tax rate on profit changes, for the benchmark economy. This example shows that these measures of business destruction do not always move in the same direction. In this instance, raising the profit tax results a higher level of both N and $\hat{T}$. This results in fewer firms, but also a lower growth rate. Since the latter effect overwhelms the former, the value of $\hat{T}$ rises.

This result is important for another reason. It seems to be an interesting but open question as to whether there is a "cleansing effect" of recessions, in that a recession may have a beneficial effect of reducing the economy of low-productivity firms. To the extent that comparative dynamics exercises should be taken seriously, an increase in the tax rate on profit will reduce the growth rate, and so could have a similar observed effect to that of a recession, since the growth rate falls. Suppose one were to take the level of 'N' as the measure of business destruction, since as N rises the number of firms falls. Figure 3b suggests that the rate of business destruction could then increase, as the low-productivity firms that were operating under the benchmark economy now would shut down earlier. However, it is not clear that this should be interpreted as a cleansing effect.

In contrast, in Figure 3a, by raising the labor tax, which causes the growth rate to rise, this lowers the rate of destruction. Through this channel there would seem to be a negative relationship between the rate of growth and the rate of business destruction.

5. Optimal and Equilibrium Levels of Creation and Destruction

It is possible to construct a measure of welfare that weighs the welfare (i.e. value functions) of each of the agents in the economy, and then to use this as a measure of welfare when making comparisons across different decision rules, or government policies. This measure can also be used to construct a social planning problem for this economy. In Huffman 2018 a planning problem for this economy is studied in order to investigate all of the channels through which the creation and destruction decisions influence welfare, and to scrutinize why the equilibrium decisions might not be socially optimal. This analysis shows that these creation and destruction decisions have a multitude of effects on the growth rate, factor prices, equilibrium conditions, as well as on each other. However, it seems that whether the equilibrium levels of creation or destruction are too high or low, relative to some optimum, would seem to rather case-sensitive.[24]

Therefore, the remainder of this analysis will focus on how a system of taxes might influence welfare, as well as the growth rate.

6. The Model with Linear Taxation and Lump-Sum Transfers

It is important to study the effect of simple linear government taxes with lump-sum transfers. It is a convenient property that the welfare functions always seem to exhibit single-peakedness, and frequently have an "inverted-U" shape over various tax rates.

Panels (a) and (b) of Figure 4 present the results from a varying the labor tax rate, while the profit tax is zero, for the benchmark economy. The welfare function here is the value function of a worker (W), who would be the median voter. As the figure shows, welfare is maximized by having a labor tax of 28%. This policy of transferring revenue from workers to firm-owners raises the growth rate, and the number of firms. Raising the labor tax above zero also lowers inequality, in spite of the fact that the transfer is going from the poorer workers to the richer firm-owners. Lastly, for this economy the growth rate is non-montonic in the labor tax: for modest labor taxes, further increases will raise the growth rate, while at higher levels, an increase will lower the growth rate. For this economy, even workers prefer a negative profit tax because this results in higher growth.

[24] In Huffman 2018 it is also shown that the Lorenz curve and Gini coefficients can be studied, and more inequality-related experiments are presented. In addition, the model is capable of explaining the Great Gatsby curve. It is also shown that the price-earnings ratios of younger firms is greater than that of older firms, even though the equilibrium rate of return in the economy is fixed at r. This feature seems to conform with what is observed about these ratios. Lastly, it is shown that the tax rates can influence these price-earnings ratios in a non-trivial manner.

Panels (c) and (d) of Figure 4 show how worker-welfare and inequality change for different tax rates, where $\mu = .035$, $\gamma = .0143$, and $\omega = 10$. These parameter values also produce a steady-state growth rate of 3% when taxes are zero. In this case (worker's) welfare is maximized by having a tax rate on *profit* of 15.4%. In the first example workers benefit from growth so much that they would never wish to tax profit, but in this second example they are willing to do so. The reason welfare is increasing in the tax rate is not because growth is not important - it is as critical as ever to workers. Instead a higher value of ω implies that research (z) is relatively unresponsive to an increase in the tax rate. However, as the tax rate rises the number of workers (N) rises because owning a firm is less attractive, and this results in a modest increase in the growth rate, through equation (18).

7. Welfare Improvements Through Productivity-Dependent Government Taxation and Transfers

As indicated earlier, in Huffman 2018 a planning problem is constructed for this model in order to establish whether the equilibrium decision rules and welfare are optimal. Within this setup it is possible to see that a system of non-linear, or state-dependent taxes and transfers, that may raise welfare welfare. This will be illustrated below through the use of several examples.[25] Here a system of labor taxes (τ_n), and tax rates that depend on firm productivity ($\tau_\pi(\theta)$) are derived in order to maximize the welfare function, which is defined to be the equally-weighted function of all of the value functions: $NW + \int V(\theta) f_\theta(\theta) d\theta$. Since the government budget constraint is continuously balanced, some of these taxes must necessarily be negative.

Example 9 Consider the parameterization of the benchmark economy described in Section 4. Figure 5 shows the tax and subsidy policy, of the sort described above, that results in

[25] The parameter values used henceforth will be the same as in the benchmark with the exception that the growth equation (18) will now by determined as $g = \delta(zN)^{1/4}$, where δ is chosen so as to imply a value for μ equal to the benchmark value of 0.10.

a higher level of welfare for this economy. In this case welfare can be increased by having the government tax labor at a rate of 12%, and then use this revenue to subsidize firms according to the schedule in Figure 5. This policy implies that the high-productivity firms should be *subsidized* at rate of 55%, while the low-productivity firms are *taxed* at a rate of 11.2%. This shifts resources from the workers, and owners of low-productivity firms (who will soon become workers), to the owners of high-productivity firms. The benchmark model had a growth rate of 3%, while under this alternative policy the growth rate is 3.27%.

To understand why this policy improves welfare, note that relative to the equilibrium level, research effort and employment both need to be increased in order to raise welfare. This can certainly be done by shifting resources from the workers to the firms, with a larger subsidy given to the high-productivity firms. As the firms age, however, this subsidy is curtailed until it eventually becomes a tax. Since the reward to being a new firm-owner is so high, this raises the level of research (z). But taxing owners of low productivity firms will raise the level of destruction, as measured by either N or $\left(1/\hat{T}\right)$.

It is of interest to assess the welfare improvement from such a policy. Relative to the benchmark, the increase in utility from the tax/subsidy policy is a welfare increase of 1.6%. Since utility is linear in consumption, it seems appropriate to view this as equivalent to an increase of 1.6% in initial consumption for all agents. Note that because this welfare improvement derives from taxing relatively poor workers and firm owners, and transferring subsidies to richer owners of young firms, this results in a substantial increase in inequality.

Example 10 Now consider the very same parameterization as in the previous example, but now let $\mu = .05$. In this case, with no taxation the equilibrium growth rate is 1.36%. The solution to the problem of maximizing welfare with the system of non-linear taxes, described above, results in a growth rate of 1.30%, so the equilibrium growth rate is too high. In this equilibrium there is too much of research, and also too much employment (or firm destruction) in equilibrium.

Figure 6 shows the implied tax and subsidy policies that result from this constrained planning problem. In this case welfare can be raised by having the government impose a labor subsidy, or negative tax, of 4.6%. The tax on firms, shown in the figure ranges from -3.3% on the owners of the low productivity firms, to a tax of 17% on the owners of the high productivity firms. As can be seen in the figure, this tax scheme is not linear, and has a slightly concave feature. Such a tax scheme certainly reduces the amount of research effort, since the benefit of being a firm-owner is reduced. Similarly, the subsidy to low-productivity firms helps raise the overall number of firms, and hence lowers the level of firm destruction (N or $\left(1/\hat{T}\right)$).

The welfare increase resulting from this system of taxes and subsidies, relative to the equilibrium is 0.25%. Because this welfare improvement derives from subsidizing relatively poor workers and firm owners, and taxing richer owners of young firms, this reduces inequality.

These examples are instructive for several reasons. First, suppose the welfare-enhancing tax policies resulting from this last example were imposed on such an economy. An independent observer of this economy would see that the government is certainly imposing a distortional tax/transfer policy between firms that certainly looks like the government is "picking winners and losers".[26] Not only that, but this policy would *reduce* the growth rate. All of this is true, but it results from the government trying to maximize welfare. The reason this policy improves welfare is that the planner recognizes that the level of research, as well as the rate of firm exit (or destruction) are decisions that need to be altered.

Additionally, this last example illustrates other novel features. In most models with intertemporal spillovers for research, the optimal policy is to subsidize research to take advantage of this externality. However, in this last example there is such a spillover, but nevertheless it is welfare-enhancing to *reduce research*. What is missing from other models in the existing literature is that they do not have an autonomous and endogenous destruction

[26] But since such a government policy is known in advance, it no more constitutes "picking winners and losers" than does a progressive or regressive tax code.

(or firm-exit) decision. In this last example the planner is using this feature, but reducing the amount of destruction, and to some extent this offsets the reduction in research, and changes the incentive to engage in research. This example shows that by ignoring the endogenous exit behavior of firms, or omitting the destruction feature, much of the existing literature is ignoring an important feature that contributes to the incentives for innovation and growth.

Another noteworthy feature of these examples is that the welfare improvements are not linked only to the growth rate, in spite of the fact that preferences are linear in consumption. In the first example, the non-linear taxes raise the growth rate from 3% to 3.27%, and this results in a welfare benefit of 1.6%. In the second example, the non-linear taxes *lower* the growth from 1.36% to 1.3%, and this raises welfare by .25%. We are accustomed to assuming that there are substantial welfare benefits from raising the growth rate. These examples show that these benefits may be much different than previously thought.

8. Final Remarks

It is an accepted fact that a growing economy is organic in nature, and exhibits a continual birth and mortality of products and technologies. Yet most studies of economic growth fail to model the separate decisions that give rise to these distinctive phenomena, and therefore cannot assess whether these decisions are made optimally.

Integral to the study of optimal growth is the determination of the appropriate incentives for agents to seek innovations of new technologies. Some of these incentives reflect the ability for innovators to capture some of the market share, or resources of older incumbents. This frequently means that the innovation process leads to the eventual termination of older technologies. It can then be a mistaken step of logic to conclude that the destruction of older technologies is an unfortunate by-product of innovation. The analysis presented here uses a simple model to show why this is not the case, and instead both the creation and the destruction effects have mutually beneficial *and* detrimental effects. The study of optimal growth, and the development of the optimal incentives to obtain this growth rate, must weigh the different impacts of these autonomous decisions.

Much of the existing literature focuses on developing the proper incentives for innovation alone, in determining the optimal growth rate. What this literature ignores is that it is equally important to provide the proper incentives for the optimal retirement or exit of older firms or technologies, since the exit and innovation decisions are interrelated. This analysis also suggests that the ideal government policy in this model may be quite different from that is most existing growth models. There may be good reasons for imposing tax or subsidies that depend on the productivity (or profit) of the firm, in order to provide the correct incentives for innovation or exit. Also, the presence of an intertemporal spillover need not necessarily imply that there is too little innovation (and growth) in equilibrium.

References

Acemoglu, D., and Cao, D., 2015. Innovation by entrants and incumbents. Journal of Economic Theory 157, pp. 255-294.

Aghion, P., and Howitt, P., 1992. A model of growth through creative destruction. Econometrica 60, pp. 323-351.

Cao, D., and Luo, W., 2017. Persistent heterogeneous returns and top end wealth inequality. Review of Economic Dynamics 26, pp. 301-326.

Decker, R., Haltiwanger, J., Jarmin, R., and Miranda, J., 2014. The role of entrepreneurship in US job creation and economic dynamism. Journal of Economic Perspectives, 28 (3), pp. 3-24.

Greenwood, J., and Jovanovic, B., 1990. Financial development, growth, and the distribution of income. Journal of Political Economy 98 (5), pp. 1076-1107.

Grossman, G. M., and Helpman, E., 1991a. Innovation and growth in the global economy. Cambridge MA: MIT Press.

Grossman, G. M., and Helpman, E., 1991b. Quality ladders in the theory of growth. Review of Economic Studies 58 (1), pp. 43-61.

Huffman, G. W., 2018. An analysis of the importance of both destruction and creation to economic growth. Manuscript, Vanderbilt University.

Jaimovich, N., and Rebelo, S., 2017. Nonlinear effects of taxation on growth. Journal of Political Economy 125 (1), pp. 265-291.

Jones, C. I., and Kim, J., 2018. A schumpeterian model of top income inequality. Journal of Political Economy 126 (5), pp. 1785-1829.

Isaacson, W., 2015. The innovators: how a group of hackers, geniuses, and geeks created the digital revolution. Simon & Schuster.

Klette, J., and Kortum, S., 2004. Innovating firms and aggregate innovation. Journal of Political Economy 112 (5), pp. 986-1018.

Luttmer, E. G. J., 2011. On the mechanics of firm growth. Review of Economic Studies 78, pp. 1042-1068.

Luttmer, E. G. J., 2007. Selection, growth, and the size distribution of firms. Quarterly Journal of Economics 122 (3), pp. 1103-1144.

Luttmer, E. G. J., 2012. Technology diffusion and growth. Journal of Economic Theory 147, pp. 602-622.

Stokey, N. L., and Rebelo, S., 1995. Growth effects of flat taxes. Journal of Political Economy 103 (3), pp. 519-550.

Figure 1

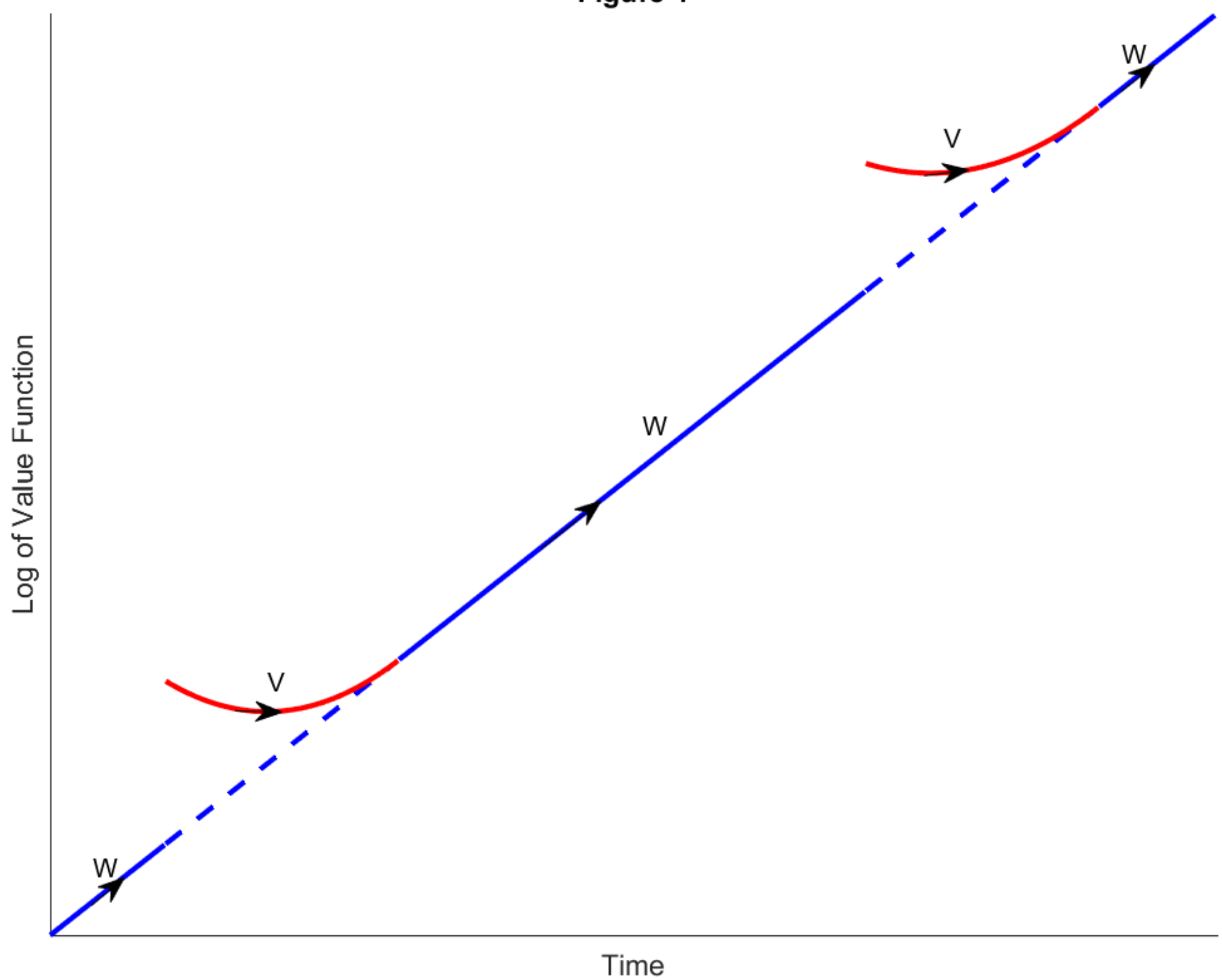


Figure 2

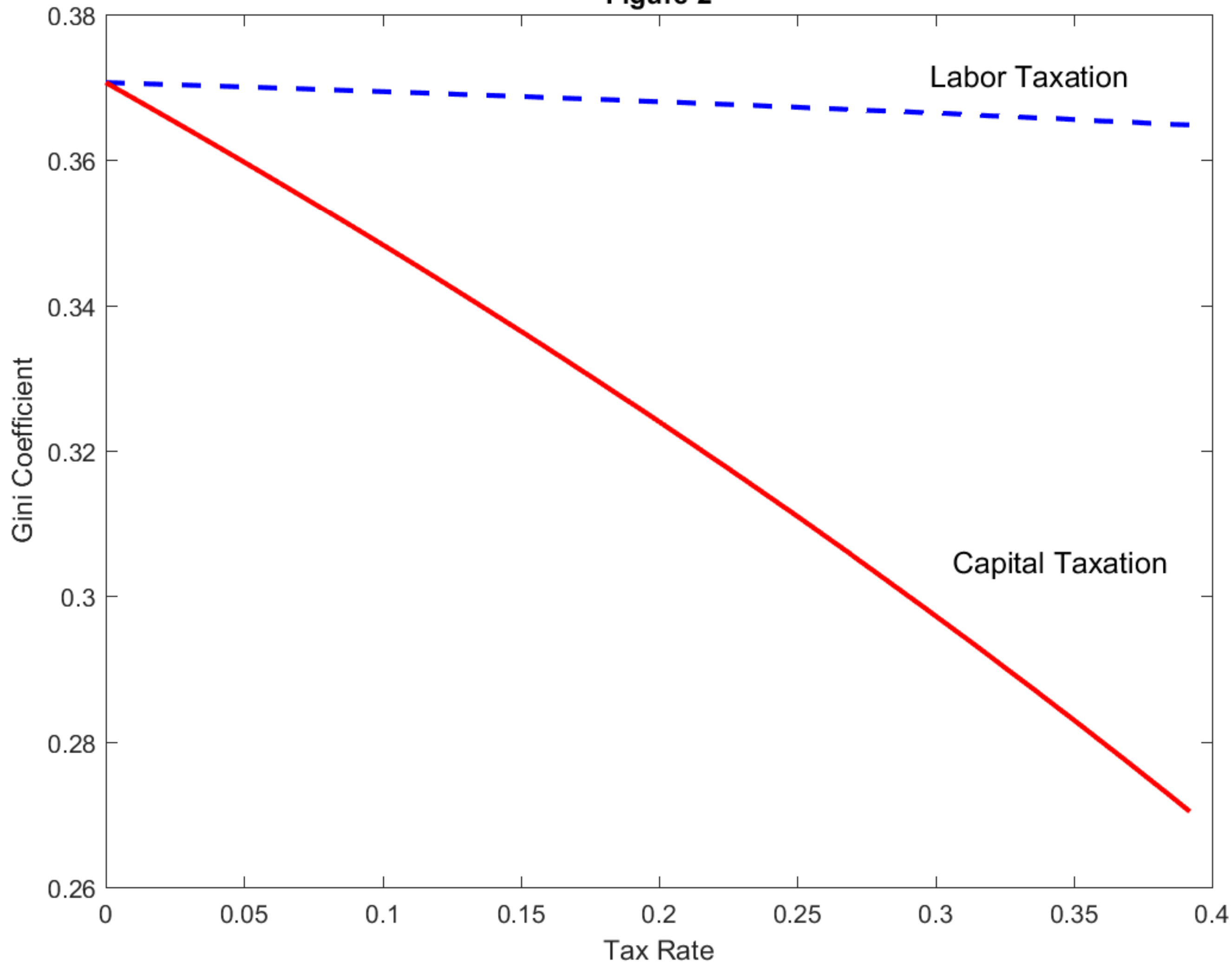

0.38
0.36
0.34
0.32
0.3
0.28
0.26
Gini Coefficient
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
Tax Rate
Labor Taxation
Capital Taxation

Figure 3a

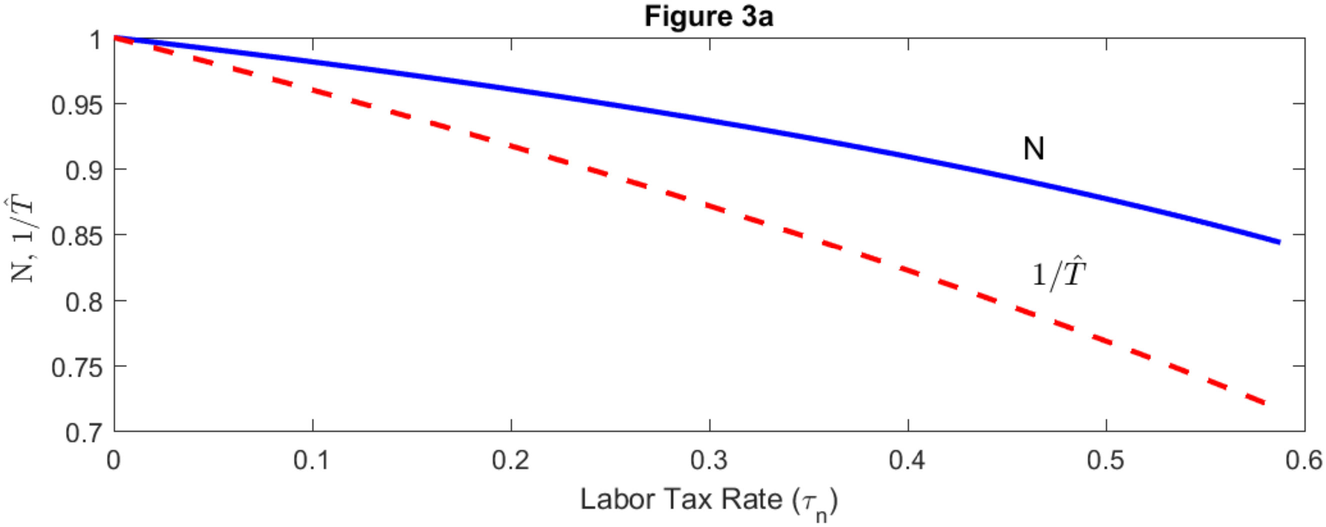
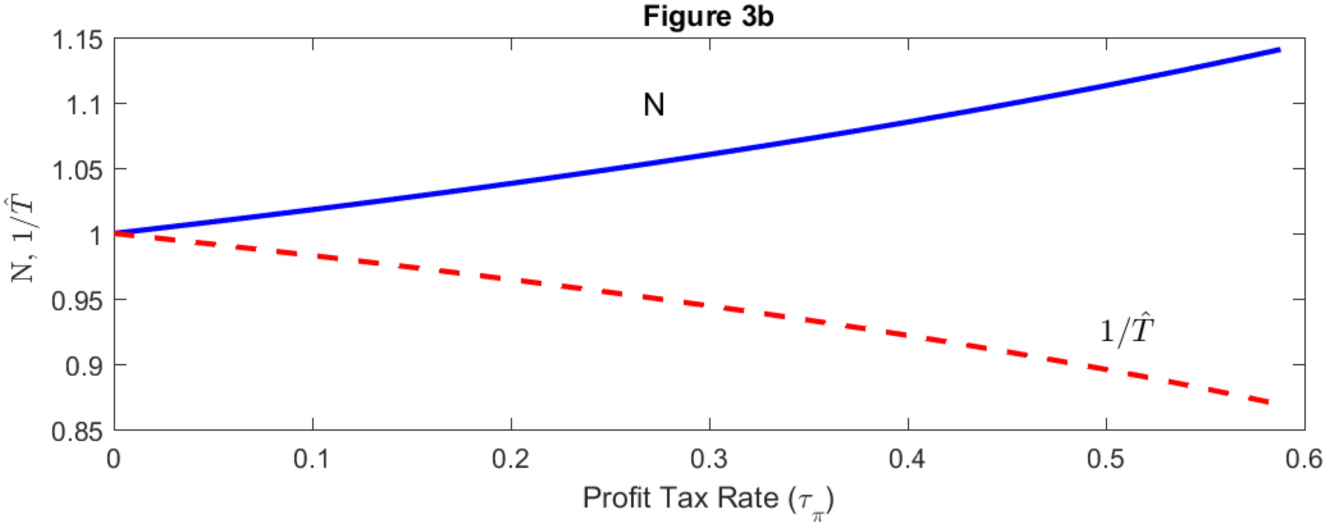


Figure 3b



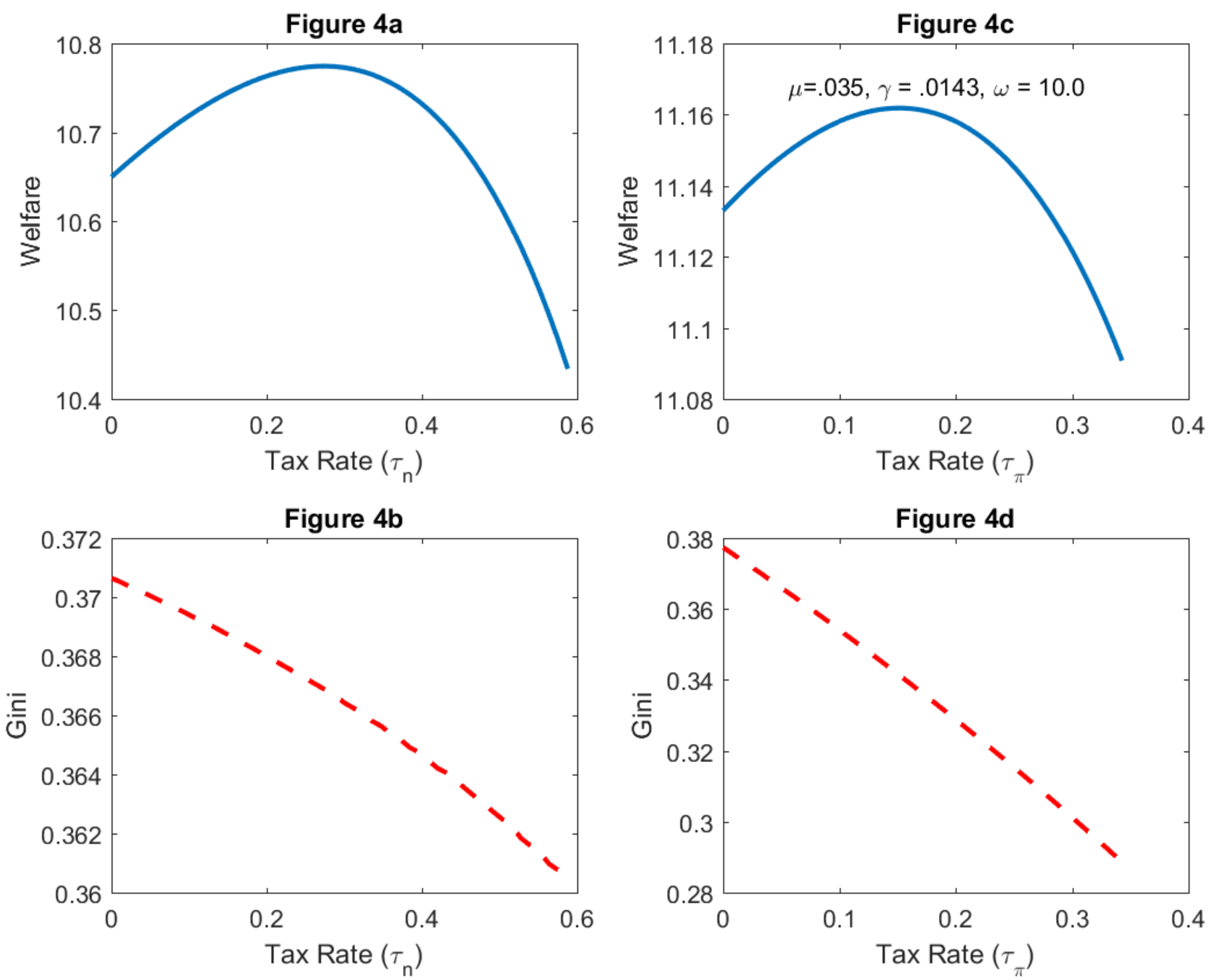
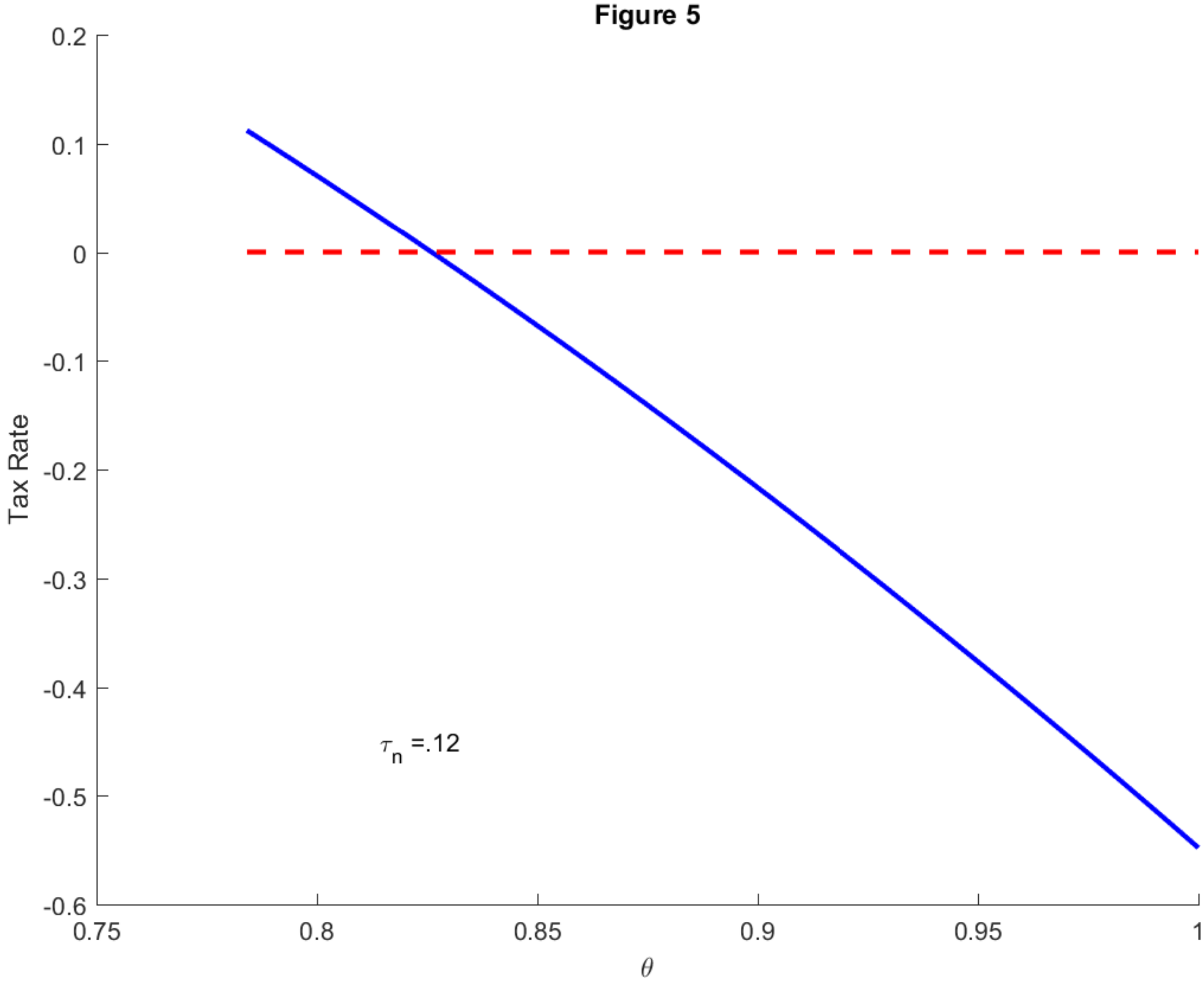


Figure 4a
Welfare
Tax Rate (τ_n)
10.8
10.7
10.6
10.5
10.4
0
0.2
0.4
0.6
Figure 4c
μ=.035, γ = .0143, ω = 10.0
Welfare
Tax Rate (τ_π)
11.18
11.16
11.14
11.12
11.1
11.08
0
0.1
0.2
0.3
0.4
Figure 4b
Gini
Tax Rate (τ_n)
0.372
0.37
0.368
0.366
0.364
0.362
0.36
0
0.2
0.4
0.6
Figure 4d
Gini
Tax Rate (τ_π)
0.38
0.36
0.34
0.32
0.3
0.28
0
0.1
0.2
0.3
0.4

Figure 5



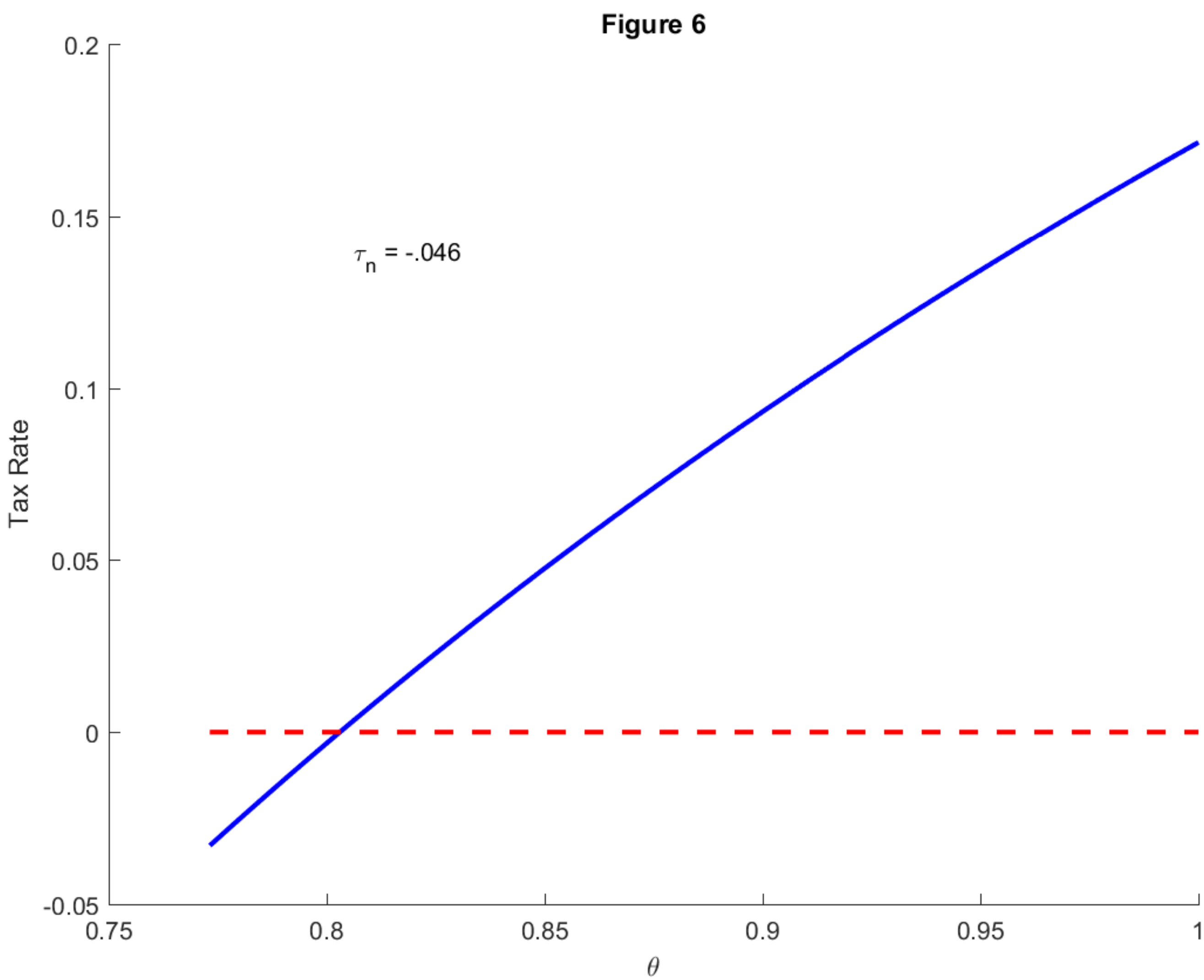


Figure 6
0.2
0.15
0.1
0.05
0
-0.05
Tax Rate
τ_n = -.046
0.75
0.8
0.85
0.9
0.95
1
θ

Appendix to "An Analysis of the Importance of Both Destruction and Creation to Economic Growth"

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March, 2019

1 Introduction

This appendix shows the proofs for the statements or propositions in the main body of the current version of the paper. First, an encapsulation of the main equations of the model are presented. Later, an analysis of the value function of the firm-owner will be presented.

2 Main Equations of the Model

The main equations that characterize the steady-state balanced growth path the model are as follows:

$$1 - N = \int_{\underline{\theta}}^{1} \left(\frac{1}{\theta}\right) d\theta = -\ln\left(\underline{\theta}\right). \tag{1}$$

$$A_w = \alpha \left[\frac{1}{N}\int_{\underline{\theta}}^{1} (\theta)^{\frac{1}{1-\alpha}} f_\theta(\theta)\, d\theta\right]^{1-\alpha} = \alpha\left[\left(\frac{1-\alpha}{N}\right)\left(1-\underline{\theta}^{\frac{1}{1-\alpha}}\right)\right]^{1-\alpha}. \tag{2}$$

$$A_\pi = (1-\alpha)\left[\frac{1}{N}\int_{\underline{\theta}}^{1} (\theta_t)^{\frac{1}{1-\alpha}} f_\theta(\theta)\, d\theta\right]^{-\alpha} = (1-\alpha)\left[\left(\frac{1-\alpha}{N}\right)\left(1-\underline{\theta}^{\frac{1}{1-\alpha}}\right)\right]^{-\alpha}. \tag{3}$$

$$V(\theta) = v_1(\theta)^{\frac{1}{1-\alpha}} + v_2(\theta)^{-(r/g)+1} \tag{4}$$

where

$$v_1 = \frac{A_\pi}{r + \left(\frac{\alpha g}{1-\alpha}\right)}, \text{ and } v_2 = \left[W - v_1\left[(\underline{\theta})^{\left(\frac{1}{1-\alpha}\right)}\right]\right](\underline{\theta})^{(1/g)(r-g)} > 0. \tag{5}$$

$$W = \frac{A_w - h(z^*) + \mu(z^*)V(1)}{[r - g + \mu(z^*)]}. \tag{6}$$

$$h'(z) = \mu'(z)\left[V(1) - W\right] \tag{7}$$

$$A_\pi(\underline{\theta})^{\left(\frac{1}{1-\alpha}\right)} = (r-g)W. \tag{8}$$

$$g = \frac{\dot{\bar{\lambda}}_t}{\bar{\lambda}_t} = N\mu(z). \tag{9}$$

3 Statement of Propositions and Proofs

Proposition 1 *The function $V(\theta)$, from equation (4), consists of two terms, one of which is increasing in θ while the other is decreasing. Nevertheless, it is possible to establish that $\frac{dV(\theta)}{d\theta} > 0$, for $\underline{\theta} < 1$, and $\frac{dV(\theta)}{d\theta} \to 0$, as $\underline{\theta} \to 1$.*

Proof.

$$\frac{\partial V(\theta)}{\partial \theta} = \left(\frac{v_1}{1-\alpha}\right)(\theta)^{\frac{1}{1-\alpha}-1} + \left(-(r/g)+1\right) v_2 (\theta)^{-(r/g)} \tag{10}$$

However, note that from equation (5) we have

$$v_2 = \left[W - v_1 \left[(\underline{\theta})^{\left(\frac{1}{1-\alpha}\right)}\right]\right] (\underline{\theta})^{(1/g)(r-g)}$$

But equation (8) says

$$W = \frac{A_\pi (\underline{\theta})^{\left(\frac{1}{1-\alpha}\right)}}{r-g}.$$

Using the last two equations it is possible to establish the following:

$$\begin{aligned}
v_2 &= \left[\frac{A_\pi (\underline{\theta})^{\left(\frac{1}{1-\alpha}\right)}}{r-g} - \left[\frac{A_\pi}{r+\left(\frac{\alpha g}{1-\alpha}\right)}\right]\left[(\underline{\theta})^{\left(\frac{1}{1-\alpha}\right)}\right]\right](\underline{\theta})^{(1/g)(r-g)} \\
&= A_\pi (\underline{\theta})^{\left(\frac{1}{1-\alpha}\right)} \left[\frac{1}{r-g} - \left[\frac{1}{r+\left(\frac{\alpha g}{1-\alpha}\right)}\right]\right](\underline{\theta})^{(1/g)(r-g)} \\
&= A_\pi (\underline{\theta})^{\left(\frac{1}{1-\alpha}\right)} \left[\frac{g}{(1-\alpha)\left(r+\left(\frac{\alpha g}{1-\alpha}\right)\right)(r-g)}\right](\underline{\theta})^{(1/g)(r-g)} \\
&= \frac{A_\pi}{\left(r+\left(\frac{\alpha g}{1-\alpha}\right)\right)}\left[\frac{g}{(1-\alpha)(r-g)}\right](\underline{\theta})^{(1/g)(r-g)}(\underline{\theta})^{\left(\frac{1}{1-\alpha}\right)} \\
&= v_1\left[\frac{g}{(1-\alpha)(r-g)}\right](\underline{\theta})^{(1/g)(r-g)}(\underline{\theta})^{\left(\frac{1}{1-\alpha}\right)}
\end{aligned} \tag{11}$$

Now using the expression for v_2 from equation (11) in equation (10) we have

$$\begin{aligned}
\frac{\partial V(\theta)}{\partial \theta} &= \left(\frac{v_1}{1-\alpha}\right)(\theta)^{\frac{1}{1-\alpha}-1} + \left(-(r/g)+1\right) v_1 \left[\frac{g}{(1-\alpha)(r-g)}\right](\underline{\theta})^{(1/g)(r-g)}(\underline{\theta})^{\left(\frac{1}{1-\alpha}\right)}(\theta)^{-(r/g)} \\
&= \left(\frac{v_1}{1-\alpha}\right)(\theta)^{\frac{\alpha}{1-\alpha}} - v_1\left[\frac{1}{(1-\alpha)}\right](\underline{\theta})^{(1/g)(r-g)}(\underline{\theta})^{\left(\frac{1}{1-\alpha}\right)}(\theta)^{-(r/g)} \\
&= \left(\frac{v_1}{1-\alpha}\right)\left[(\theta)^{\frac{\alpha}{1-\alpha}} - (\underline{\theta})^{(1/g)(r-g)}(\underline{\theta})^{\left(\frac{1}{1-\alpha}\right)}(\theta)^{-(r/g)}\right] \\
&= \left(\frac{v_1}{1-\alpha}\right)(\theta)^{\frac{\alpha}{1-\alpha}}\left[1 - (\underline{\theta})^{(1/g)(r-g)}(\underline{\theta})^{\left(\frac{1}{1-\alpha}\right)}(\theta)^{-(r/g)}(\theta)^{\frac{-\alpha}{1-\alpha}}\right] \\
&= \left(\frac{v_1}{1-\alpha}\right)(\theta)^{\frac{\alpha}{1-\alpha}}\left[1 - \left(\frac{\underline{\theta}}{\theta}\right)^{(r/g)+\frac{\alpha}{1-\alpha}}\right].
\end{aligned}$$

Since$\left(\frac{\underline{\theta}}{\theta}\right) \leq 1$, this last expression is non-negative, but it is negative when $\theta = \underline{\theta}$, and so $V'(\theta) > 0$. ■

Proposition 2 *From equation (4) it is possible to establish the following:*

$$\frac{\partial V(1)}{\partial v_1}\frac{\partial v_1}{\partial g} < 0, \textit{ and } \frac{\partial V(1)}{\partial g} > 0.$$

Proof. Obvious from equations 4 and 5. ■

Proposition 3 *Equations (4) and (6) imply that*

$$\frac{\partial (V(1) - W)}{\partial A_\pi} > 0, \frac{\partial (V(1) - W)}{\partial A_w} \leq 0,$$

with the latter derivative holding strictly when $\underline{\theta} < 1$.

Proof. From equation (6) it is easy to see that

$$\frac{\partial W}{\partial A_\pi} = \left(\frac{\mu}{r-g+\mu}\right)\frac{\partial W}{\partial A_\pi} < \frac{\partial W}{\partial A_\pi}$$

which then implies that

$$\frac{\partial V(1)}{\partial A_\pi} - \frac{\partial W}{\partial A_\pi} > 0.$$

Hence a marginal increase in A_π raises $(V(1) - W)$, which then raises the return to innovation, resulting in a higher value of z. Now note that repeated substitution of equation (6) into (4) yields the following

$$V(1) = \frac{\left[\frac{A_\pi}{r+\left(\frac{\alpha g}{1-\alpha}\right)}\right]\left[1 - (\underline{\theta})^{\left(\frac{r}{g}+\left(\frac{\alpha}{1-\alpha}\right)\right)}\right] + (\underline{\theta})^{(1/g)(r-g)}\frac{[A_w - h(z^*)]}{[r-g+\mu(z^*)]}}{1 - \left(\frac{\mu(z^*)}{[r-g+\mu(z^*)]}\right)(\underline{\theta})^{(1/g)(r-g)}} \tag{12}$$

From equation (6) it is easy to see that the change in A_w, operating through both W and $V(1)$ yields the following

$$\frac{\partial W}{\partial A_w} = \left(\frac{1}{r-g+\mu}\right) + \left(\frac{\mu}{r-g+\mu}\right)\frac{\partial V(1)}{\partial A_w}. \tag{13}$$

Equation (12) implies that

$$\frac{\partial V(1)}{\partial A_w} = \frac{(\underline{\theta})^{(1/g)(r-g)}\frac{1}{[r-g+\mu]}}{1 - \left(\frac{\mu}{[r-g+\mu]}\right)(\underline{\theta})^{(1/g)(r-g)}}$$

For now let us abbreviate $\mu(z^*)$ as μ. From equation (13) we then have

$$\begin{aligned}
\frac{\partial V(1)}{\partial A_w} - \frac{\partial W}{\partial A_w} &= \left(1 - \frac{\mu}{r-g+\mu}\right)\frac{\partial V(1)}{\partial A_w} - \left(\frac{1}{r-g+\mu}\right) \\
&= \left[(r-g)\frac{\partial V(1)}{\partial A_w} - 1\right]\left(\frac{1}{r-g+\mu}\right) \\
&= \left[(r-g)\left[\frac{(\underline{\theta})^{(1/g)(r-g)}\frac{1}{[r-g+\mu]}}{1 - \left(\frac{\mu}{[r-g+\mu]}\right)(\underline{\theta})^{(1/g)(r-g)}}\right] - 1\right]\left(\frac{1}{r-g+\mu}\right) \\
&= \left[\frac{1}{1 - \left(\frac{\mu}{[r-g+\mu]}\right)(\underline{\theta})^{(1/g)(r-g)}}\right]\left(\frac{1}{r-g+\mu}\right) \cdot \\
&\quad \left[\left[(\underline{\theta})^{(1/g)(r-g)}\frac{(r-g)}{[r-g+\mu]}\right] - 1 + \left(\frac{\mu}{[r-g+\mu]}\right)(\underline{\theta})^{(1/g)(r-g)}\right] \\
&= \left[\frac{1}{1 - \left(\frac{\mu}{[r-g+\mu]}\right)(\underline{\theta})^{(1/g)(r-g)}}\right]\left(\frac{1}{r-g+\mu}\right)\left[-1 + (\underline{\theta})^{(1/g)(r-g)}\right] \leq 0.
\end{aligned}$$

since $(\underline{\theta}) \in (0,1)$. Hence a marginal increase in A_w lowers $(V(1) - W)$, which then raises the return to innovation, resulting in a higher value of z. ■

Proposition 4 *The ratio of incomes (A_π / A_w) is increasing in N (or equivalently $\underline{\theta}$). Additionally, along a balanced growth path, if $h(0) = 0$, and $h'(0) = 0$, then $A_\pi (\underline{\theta})^{\left(\frac{1}{1-\alpha}\right)} > A_w$.*

Proof. Taking the ratio of equations (2) and (3) yields

$$\frac{A_\pi}{A_w} = \frac{N}{\alpha} \left[\frac{1}{1 - \underline{\theta}^{\frac{1}{1-\alpha}}} \right].$$

Obviously this is increasing in N or $\underline{\theta}$. Also, equations (6) and (7) imply that

$$\begin{aligned} A_\pi (\underline{\theta})^{\left(\frac{1}{1-\alpha}\right)} &= (r - g) W \\ &= A_w + [\mu(z^*) (V(1) - W) - h(z^*)]. \end{aligned}$$

But since $h(0) = 0$, and $h'(0) = 0$, then the term in square brackets must be positive, which establishes the result. ■

Proposition 5 *If $h(0) = 0$, $h'(0) = 0$, $h'' > 0$, and $\mu'(0) = 0$then on a balanced growth path $g > 0$.*

Proof. Suppose $g = 0$. Then either $z = 0$, or $N = 0$. Suppose the former is true. Then equation (7) implies that $V(1) = W$. Equations (4) and (5) imply that $\underline{\theta} = 1$, which implies that $N = 1$. But this implies that $W = \frac{A_w}{r} < \infty$, while $\lim_{N \nearrow 1} \left(V = \frac{A_\pi}{r}\right) = +\infty$, which is a contradiction. Similarly, if $N = 0$, and $\underline{\theta} = e^{-1}$,this would imply that $V < +\infty$,while $W = +\infty$, because $A_w = +\infty$ This means that many firm-owners could improve their utility by shutting down their firms and becoming workers. But this necessitates having $N > 0$. ■

Proposition 6 *If $h'(0) > h(0) = 0$, or if $\mu'(0) = 0$, then there may exist an equilibrium in which $g = 0$. In this case if $h'(0) > \mu'(0)[V - W]$ then the marginal cost of engaging in research (z) can be greater than the benefit, and hence no research takes place $(z = 0)$, even if $V - W > 0$, and so $g = 0$. In this case the system has 6 equations in 6 unknowns (with $z = g = 0$).*

Proof. If $h'(0) > 0$, then there may exist an equilibrium in which $g = 0$.[1] In this case if

$$h'(0) > \mu'(0)[V - W] \tag{14}$$

then the marginal cost of engaging in research (z) can be greater than the benefit, and so no research takes place $(z = 0)$, even if $V - W > 0$, and so $g = 0$. In this case the firm owners will never shut down their firm and so equation (8) holds as follows:

$$A_\pi (\underline{\theta})^{\left(\frac{1}{1-\alpha}\right)} \geq rW. \tag{15}$$

If this last equation holds with equality then the marginal firm owner has utility just equal to that of a worker. If it holds with an inequality, he is better off than a worker. In this case there can be an interval or a continuum of possible values of N or $(\underline{\theta})$ for which there can exist an equilibrium without growth. Workers have the following value function:

$$W = \frac{A_w}{r}$$

and

$$V = \frac{A_\pi}{r}.$$

In this case the system has 6 equations in 6 unknowns (with $z = g = 0$). ■

[1] The same sort of "no-growth" equilibria can exist if, for example, $\mu'(0) = 0$,but $\mu'(z) > 0$ for some $z > 0$.

Proposition 7 *If, in addition to the conditions of this no-growth equilibrium, it is the case that* $A_\pi\left(\underline{\theta}\right)^{\left(\frac{1}{1-\alpha}\right)} > rW$*, then there are a continuum of equilibria with* $g = z = 0$*, with different values of* N*,* $(\underline{\theta})$*,* A_w*,* A_π*,* V*,* W*.*

Proof. Assume equation (15) holds with strict inequality. From the existing equilibria, consider an arbitrarily small decrease in the value of N, and therefore a slightly smaller value of $\underline{\theta}$. This will result in a slight decrease in A_π and a rise in A_w. This will result in a slightly lower the value of $(V - W)$. This lowers the left side of equation (15) and raises the right side. However, for sufficiently small changes in N, equations (14) and (15) will still hold with an inequality. Another way to see this is that if equations (14) and (15) hold strictly, then equations (1), (2), (3), (4), (6) are five equations in 6 unknowns (A_w, A_π, V, W, $\underline{\theta}$, N, with $z = g = 0$. In other words, from an initial equilibrium, with equations (14) and (15) holding strictly, for some equilibrium levels of variables, one could vary the values of N and $\underline{\theta}$ slightly, resulting in small changes in A_w, A_π, V, W, through the necessary equations, while still getting equations (14) and (15) to hold. Keep in mind that equation (15) means that the marginal firm (i.e. the one with the worst technology, and therefore the lowest profit) has an income that is higher than the wage of a worker. In economic terms what this means is that from an initial equilibrium, one could then lower N and $\underline{\theta}$ slightly. This would also raise W slightly also. This would have a marginal effect on A_w, A_π, V, W, $\underline{\theta}$, N, but equations (14) and (15) would still hold, and so $z = g = 0$. At the point at which equation (15) holds with equality, then you cannot lower N any further. However, having equation (15) hold with equality does not imply growth ($g > 0$), it only implies that equilibrium wage equals the profit from the least-profitable firm. It is not possible for equation (14) to hold with equality while equation (15) to hold with inequality. The reason is that this would imply that there is growth in the wages, and so the value function of the worker is rising, while the profitability of the marginal firm is falling, while not shutting down. These facts are not compatible. ■

Proposition 8 *The length of time that a firm is operational is calculated as follows:* $\hat{T} = \frac{-\ln(\underline{\theta})}{g} = \frac{1-N}{g}$*. The average time it takes the worker to cycle through from initially becoming a worker, to becoming a firm-owner, and finally shutting it down, is* $T = \frac{1}{g}$*.*

Proof. Now for a firm with a fixed value of λ, when the best technology is $\overline{\lambda}_t$, it is the case that $\theta_t = \lambda/\overline{\lambda}_t$. Using the fact that $\left(\dot{\theta}/\theta\right) = -g$, this implies that since θ_t starts out at 1, and falls to $\underline{\theta}$, and so the lifespan of a firm ($\hat{T}$) must satisfy the following:

$$e^{-g\hat{T}} = \underline{\theta}.$$

This in turn implies that the length of time that a firm is operational is calculated as follows:

$$\hat{T} = \frac{-\ln\left(\underline{\theta}\right)}{g} = \frac{1-N}{g}. \tag{16}$$

However, g, N and $\underline{\theta}$ are all functions of the parameters, and the policy variables in the economy. Let us use the short-hand notation of $\mu(z) = \mu z$. First, let us establish the expected waiting time for each worker to find an innovation. In this case the probability distribution over waiting a length of time s for an innovation is written as

$$F(s) = 1 - e^{-\mu z s}.$$

For a worker, the expected time to an innovation is then written as

$$E(s) = \int_0^\infty s\mu z e^{-\mu z s} ds = 1/\mu z.$$

Equation (16) shows the average amount of time a worker spends in the workforce. Adding these two quantities together delivers the average amount of time an agent will spend in the two activities:

$$T = \frac{-\ln\left(\underline{\theta}\right)}{g} + \left(\frac{1}{\mu z}\right).$$

Using equation (9), we then obtain the necessary expression. ■

The degree of income mobility can then be measured as the inverse of the average time to cycle through these two activities:

$$\frac{1}{T} = g. \tag{17}$$

4 Analysis of the Value Function of a Firm-Owner

Here it is shown that how to characterize the value function of a firm-owner in the model. First, note that since $\frac{d}{dt}\log\left(\overline{\lambda}_t\right) = g$, and $\pi\left(\theta_t\right) = A_\pi\left(\theta_t\right)^{\frac{1}{1-\alpha}}$. As described in the text, since the profit functions, the wage function, and the function $h(z,\overline{\lambda}_t)$ are all homogenous of degree one in $\overline{\lambda}_t$, it follows that the value functions for the optimization problems will then be homogeneous as well. It follows that the value function for a firm-owner with relative technology (θ_t) can be written as follows

$$V\left(\theta_t\right)\left(\bar{\lambda}_t\right) = \int_t^T e^{-r(s-t)}\left(\left(\bar{\lambda}_t\right)e^{g(s-t)}\right)A_\pi\left(\theta_s\right)^{\frac{1}{1-\alpha}}ds + e^{-r(T-t)}W\left(\bar{\lambda}_t e^{g(T-t)}\right), \tag{18}$$

and so dividing by $\bar{\lambda}_t$ results in

$$V\left(\theta_t\right) = \int_t^T e^{-r(s-t)}\left(e^{g(s-t)}\right)A_\pi\left(\theta_s\right)^{\frac{1}{1-\alpha}}ds + e^{-r(T-t)}W\left(e^{g(T-t)}\right). \tag{19}$$

Since $\left(\frac{\dot{\theta}}{\theta}\right) = -g$, this last expression can be written as

$$V\left(\theta_t\right) = \frac{A_\pi\left(\theta_t\right)^{\frac{1}{1-\alpha}}}{r+\left(\frac{\alpha g}{1-\alpha}\right)}\left[1-e^{-(T-t)\left(r+\left(\frac{\alpha g}{1-\alpha}\right)\right)}\right] + W\left(e^{(-r+g)(T-t)}\right), \tag{20}$$

where T is the exit date for the firm. By choosing this date T optimally, this yields the following exit condition:

$$A_\pi\left(\theta_T\right)^{\frac{1}{1-\alpha}} = (r-g)\,W. \tag{21}$$

The remaining lifetime of a firm with relative technology (θ_t) must satisfy

$$T - t = \frac{\ln\left(\theta_t\right) - \ln\left(\underline{\theta}\right)}{g}.$$

In general for a firm with relative technology (θ) equation (20) must then satisfy

$$V\left(\theta\right) = \frac{A_\pi}{r+\left(\frac{\alpha g}{1-\alpha}\right)}\left[(\theta)^{\frac{1}{1-\alpha}}\right] + W\left(\frac{\theta}{\underline{\theta}}\right)^{-(1/g)(r-g)} - \frac{A_\pi}{r+\left(\frac{\alpha g}{1-\alpha}\right)}\left[(\underline{\theta})^{\left(\frac{r}{g}+\left(\frac{\alpha}{1-\alpha}\right)\right)}\right](\theta)^{-(r/g)+1} \tag{22}$$

which then can be written as

$$V\left(\theta\right) = v_1\left(\theta\right)^{\frac{1}{1-\alpha}} + v_2\left(\theta\right)^{-(r/g)+1}$$

where for any $\theta \in [\underline{\theta}, 1]$

$$\begin{aligned} v_1 &= \frac{A_\pi}{r+\left(\frac{\alpha g}{1-\alpha}\right)} \\ v_2 &= \left[W - v_1\left[(\underline{\theta})^{\left(\frac{1}{1-\alpha}\right)}\right]\right](\underline{\theta})^{(1/g)(r-g)} > 0. \end{aligned}$$

It is easy to see from equation (22) that the following *value matching condition* must hold

$$V\left(\underline{\theta}\right) = W. \tag{23}$$

The optimal exit condition (equation (21)) can also derived by choosing the optimal value of $(\underline{\theta})$. Maximizing the value function in equation (22) with respect to $(\underline{\theta})$ also leads to the condition in equation (21).

The *smooth-pasting* condition is derived by taking the derivative of equation (19) with respect to t, and evaluate the result at $t = T$.[2] Then, using equation (21) yields the fact that $\dot{V}_T = \dot{W}_T$. The latter functions are the technology-normalized value functions (i.e. divided by $\overline{\lambda}_t$). Figure (1) may be a little deceptive because this figure shows the actual value functions: $\overline{\lambda}_t V(\theta)$, and $\overline{\lambda}_t W$, rather than $V(\theta)$, and W. These functions are growing at the rate of g, when the two functions equal each other.

Another way to characterize the value function of a new firm-owner is to note that since $\left(\frac{\dot{\theta}}{\theta}\right) = -g$, for the case of a firm-owner with a new technology at $t = 0$, equation (19) can be written as follows:

$$\begin{aligned}
V\left(1\right) &= \int_0^\infty e^{-rs}\left(e^{gs}\right)A_\pi\left(\theta_s\right)^{\frac{1}{1-\alpha}}ds + e^{-rT}W\left(e^{-gT}\right) - \int_T^\infty e^{-rs}\left(e^{gs}\right)A_\pi\left(\theta_s\right)^{\frac{1}{1-\alpha}}ds \\
&= \int_0^\infty e^{-rs}\left(e^{gs}\right)A_\pi\left(e^{-gs}\right)^{\frac{1}{1-\alpha}}ds + e^{-rT}W\left(e^{-gT}\right) - \left(\theta_T\right)^{\frac{1}{1-\alpha}}\int_T^\infty e^{-rs}\left(e^{gs}\right)A_\pi\left(e^{-gs}\right)^{\frac{1}{1-\alpha}}ds \\
&= \int_0^\infty e^{-rs}A_\pi\left(e^{gs}\right)^{\frac{-\alpha}{1-\alpha}}ds + e^{-rT}W\left(e^{-gT}\right) - e^{-rT}\left(e^{gT}\right)\left(\theta_T\right)^{\frac{1}{1-\alpha}}\int_0^\infty e^{-rs}A_\pi\left(e^{gs}\right)^{\frac{-\alpha}{1-\alpha}}ds \\
&= \frac{A_\pi}{r + \left(\frac{\alpha g}{1-\alpha}\right)} + e^{-(r-g)T}\left[W - \frac{A_\pi\left(\theta_T\right)^{\frac{1}{1-\alpha}}}{r + \left(\frac{\alpha g}{1-\alpha}\right)}\right].
\end{aligned}$$

The first term on the right side of this last expression is the discounted value of profits from running the firm *forever*, given that the profits are falling at the rate of $\left(\frac{\alpha g}{1-\alpha}\right)$. Next, the term $e^{-(r-g)T}$, reflects that fact that at some future date T, which is chosen optimally, the firm will be shut down. At that date the firm will have relative technology denoted by θ_T. By shutting down the firm at that date the firm-owner will be giving up a future profit stream, the value of which is $\frac{A_\pi(\theta_T)^{\frac{1}{1-\alpha}}}{r+\left(\frac{\alpha g}{1-\alpha}\right)}$. But the firm-owner benefit from switching to becoming a worker because the value of doing so exceeds that of keeping the firm operational forever (i.e. $W > A_\pi\left(\theta_T\right)^{\frac{1}{1-\alpha}}$).

[2] Equivalently, one could take the derivative of equation (22) with respect to θ, and evaluate the result at $\theta = \underline{\theta}$.